Topological Gravity and Equivariant BRST Cohomology

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Geneva, Dec 4, 2015

Raymond Stora's Memorial Day

Witten's topological gravity

- Witten's original formulation of topological gravity (1988) was utterly simple

$$s_0\, g_{\alpha\beta} = \psi_{\alpha\beta} \qquad s_0\, \psi_{\alpha\beta} = 0 \qquad s_0^2 = 0$$

- He interpreted s_0 as De Rham differential on the space of space-time metrics.
- Witten's idea was that the BRST cohomology of s_0 — i.e. the physical observables of topological gravity— would correspond to non-trivial closed forms on the space of moduli of metrics.

Mumford-Deligne intersection theory on moduli space of Riemann surfaces

- Witten knew of interesting candidates in 2d for de Rham cohomology classes on the moduli space of (punctured) Riemann surfaces.
- These were classes of even degree obtained by taking powers of first Chern-classes $c_1(\mathcal{L}_{x_i})$ of certain holomorphic line bundles associated to a puncture x_i on the surface.

The conjecture

- Witten's conjecture was therefore that BRST observables of s_0 in 2d would correspond to the Mumford-Deligne classes and that QFT correlators would reproduce intersection theory on moduli space

$$\langle \sigma_{n_1}(x_1) \cdots \sigma_{n_k}(x_k) \rangle = \int_{\mathcal{M}_{g,k}} c_1^{n_1}(\mathcal{L}_{x_1}) \wedge \cdots \wedge c_1^{n_k}(\mathcal{L}_{x_k})$$

The lost observables

An obvious problem and a question

- There was an obvious problem with this conjecture: in physics what counts is the local BRST cohomology. And the local cohomology of s_0 is clearly empty.
- Where were then the topological gravity observables?

Raymond's (and others) answer

- The answer to this question (Baulieu & Singer, Ovry, Stora, &v Baal, in the context of topological YM) introduced physicists to the concept of equivariant BRST cohomology.
- Raymond and others explained one had to modify Witten's "naive" BRST operator s_0

$$
\begin{aligned}
s\, g_{\mu\nu} &= \psi_{\mu\nu} - \mathcal{L}_c\, g_{\mu\nu} \\
s\, \psi_{\mu\nu} &= -\mathcal{L}_c\, \psi_{\mu\nu} + \mathcal{L}_\gamma\, g_{\mu\nu}
\end{aligned}
$$

- c^μ is the ghost of diffeomorphisms of ghost number +1 and γ^μ is a ghost-for-ghost of ghost number +2

$$
s\ c^\mu = -\frac{1}{2}\mathcal{L}_c\, c^\mu + \gamma^\mu \qquad s\,\gamma^\mu = -\mathcal{L}_c\,\gamma^\mu
$$

$$
s^2 = 0
$$

Equivariant BRST cohomology

- At first sight this seems silly: The new s is related to the "naive" s_0 by a field redefinition: $\psi_{\alpha\beta} \to \psi_{\alpha\beta} - \mathcal{L}_c\, g_{\alpha\beta}$.
- Hence the simple-minded (absolute) cohomology of s is as empty as the cohomology of s_0.
- However for s one can define a different cohomology: the cohomology on local field functionals which do not depend on the reparametrization ghost $c^\mu(x)$.
- This is called the equivariant (with respect to diffeomorphisms) cohomology of s.

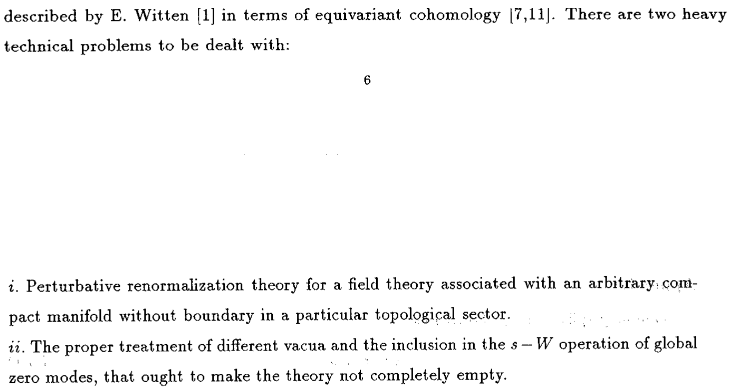
The observables regained

- In 2 dimensions one finds that the equivariant cohomology of s is generated by an operator of ghost number +2 in precise correspondence with the basic Mumford-Deligne 2-form on moduli space of (punctured) Riemann surfaces

$$\sigma_2^{(0)} = \frac{1}{2\,\pi}\left[\sqrt{g}\,\epsilon_{\mu\nu}\left(D^{\nu}\,\gamma^{\mu} - \frac{1}{2}\,\psi^{\mu\lambda}\,\psi^{\nu}_{\lambda}\right)\right] \leftrightarrow c_1(\mathcal{L}_x)$$

Still...Raymond did not think everything was clear...

- Raymond's conclusion at the end of his paper



described by E. Witten [1] in terms of equivariant cohomology [7,11]. There are two heavy technical problems to be dealt with:

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i. Perturbative renormalization theory for a field theory associated with an arbitrary compact manifold without boundary in a particular topological sector.
ii. The proper treatment of different vacua and the inclusion in the $s-W$ operation of global zero modes, that ought to make the theory not completely empty.

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What about the Slavnov-Taylor identity?

- Equivariant observables are BRST-trivial in the simple-minded s_0-cohomology.
- In the BRST framework one expects BRST-trivial operators to decouple from physical correlators

$$\langle (s_0\, X)\, O \rangle = \langle s_0\, (X\, O) \rangle = 0$$

- Why this is not so (if correlators of equivariant observables were indeed non-trivial)?
- In other words, how does the theory “know” that the s_0-trivial operator must not decouple, while those trivial in the equivariant cohomology do?

Gribov's problem

- The answer to this question turn out to be related to Gribov's horizon.
- To compute correlators of gauge theories one needs to fix gauge-invariance, including the "[...] global zero modes which ought to make the theory non-empty" .
- For non-abelian theories one cannot do this globally on field space, due to Gribov's horizon.
- One can fix the gauge locally on orbit space and try to "glue" together the correlators obtained on the different patches.

Gribov's problem in 2d topological gravity

- The gauge orbit space of 2d topological gravity is the moduli space of Riemann surfaces.
- Fixing the gauge locally means in this context to choose, on each patch U_α of the moduli space, a section $g_{\mu\nu}(x; m)$ depending on the local coordinates m^a on the moduli space.
- Gauge-fixed correlators of topological gravity observables O_n of ghost number n are local n-forms on the moduli space

$$\langle O \rangle_{U_\alpha} = \Omega^{(n)}_{a_1 \ldots a_n; \alpha}(m)\, dm^{a_1} \wedge \ldots \wedge dm^{a_n} \equiv \Omega^{(n)}_\alpha$$

The un-integrated Slavnov-Taylor identity

- BRST-invariance of the observable O_n of ghost number n translates into closeness of the local n-form $\Omega_\alpha^{(n)}$:

$$0 = \langle s\, O_n \rangle_{U_\alpha} = d_m \langle O_n \rangle_{U_\alpha}$$

- BRST-triviality of the operator $s\, X_{n-1}$ translates into local exactness of the form $\langle s\, X_{n-1} \rangle_{U_\alpha}$

$$\langle s\, X_{n-1} \rangle_{U_\alpha} = d_m \langle X_{n-1} \rangle_{U_\alpha}$$

Global BRST properties

- One needs to understand under which condition one can reconstruct a global form from the local ones

$$\langle O_n \rangle_{U_\alpha} = \langle O_n \rangle_{U_\beta} \quad \text{on} \quad U_\alpha \cap U_\beta$$
$$\langle X_{n-1} \rangle_{U_\alpha} = \langle X_{n-1} \rangle_{U_\beta} \quad \text{on} \quad U_\alpha \cap U_\beta$$

- If the first condition holds $\langle O_n \rangle$ gives rise to a globally defined closed form on the moduli space.
- If the second condition holds $\langle s\, X_{n-1} \rangle$ does correspond to a globally defined exact form.

Equivariance explained

One finds that (C. Becchi & C.I):

X_{n-1} does give rise to a globally defined form on the moduli space if and only if it does not depend on the ghost c^μ, i.e. if it an equivariant operator.

- This explains why only operators which are BRST-trivial in the equivariant sense decouple, while operators which are BRST-trivial in the “naive” absolute cohomology do not in general decouple.

Local gauges and contact terms

- However one understands also something more...
- One also finds that, for generic local gauges U_α, even the correlators of equivariant observables do jump by exact terms when going from one patch to another

$$\langle O_n \rangle_{U_\alpha} - \langle O_n \rangle_{U_\beta} = d_m \langle O_{n-1}^{(\alpha\beta)} \rangle \quad \text{on} \quad U_\alpha \cap U_\beta$$

- If the observables are equivariant, nevertheless, one can show that it is possible to compensate these jumps by adding uniquely defined contact terms.

Gauge independence

- These contact terms are obtained by integrating on the intersections $U_\alpha \cap U_\beta$ correlators of local “descendant” operators of ghost number $n-1$ (and also by integrating “descendant” operators of ghost number $n-2$ on the intersections of intersections $U_\alpha \cap U_\beta \cap U_\gamma$ etc.).
- The contact terms so defined reestablish complete gauge-independence of the class $\langle O_n \rangle$ associated to equivariant observables.

Conclusions

- Once again Raymond's mastery and profound knowledge of both physics and mathematics has given us, physicists, the correct framework and language to understand — in a simple and yet rigorous way — aspects of quantum field theory that are delicate and subtle.
- Still, it was my experience that Raymond was often unsatisfied even with the same beautiful and elegant solutions to physics questions that he himself had earlier offered: his incessant striving for clarity and his inexhaustible intellectual curiosity made him suspicious of solutions which might turn out to be merely formal and/or superficial answers and prompted him to go beyond that surface and to search unrelentlessly searching for more complete and compelling understandings.