TOPOLOGICAL FIELD THEORY AND EQUIVARIANT COHOMOLOGY

A tribute to Raymond Stora

A few observations that interested Raymond, and some unanswered questions

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When I first met Raymond, (about 1980), it was about intriguing properties of various algebraic "Russian formulae" and their possible geometrical interpretations, in particular in the case of the antifield formalism that Marc will review.

I was asking Raymond to explain me the geometry of BV antifields. I was trying to incorporate these fields as part of generalised gauge fields.

We had many discussions on this, using the theories of charged 2-forms, 2d gravity, Witten open-string field, supergravity and, most instructively, topological supersymmetric QFTs.

Eventually, the pieces of the puzzles assembled themselves, with the idea of equivalent cohomology and the use of shadow fields. As an unexpected by-product, came the idea that all Poincaré supersymmetries derive from certain topological symmetries.

We have yet to probably discover new pieces of the puzzle.

The basic features of the Batalin and Vilkovisky formalism are the doubling of all fields $\phi \to \phi, \phi^*$ and

1) A local action $S[\phi, \phi^*]$ with a B-V graded master equation

$$\frac{\delta S}{\delta \phi}\frac{\delta S}{\delta \phi^*} \pm \frac{\delta S}{\delta \phi}\frac{\delta S}{\delta \phi^*} = 0$$

2) The definition of the BRST operator s, with $s^2 = 0$

$$s\phi = \frac{\delta S[\phi, \phi^*]}{\delta \phi^*} \qquad s\phi^* = \frac{\delta S[\phi, \phi^*]}{\delta \phi}$$

3) The elimination of the antifields by a local gauge function $Z(\phi)$

$$\phi^* = \frac{\delta Z[\phi]}{\delta \phi}$$

The mean values of local operators belonging to the cohomology of s are independent on the choice of the gauge function.

The issue was to get a geometrical interpretation, at least in few cases.

I wanted to define a nilpotent s acting on fields and anti-fields with a "Russian formula".

Only afterward the BV action and its anomalies should be computed.

What convinced Raymond for the possible unification between fields and anti-fields for forms came much later, in the case of the topological Yang–Mills theory, and the picture we draw with I.M. Singer.

The really instructive example is a Yang-Mills field A with a 2 form B_2 in D=4 (or D=8 with a breaking of SO(8) into Spin(7) or SU(4)):

$$\tilde{A}(x) = A_4^{-3} + A_3^{-2} + A_2^{-1} + A + c$$

$$\tilde{B}_2(x) = B_3^{-1} + B_4^{-2} + B_0^2 + B_1^1 + B_2$$

$$D^{\tilde{A}} = d + [\tilde{A}, \quad]$$

$$F^{\tilde{A}} = d\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] \qquad\qquad G^{\tilde{A}} = D^{\tilde{A}}\tilde{B}_2$$

and

$$\mathcal{D} = s + D^{\tilde{A}} = d + s + [\tilde{A}, \quad]$$

$$\mathcal{F} = (s+d)\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] = s\tilde{A} + F^{\tilde{A}}$$

To determine s with $(s+d)^2 = 0$, we only need to put constraints on the curvatures $\mathcal{F}$ and $\mathcal{D}\tilde{B}_2$ of $\tilde{A}$ and $\tilde{B}_2$ respectively, consistently with the Bianchi identities.

Unfortunately , imposing the "too naive" Russian formula, which I did firstly

$$\mathcal{F}^{\tilde{A}} = F^{A} \quad and \quad \mathcal{D}^{\tilde{A}}\tilde{B}_2 = D^{A}B_2$$

gives a violation of the Bianchi identity proportionally to $[F, B_0^2]$, there is no nilpotent symmetry for the 2-form, and the action $\int d^4x \; Tr(G_{\mu\nu\rho})^2$ is not invariant for a gauge symmetry of the 2 form, $sB_2 = DB_1^1 + [B_2^{-1}, B_0^2$.

The only consistent possibility was weird (I rejected it....) :

$$\mathcal{F}^{\tilde{A}} = \tilde{B}_2$$

$$\mathcal{D}^{\tilde{A}}\tilde{B}_2 = 0$$

But it looked absurd because it gives a BV action

$$\int Tr(\tilde{B}_2 \wedge \tilde{B}_2 + \tilde{B}_2 \wedge \mathcal{F}^{\tilde{A}})$$

that is, an "empty" classical action

$$\int Tr(B_2 \wedge B_2 + B_2 \wedge F^A) \sim \int Tr(F^A \wedge F^A)$$

and BRST variations like

$$sA = B_1^1 - D^A c$$

By some detours it was useful to understand the large gauge symmetry of Witten open string field theory (with E. Bergshoeff and I. Sezgin), as it was done by Siegel, Bocchichio, and Thorn and Neveu and West. I was really contemplating the fact that the BV action contained what is needed to "gauge-fix" $Tr(F^A \wedge F^A)$ **toward a ghost dependent renormalisable QFT.**

It was an interesting symmetry, with a very nice algebraic structure under the form of a Russian formula. Moreover, the structure was interesting in the Abelian case, with the Chapline-Manton coupling to a Chern-Simons form - something that also interested Raymond - and the Green-Schwartz mechanism.

In fact I was really blind, but Raymond had taught me to be very patient.

The topological Yang–Mills (TQFT) symmetry and its equivariant cohomology

Later on, Witten twisted the $N = 2SYM$, by $\lambda \equiv (\Psi_\mu, \chi_{\mu\nu}, \eta)$ and $\phi \equiv (\Phi, \bar\Phi)$ into

$$\int Tr\Big(F^2_{\mu\nu} - \chi_{\mu\nu} D_{[\mu}\Psi_{\nu]} - \eta D_\mu \Psi_\mu + \bar\Phi D^2 \Phi + [\bar\Phi, \Phi]^2 + \dots\Big)$$

And then introduced a single generator "twisted new supersymmetry"

$$QA_\mu = \Psi_\mu$$

$$Q\Psi_\mu = D_\mu \Phi$$

$$Q\Phi = 0$$

$$Q\bar\Phi = \bar\eta$$

$$Q\bar\eta = [\Phi, \bar\Phi]$$

$$Q\chi_{\mu\nu} = F^+_{\mu\nu}$$

For A, Ψ, Φ, $\bar{\Phi}$, η one has

$$Q^2 = \delta_{gauge}(\Phi)$$

Furthermore, Witten introduced series of cocycle equations such as

$$Q\Big(Tr(\Psi \wedge \Psi + F \wedge \Phi)\Big) = d\Big(Tr(\Psi \wedge \Phi)\Big)$$

He showed that such cocycles have very interesting expectation values (mathematically), as foreseen by Atyiah, who wanted a Yang–Mills QFT generalisation of supersymmetric quantum mechanics adapted to a field theory computational tool for the Donaldson invariants.

Was there something conceptually new ?

With I. M. Singer, we changed Q into Q_s, with $Q_s^2 = 0$.

$$
\begin{aligned}
Q_s A_\mu &= \Psi_\mu + D_\mu c & \qquad Q_s \Psi_\mu &= D_\mu \Phi - [c, \Psi_\mu] \\
Q_s c &= \Phi - \frac{1}{2}[c, c] & \qquad Q_s \Phi &= -[c, \Phi] \\
s\chi_{\mu\nu} &= H_{\mu\nu} - [c, \chi_{\mu\nu}] & \qquad sH_{\mu\nu} &= -[c, H_{\mu\nu}] + [\Phi, \chi_{\mu\nu}] \\
Q_s \bar{\Phi} &= \bar{\eta} - [c, \bar{\Phi}] & \qquad Q_s \eta &= [\Phi, \bar{\Phi}] - [c, \eta]
\end{aligned}
$$

The right interpretation of the system is

$$(d + Q_s)(A + c) + (A + C)^2 = F + \Psi + \Phi$$

$$(d + Q_s)(F + \Psi + \Phi) + [A + C, F_\Psi + \Phi] = 0$$

So $I_{Witten} = \int TrF \wedge F + Q_s(localising\ functional)$ **and one has obviously**

$$(d + Q_s)Tr(F + \Psi + \Phi)(F + \Psi + \Phi) = 0$$

A TQFT is just about an action derived from a huge symmetry, eg

$$\delta_{topological} A_\mu = \epsilon_\mu + D_\mu \alpha \qquad \leftrightarrow \qquad Q_s A_\mu = \Psi_\mu + D_\mu c$$

It is gauge-fixed by a clever BRST invariant localisation around field configurations, and permits to explore moduli spaces by computing the mean values of some observables using quantum fluctuations around a localising functional.

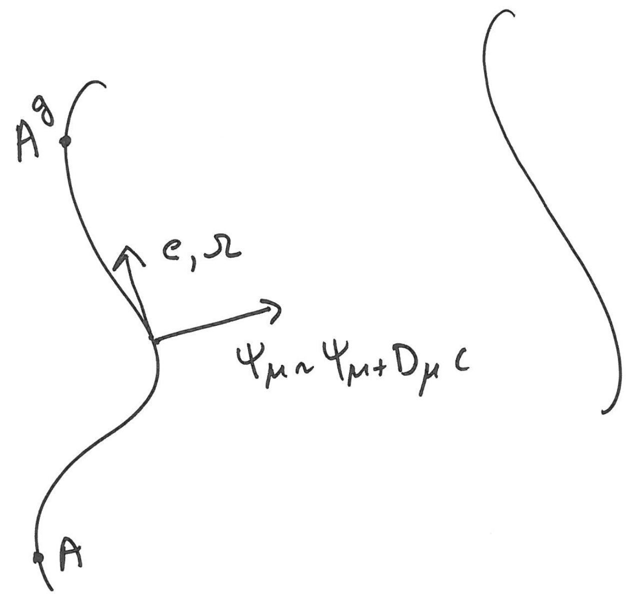
It is called equivariant because we are only interested in the symmetry governed by

$$\epsilon_\mu \sim \epsilon_\mu + D_\mu \alpha$$

We found that the choice of the localisation is often governed by extra supersymmetries, such as vector symmetry.

$$\int [dA_\mu](\ldots) \exp \int \mathrm{Tr}\, F \wedge F = \,?$$

$$\delta A_\mu = \epsilon_\mu + D_\mu \epsilon \quad \longleftrightarrow \quad Q_S A_\mu = \Psi_\mu + D_\mu c$$



localize at $\begin{cases} F^-_{\mu\nu} = 0 \\ D_\mu \Psi_\mu = 0 \end{cases}$

and, of course, $\partial_\nu A_\mu = 0$.

Later on, we called the field c a shadow, with G. Bossard.

Further work had to be done to understand how Q_s truly commutes with the standard BRST operator of the gauge symmetry s_{gauge}, and find the action of Q_s on the FP ghost Ω and of s_{gauge} on the shadow c. This is

$$(d + Q_s + s_{gauge})(A + c + \Omega) + (A + C + \Omega)^2 = F + \Psi + \Phi$$

$$(d + Q_s + s_{gauge})(F + \Psi + \Phi) + [A + C + \Omega, F + \Psi + \Phi] = 0$$

and $(d + Q_s + s_{gauge})^2 = 0$. In particular, the "susy" Q_s-transformation of the Faddeev–Popov ghost is

$$Q_s\Omega = \mu \qquad\qquad sc = -\mu - [c, \Omega]$$

The unique renormalisable action that realises the localisation is

$$\int trF \wedge F + Q_s \; Tr\Big(\chi F^- + \bar{\Phi}(D_\mu \Psi_\mu - [\Phi, \eta])\Big) + s_{gauge} Q_s \; \Big(Tr A_\mu^2\Big)$$

The observables are the element of the cohomology of Q_s within the cohomology of s_{gauge} with ghost number zero. They are classified by their form degree and shadow number. We have two Ward identities for the TQFT.

This set-up with shadows *and* ghosts realised the scheme of "basic cohomology" that had been shown to be necessary in a beautiful paper by Raymond, with S. Ouvry and P. Van Baal, using results of a superfield approach by J.H. Horne. In fact, Raymond had noticed in his unique way the question that we left a little bit ambiguous with I. M. Singer, for separating neatly ψ_μ from $\Psi_\mu + D_\mu c$ in quantum field theory.

Now, Camillo will use these tools of equivalent cohomology for a very interesting case, in 2d gravity.

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