The power of cohomology: the role of the antifields in BRST theory

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which is crucial for computing explicitly the BRST cohomology.

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This different point of view turns out to be crucial for computing the BRST cohomology.

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(In practical terms, the "jet space" J^k of order k is the space coordinatized by the fields and their successive derivatives up to order k.)

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The phase space Π of a gauge theory can be covariantly described as the space of solutions to the equations of motion modulo the gauge transformations.

Formally, Π is the quotient space $\Pi = \Sigma/\mathscr{O}$ of the stationary surface Σ by the gauge orbits $\mathscr{O}$ generated by the gauge transformations.

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Both differentials appear in s.

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Nilpotency of s implies $\delta^2 = 0$, $\delta\gamma + \gamma\delta = 0$, $\gamma^2 = 0$.

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We shall for this reason only focus here on the Koszul-Tate differential δ.

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Thus, the Koszul-Tate complex provides a resolution of the algebra $C^\infty(\Sigma)$ of smooth functions on the stationary surface.

(If one includes the ghosts, one gets $C^\infty(\Sigma) \otimes \Lambda(C^a, \partial_\mu C^a, \cdots)$.)

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(2) When the gauge transmormations are “open" (on-shell closure only"), the construction is more elaborate because $\gamma^2 \neq 0$, but $\gamma^2 \approx 0$ (only on-shell). This requires additional terms in s,

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This is the Batalin-Vilkovisky construction, which works because the Koszul-Tate complex is a resolution.

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A local p-form is a p-form with coefficients that are local functions.

So a local functional is

$$F = \int \omega, \quad \omega = f d^n x,$$

where f is a local function.

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This defines the mod-d cohomology $H^m(s|d)$.

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it turns out not to be true that $H_m(s|d) = 0$ for $m > 0$.

For instance an abelian ghost C^* fulfills $\delta C^* = \partial_\mu A^{*\mu}$ and defines a non-trivial element of $H_2(\delta|d)$.

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Using the Koszul-Tate differential, one easily sees that the characteristic cohomology is just $H_0^k(d|\delta)$.

Reading the cocycle condition $\delta\omega + da = 0$ in both directions, one easily proves the isomorphisms

$$H^i_j(\delta|d) \simeq H^{i-1}_{j-1}(d|\delta), \ i,j > 1, \ (i,j) \neq (1,1); \ H^1_1(\delta|d) \simeq \frac{H^0_0(d|\delta)}{\mathbb{R}}$$

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The details can be found in G. Barnich, F. Brandt, MH Physics Reports 2000.

Conclusions and comments

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The antifields can indeed also be viewed as the generators of the Koszul-Tate "resolution" that implements the equations of motion in cohomology.

This point of view turns out to be crucial for computing the BRST cohomology.