Cartan operations and Weil algebras
Applications and noncommutative generalization

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Raymond Stora, le physicien, l'homme, l'ami (Memorial meeting),
CERN, Dec. 4, 2015

Cartan operations

$\mathfrak{g}$ Lie algebra, Ω d.g.a.

An operation of $\mathfrak{g}$ *in* Ω or *a* $\mathfrak{g}$*-operation in* Ω is a linear mapping

$$X \mapsto i_X$$

of $\mathfrak{g}$ into $\mathrm{Der}^{-1}_{\mathrm{gr}}(\Omega)$ such that

$$L_X = i_X d + d i_X \quad (\textit{Lie derivative})$$

satisfies

$$[i_X, L_Y] = i_{[X,Y]} \quad (\textit{Cartan relation})$$

$$\Rightarrow \begin{cases} [L_X, d] = 0 \\ [L_X, L_Y] = L_{[X,Y]} \end{cases}$$

$\alpha \in \Omega$ *invariant* $\Leftrightarrow L_X(\alpha) = 0 \quad \forall X \in \mathfrak{g}$

horizontal $\Leftrightarrow i_X(\alpha) = 0 \quad \forall X \in \mathfrak{g}$

basic $\Leftrightarrow$ invariant and horizontal

Classical example (principal bundle, etc.) $\Rightarrow$ terminology

Category of $\mathfrak{g}$-operations.

Algebraic connections (classical)

Operation of $\mathfrak{g}$ in Ω **graded commutative**
Algebraic connection

$$\alpha : \mathfrak{g}^* \to \Omega^1$$

such that $\forall X \in \mathfrak{g}, \theta \in \mathfrak{g}^*$

$$\begin{cases} i_X(\alpha(\theta)) & = & \theta(X) \\ L_X(\alpha(\theta)) & = & \alpha(\theta \circ \mathrm{ad}(X)) \end{cases}$$

$\Rightarrow \alpha : \wedge\mathfrak{g}^* \to \Omega$
Curvature of $\alpha = \varphi : \mathfrak{g}^* \to \Omega^2$

$$\varphi(\theta) = (d\alpha - \alpha d)(\theta)$$

i.e. obstruction for α to be a d.g.a.homomorphism

$$\Rightarrow \begin{cases} i_X(\varphi(\theta)) & = & 0 \\ L_X(\varphi(\theta)) & = & \varphi(\theta \circ \mathrm{ad}(X)) \end{cases}$$

Other formulation

$$A \in \mathfrak{g} \otimes \Omega^1 : \alpha(\omega) = (\omega \otimes I_{\Omega^1})A$$

$$\begin{cases} i_X(A) & = & X \\ L_X(A) & = & \text{ad}(X)A \end{cases}$$

$$F \in \mathfrak{g} \otimes \Omega^2 : \varphi(\omega) = (\omega \otimes I_{\Omega^2})F$$

$$\begin{cases} i_X(F) & = & 0 \\ L_X(F) & = & \text{ad}(X)F \end{cases}$$

$$F = dA + \frac{1}{2}[A, A]$$

Weil algebra $W(\mathfrak{g})$

$=$ Universal initial object in the category of $\mathfrak{g}$-operations with connections

$$W(\mathfrak{g}) = \wedge\mathfrak{g}^* \otimes S\mathfrak{g}^*$$

$$\text{graduation} : \wedge^m \mathfrak{g}^* \otimes S^n\mathfrak{g}^* \subset W^{m+2n}(\mathfrak{g})$$

$$\begin{array}{rcl} \alpha(\theta) & = & \theta \otimes \mathbb{1} \in W^1(\mathfrak{g}) \\ \varphi(\theta) & = & \mathbb{1} \otimes \theta \in W^2(\mathfrak{g}) \end{array}$$

$\exists$ unique diff. d such that $d\alpha(\theta) = \alpha(d\theta) + \varphi(\theta)$

Setting $i_X(\alpha(\theta)) = \theta(X), i_X(\varphi(\theta)) = 0$
($i_X(A) = X$, etc.) $\Rightarrow$ operation of $\mathfrak{g}$ in $W(\mathfrak{g})$ and α connection with curvature φ

Other presentation : generator $\alpha(\mathfrak{g}^*), d\alpha(\mathfrak{g}^*)$ i.e. $W(\mathfrak{g}) =$ Free graded commutative differential envelope of $\wedge\mathfrak{g}^*$

Cohomologies of $W(\mathfrak{g})$

Theorem

$$H^n(W(\mathfrak{g})) = H_I^n(W(\mathfrak{g})) = \delta^{0n}\mathbb{K}$$

Theorem

$$W_B^{2n+1}(\mathfrak{g}) = H_B^{2n+1}(W(\mathfrak{g})) = 0$$

$$W_B^{2n}(\mathfrak{g}) = H_B^{2n}(W(\mathfrak{g})) = \mathcal{I}^n(\mathfrak{g})$$
$= \textit{invariant polynomials of degree } n \textit{ on } \mathfrak{g}$

$\mathfrak{g}$-operation with connection in Ω
$\alpha \mapsto \alpha$ or $A \mapsto A \Rightarrow \text{can} : W(\mathfrak{g}) \to \Omega$ morphism (unique) of $\mathfrak{g}$-operations with connections

Weil homomorphism

Theorem

Let Ω be a $\mathfrak{g}$-operation with connection, there is a unique morphism of $\mathfrak{g}$-operation with connection can : $W(\mathfrak{g}) \to \Omega$.

Theorem

$\widetilde{can} : H_B(W(\mathfrak{g})) \to H_B(\Omega)$ *does not depend on the connection of* Ω. *It only depends on the* $\mathfrak{g}$*-operation*

$=$ *Weil homomorphism.*

Cartan map $\gamma : \mathcal{I}(\mathfrak{g}) \to \wedge_I \mathfrak{g}^*$

$\mathbb{1} \otimes P = dQ, Q \in W_I(\mathfrak{g});\ \ \rho : W(\mathfrak{g}) \to \wedge \mathfrak{g}^*$

$\rho(Q) \in \wedge_I \mathfrak{g}^*$ independent of the choice of $Q \Rightarrow \mathcal{I}^n(\mathfrak{g}) \to \wedge_I^{2n-1} \mathfrak{g}^*$.

Extension of L to $U(\mathfrak{g})$

Universal property of $U(\mathfrak{g}) \Rightarrow X \mapsto L_X$ extends uniquely as algebra-homomorphism $h \mapsto L_h$ of $U(\mathfrak{g})$ into $\mathrm{End}^0(\Omega)$.

Theorem

The canonical extension satisfies :
(a) $L_h d = d L_h, \quad \forall h \in U(\mathfrak{g})$
(b) $L_h(\mathbb{1}) = \varepsilon(h)\mathbb{1}, \quad \mathbb{1}$ *unit of* Ω
(c) $L_h(\alpha\beta) = \sum_i L_{h_i^{(1)}}(\alpha) L_{h_i^{(2)}}(\beta), \quad \Delta h = \sum_i h_i^{(1)} \otimes h_i^{(2)}.$
(d) $i_X L_h = \sum_i L_{h_i^{(1)}} i_{ad(h_i^{(2)})X}, \quad ad(h)g = \sum_i S(h_i^{(1)}) g h_i^{(2)}$

$$\Rightarrow \begin{cases} L_h d & = & d L_h \\ L_{h_1} L_{h_2} & = & L_{h_1 h_2} \\ L_{\mathbb{1}} & = & I_\Omega \\ L_h \circ m & = & m \circ (L \otimes L)_{\Delta h} \\ L_h(\mathbb{1}) & = & \varepsilon(h)\mathbb{1} \end{cases}$$

Extension of i to $U(\mathfrak{g})$?

There is no canonical extension of i to $U(\mathfrak{g})$ which can be used to define operations of general Hopf algebras, however, defining $\bar{\imath}_h$ for $h \in U(\mathfrak{g})$ by setting

$$\bar{\imath}_{X^{n+1}} = L_{X^n} i_X \qquad \text{for } X \in \mathfrak{g}$$

One has

$$\bar{\imath}_{\mathbb{1}} = 0, L_h = d\bar{\imath}_h + \bar{\imath}_h d + \varepsilon_h,\ \bar{\imath}_g L_h = \sum L_{h_i^{(1)}} \bar{\imath}_{ad(h_i^{(2)})g}$$

for $h, g \in U(\mathfrak{g})$ where $\varepsilon_h = \varepsilon(h) I_\Omega$.
This shows that the above relations are consistent.

Operations of Hopf algebras

$\mathcal{H}$ Hopf algebra, Ω d.g.a.

An operation of $\mathcal{H}$ *in* Ω is a linear mapping $h \mapsto i_h$ of $\mathcal{H}$ into $\mathrm{End}^{-1}(\Omega)$ such that

$$i_{\mathbb{1}} = 0$$

and that by setting

$$L_h = di_h + i_h d + \varepsilon_h$$

one has

$$\begin{cases} i_h(\alpha\beta) &=& \sum i_{h_i^{(1)}}(\alpha) L_{h_i^{(2)}}(\beta) + (-1)^a \alpha i_h(\beta) \\ i_g L_h &=& \sum L_{h_i^{(1)}} i_{\mathrm{ad}(h_i^{(2)})g} \\ L_h L_g &=& L_{hg} \end{cases}$$

$$\Rightarrow \begin{cases} L_{\mathbb{1}} &=& I_\Omega \quad \text{and} \quad L_h(\mathbb{1}) = \varepsilon(h)\mathbb{1} \\ L_h(\alpha\beta) &=& \sum L_{h_i^{(1)}}(\alpha) L_{h_i^{(2)}}(\beta) \\ L_h d &=& d L_h \end{cases}$$

$\Rightarrow$ category of $\mathcal{H}$-operations.

Invariants, horizontality and basicity

$\alpha \in \Omega$ *invariant* iff

$$L_h(\alpha) = \varepsilon(h)\alpha$$

$\Rightarrow \Omega_I$

$\alpha \in \Omega$ *horizontal* iff

$$i_h(\alpha) = 0$$

$\Rightarrow \Omega_H$

$\alpha \in \Omega$ *basic* if it is both i.e.

$$\begin{cases} L_h(\alpha) & = & \varepsilon(h)\alpha \\ i_h(\alpha) & = & 0 \end{cases}$$

$\Rightarrow \Omega_B$

Ω_I differential graded algebra,

Ω_H graded algebra stable by $L_h, h \in \mathcal{H}$

$\Omega_B = \Omega_I \cap \Omega_H$ differential graded algebra

$\Rightarrow H_I(\Omega) = H(\Omega_I), H_B(\Omega) = H(\Omega_B)$

The differential graded algebra $C(\mathcal{H})$

$$C(\mathcal{H}) = \oplus C^n(\mathcal{H}), C^n(\mathcal{H}) = (\mathcal{H}^{\otimes^n})^*$$

$$\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \Rightarrow -\mu^t : C^1(\mathcal{H}) \to C^2(\mathcal{H})$$

extends as antiderivation $d_0 \in \mathrm{Der}^{+1}(C(\mathcal{H}))$
$d_0(\Psi)(h_0, h_1, \ldots, h_n) = \sum(-1)^k \Psi(h_0, \ldots, h_{k-1}h_k, \ldots, h_n)$
associativity of $\mu \Rightarrow d_0^2 = 0$

Theorem

$H^0(C(\mathcal{H}), d_0) = \mathbb{K}$ and $H^n(C(\mathcal{H}, d_0) = 0$ for $n \geq 1$.

Homotopy $K(\Psi)(h_1, \ldots, h_{n-1}) = \Psi(\mathbb{1}, h_1, \ldots, h_{n-1})$ for $n \geq 1$.
Counit $\varepsilon \Rightarrow \mathbb{K}$ as $\mathcal{H}$-bimodule $\Rightarrow C(\mathcal{H})$ = Hochschild cochains with values in $\mathbb{K}$

$$d\omega = d_0\omega + \varepsilon\omega + (-1)^{n+1}\omega\varepsilon, \quad \omega \in C^n(\mathcal{H})$$

$H^\bullet(C(\mathcal{H}), d)$ is not trivial e.g. $H^\bullet(C(U(\mathfrak{g})), d) = H^\bullet(\mathfrak{g})$
d.g.a. $C(\mathcal{H}) = (C(\mathcal{H}), d)$ by definition.

Operation of $\mathcal{H}$ in $C(\mathcal{H})$

$i_h(\Psi)(g_1, \ldots, g_{n-1}) = \sum_{p=0}^{n-2}(-1)^p \sum_{i_p}$
$\Psi(g_1, \ldots, g_p, h_{i_p}^{(1)} - \varepsilon(h_{i_p}^{(1)})\mathbb{1}, \mathrm{ad}(h_{i_p}^{(2)})g_{p+1}, \ldots, \mathrm{ad}(h_{i_p}^{(n-p)})g_{n-1})$
$+(-1)^{n-1}\Psi(g_1, \ldots, g_{n-1}, h - \varepsilon(h)\mathbb{1})$

This defines an $\mathcal{H}$-operation with

$$L_h(\Psi)(g_1, \ldots, g_n) = \sum_{i_0} \Psi(\mathrm{ad}(h_{i_0}^{(1)})g_1, \ldots, \mathrm{ad}(h_{i_0}^{(n)})g_n)$$

One has $i_h d + d i_h = i_h d_0 + d_0 i_h \Rightarrow$
$h \mapsto i_h$ is also an operation in $(C(\mathcal{H}), d_0)$

Theorem

$H_I^0(C(\mathcal{H}), d_0) = \mathbb{K}$, $H_I^n(C(\mathcal{H}), d_0) = 0$ for $n \geq 1$.

Connections

A *connection on the $\mathcal{H}$-operation* Ω is a homomorphism of graded algebras

$$\alpha : C(\mathcal{H}) \to \Omega$$

such that for $\Psi \in C(\mathcal{H})$

$$\begin{cases} i_h \alpha(\Psi) & = & \alpha(i_h \Psi) \\ L_h \alpha(\Psi) & = & \alpha(L_h \Psi) \end{cases}$$

$$\Rightarrow \begin{cases} i_h(\alpha(\psi)) & = & \psi(h) - \varepsilon(h)\psi(\mathbb{1}) \\ L_h \alpha(\psi) & = & \alpha(\psi \circ ad(h)) \end{cases}$$

for $\psi \in \mathcal{H}^* = C^1(\mathcal{H})$
Example : $\Omega = C(\mathcal{H}), \alpha_C = I_{C(\mathcal{H})}$.

The $\mathcal{H}$-operation with connection $W(\mathcal{H})$

Define the differential graded algebra $W(\mathcal{H})$ by

$$W(\mathcal{H}) = \Omega_{\mathrm{gr}}(C(\mathcal{H}))$$

Theorem

Let (i, Ω, α) be a $\mathcal{H}$-operation with connection, then $\alpha : C(\mathcal{H}) \to \Omega$ extends uniquely as homomorphism $W(\alpha) : W(\mathcal{H}) \to \Omega$ of differential graded algebra.

Let $\alpha_W : C(\mathcal{H}) \to W(\mathcal{H})$ be the canonical injection and set

$$\varphi_W = d \circ \alpha_W - \alpha_W \circ d : C(\mathcal{H}) \to W(\mathcal{H})$$

Theorem

There is a unique operation $h \mapsto i_h$ of $\mathcal{H}$ in the differential graded algebra $W(\mathcal{H})$ for which α_W is a connection of curvature φ_W.

Universal property of $W(\mathcal{H})$

By combining the 2 above results $\Rightarrow$

Theorem

Let Ω be a $\mathcal{H}$-operation with connection, there is a unique morphism of $\mathcal{H}$-operation with connection from $W(\mathcal{H})$ to Ω.

$W(\mathcal{H})$ with this structure will be refered to as *The Weil algebra of the Hopf algebra $\mathcal{H}$*.

$$W(\mathcal{H}) = C(\mathcal{H}) \oplus W_\varphi(\mathcal{H})$$

$W_\varphi(\mathcal{H})$ being the ideal generated by $\varphi_W(C(\mathcal{H}))$ which is a differential graded ideal $\Rightarrow$

$$\rho : W(\mathcal{H}) \to C(\mathcal{H})$$

which turns out to be a morphism of $\mathcal{H}$-operation with connection with $C(\mathcal{H})$ equipped with its canonical flat conection given by $\alpha_C = I_{C(\mathcal{H})}$.

Cohomology and invariant cohomology of $W(\mathcal{H})$

Theorem

One has $H^0(W(\mathcal{H})) = H^0_I(W(\mathcal{H})) = \mathbb{K}$ *and* $H^n(W(\mathcal{H})) = H^n_I(W(\mathcal{H})) = 0$ *for* $n \geq 1$

Since $W(\mathcal{H}) = T_{C(\mathcal{H})}(\Omega^1_{\mathrm{gr}}(C(\mathcal{H})))$, $\exists!$ $K \in \mathrm{Der}^{-1}(W(\mathcal{H}))$ such that

$$K \circ \alpha_W = 0$$

and which satisfies

$$K \circ d \circ \alpha_W = \deg \circ \alpha_W$$

One has

$$K \circ L_h = L_h \circ K$$

$\Rightarrow$ K gives the appropriate homotopy

$$\begin{cases} (Kd + dK) \circ \alpha_W & = \deg \circ \alpha_W \\ (Kd + dK) \circ d \circ \alpha_W & = d \circ \alpha_W \circ \deg \\ & = (\deg - 1) \circ d \circ \alpha_W \end{cases}$$

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