Extended Theorems for Signal Induction in Particle Detectors

Werner Riegler, CERN

RD51 workshop, Dec. 9th 2015

- Principle of Signal Induction in Particle Detectors, Ramo's Theorem
- Quasistatic Approximation of Maxwell's Equations, Applications
- Generalized Signal Theorems
- Signals in Resistive Plate Chambers
- Signals in Silicon Strip Detectors

Available online at www.sciencedirect.com

SCIENCE DIRECT

ELSEVIER

Nuclear Instruments and Methods in Physics Research A 535 (2004) 287–293

NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH Section A

www.elsevier.com/locate/nima

Extended theorems for signal induction in particle detectors VCI 2004

W. Riegler*

CERN, PH Division, Rt. De Meyrin, Geneva 23CH-1211, Switzerland

Available online 13 August 2004

N·H

ELSEVIER

Nuclear Instruments and Methods in Physics Research A 491 (2002) 258–271

NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH Section A

www.elsevier.com/locate/nima

Induced signals in resistive plate chambers

Werner Riegler

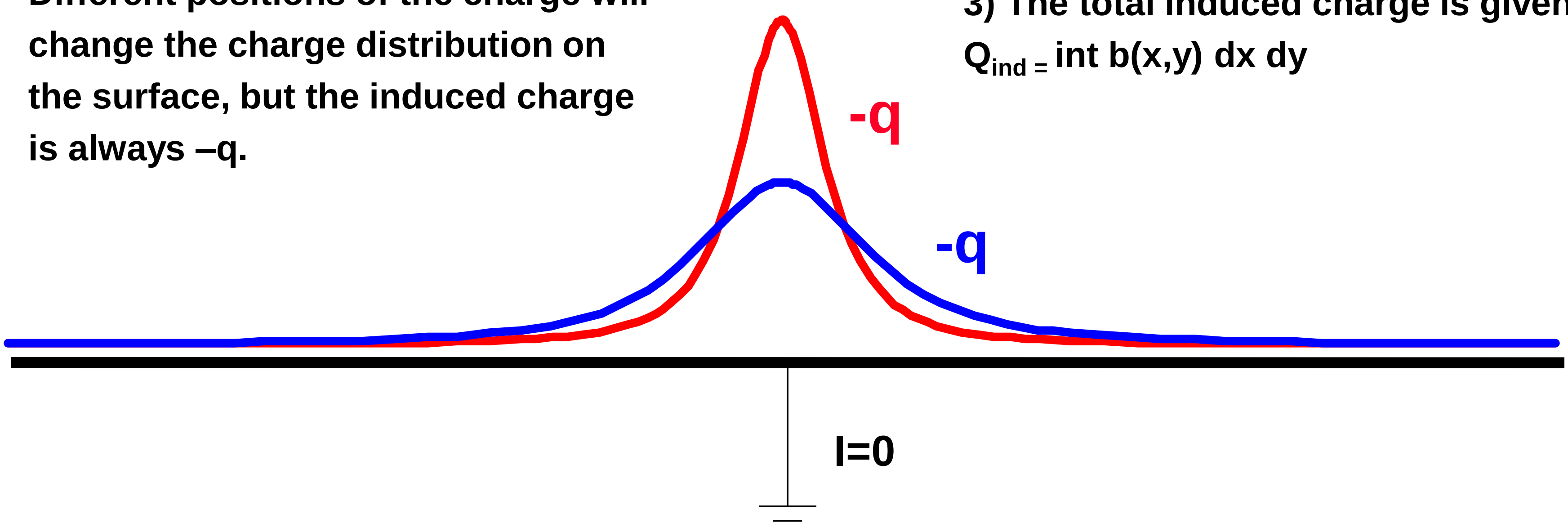
EP Division, CERN, CH-1211 Geneva 23, Switzerland

Received 10 April 2002; received in revised form 30 April 2002; accepted 2 May 2002

Charge Above a Metal Plane

A point charge in presence of an infinite grounded metal plane will induce a total charge of –q.

Different positions of the charge will change the charge distribution on the surface, but the induced charge is always –q.



Charge density

1) Calculate the electric field E(x,y,z)

2) The charge density on the plate

Is given by $b(x,y) = \varepsilon_0 E(x,y,z=0)$

3) The total induced charge is given by

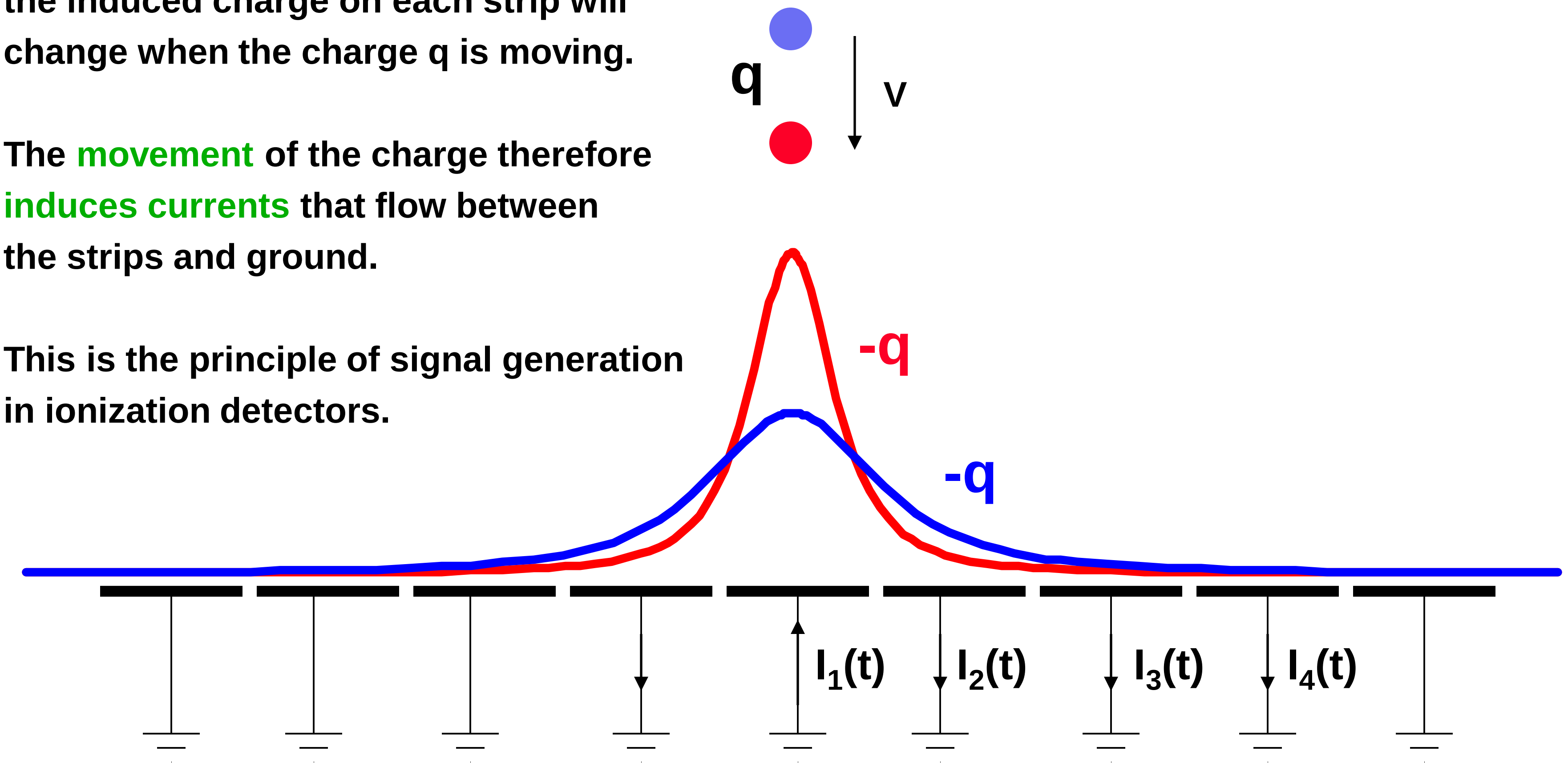
$Q_{ind} =$ int b(x,y) dx dy

Charge above Infinite Plane

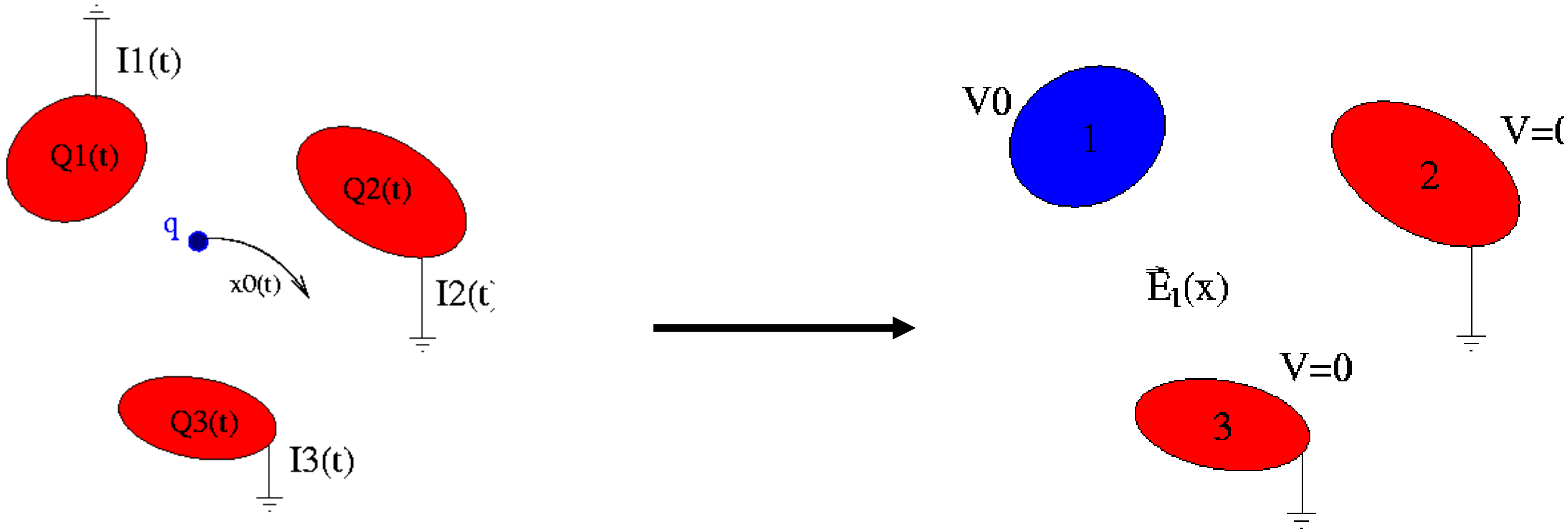
In case the strips are segmented,
the induced charge on each strip will
change when the charge q is moving.

The movement of the charge therefore
induces currents that flow between
the strips and ground.

This is the principle of signal generation
in ionization detectors.



Ramo's Theorem

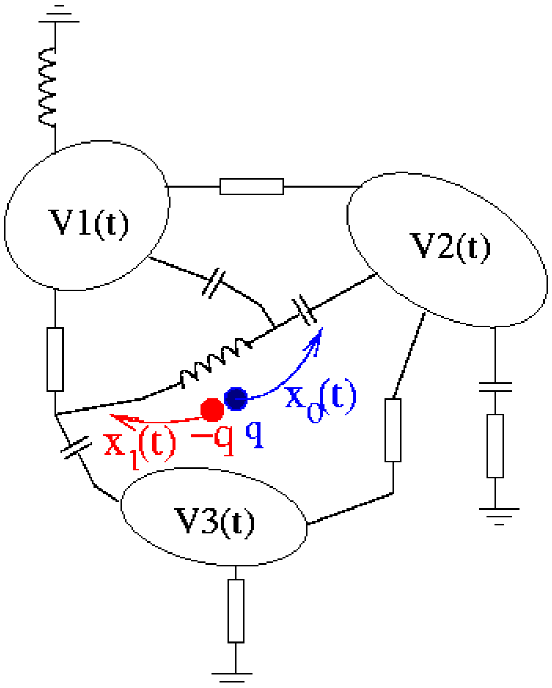


The current induced on a grounded electrode by a charge q moving along a trajectory $x_0(t)$ can be calculated the following way:

- One removes the charge, sets the electrode in question to voltage V_0 and grounds all other electrodes.
- This defines an electric field E(x), the so called weighting field of the electrode.
- The induced current is the given by $I_1(t) = q/V_0\, E_1(x_0(t))\, d/dt\, x_0(t)$

Extensions of the Theorem

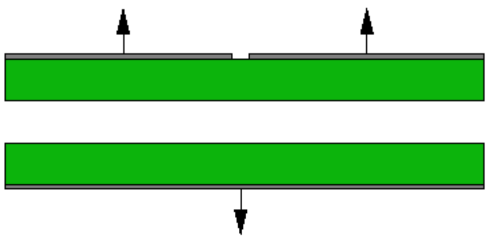
An extension of the theorem, where the electrodes are connected with arbitrary discrete impedance elements, has been given by Gatti et al., NIMA 193 (1982) 651.



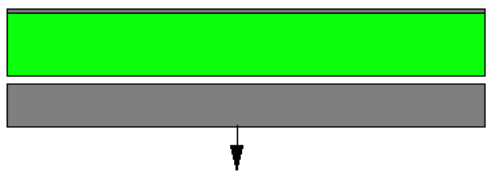
However this still doesn't include the scenario where a conductive medium is present in between the electrodes, as for example in Resistive Plate Chambers or undepleted Silicon Detectors.

Extensions of the Theorem

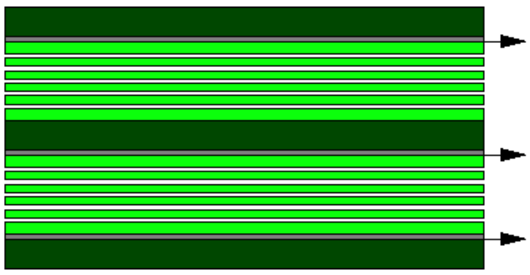
Resistive Plate Chambers



2mm Bakelite, $\rho \approx 10^{10}\ \Omega$cm

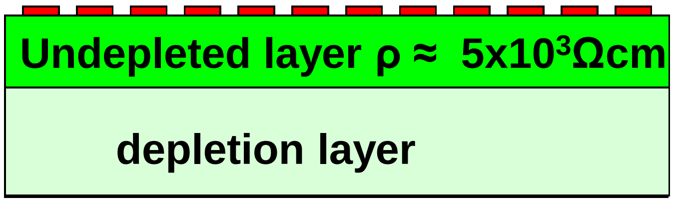


3mm glass, $\rho \approx 2 \times 10^{12}\ \Omega$cm

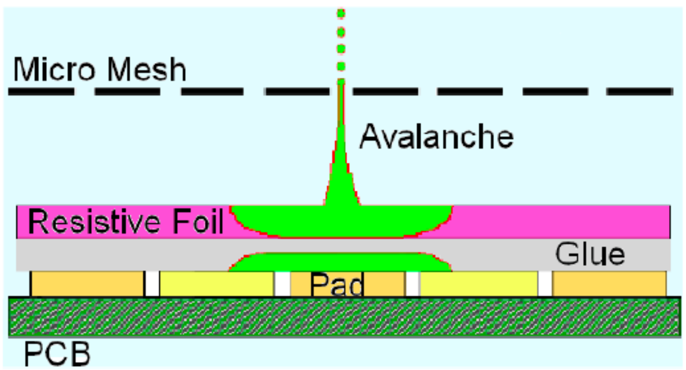


0.4mm glass, $\rho \approx 10^{13}\ \Omega$cm

Silicon Detectors



Irradiated silicon typically has larger volume resistance.



Charge spreading with resistive layers

Quasistatic Approximation of Maxwell's Equations

Pointed out by B. Schnizer et. al, NIMA 478 (2002) 444-447

In an electrodynamic scenario where Faraday's law can be neglected,
I.e. the time variation of magnetic fields induces electric fields that are small compared to the fields resulting from the presence of charges, Maxwell's equations 'collapse' into the following equation:

$$\vec{\nabla}\left[\varepsilon(\vec{x})\vec{\nabla}\right]\frac{d}{dt}\phi(\vec{x},t)+\vec{\nabla}\left[\sigma(\vec{x})\vec{\nabla}\right]\phi(\vec{x},t)=-\frac{d}{dt}\rho_{ext}(\vec{x},t) \quad \text{and} \quad \vec{E}(\vec{x},t)=-\vec{\nabla}\phi(\vec{x},t)$$

This is a first order differential equation with respect to time, so we expect that in absence of external time varying charges electric fields decay exponentially.

Performing Laplace Transform gives the equation

$$\vec{\nabla}\left[\varepsilon_{eff}(\vec{x},s)\vec{\nabla}\right]\phi(\vec{x},s)=-\rho_{ext}(\vec{x},s) \quad \text{with} \quad \varepsilon_{eff}(\vec{x},s)=\varepsilon(\vec{x})+\frac{1}{s}\sigma(\vec{x})$$

This equation has the same form as the Poisson equation for electrostatic problems.

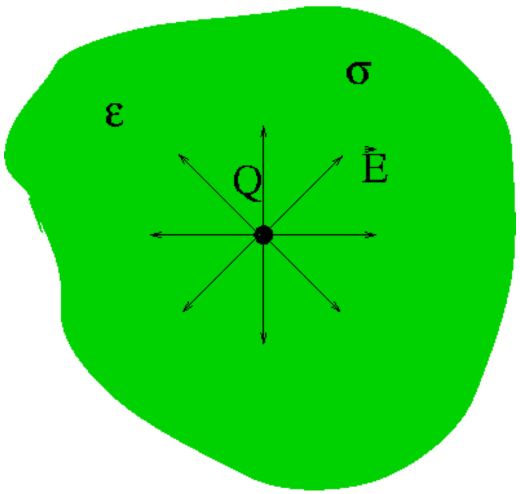
Quasistatic Approximation of Maxwell's Equations

This means that in case we know the electrostatic solution for a given ε we find the electrodynamic solution by replacing ε with ε +σ/s and performing the inverse Laplace transform.

Point charge in infinite space with conductivity σ.

$$\phi(r) = \frac{Q}{4\pi\varepsilon_r\varepsilon_0}\frac{1}{r} \quad \rightarrow \quad \phi(r,s) = \frac{Q/s}{4\pi(\varepsilon_r\varepsilon_0 + \sigma/s)}\frac{1}{r}$$

$$\phi(r,t) = \mathcal{L}^{-1}\left[\phi(r,s)\right] = \frac{Q}{4\pi\varepsilon_r\varepsilon_0}\frac{e^{-t/\tau}}{r} \quad \text{with} \quad \tau = \frac{\varepsilon_r\varepsilon_0}{\sigma}$$



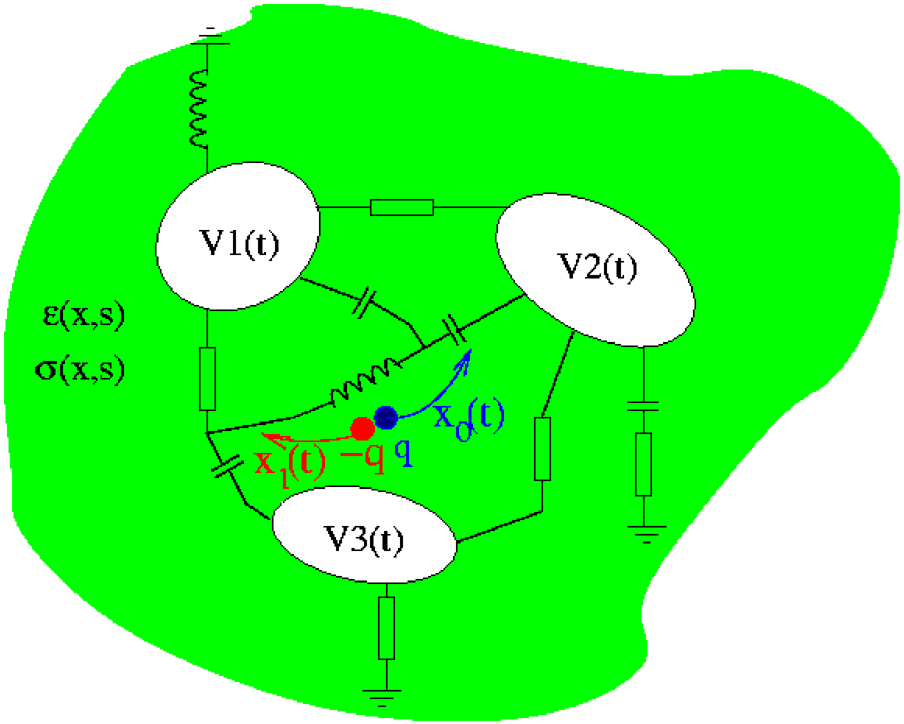
The fields decays exponentially with a time constant τ.

Formulation of the Problem

At t=0, a pair of charges +q,-q is produced at some position in between the electrodes.

From there they move along trajectories $x_0(t)$ and $x_1(t)$.

What are the voltages induced on electrodes that are embedded in a medium with position and frequency dependent permittivity and conductivity, and that are connected with arbitrary discrete elements ?



W. Riegler: NIMA 491 (2002) 258-271

Quasistatic approximation

$$\vec{\nabla}\left[\varepsilon_{eff}(\vec{x},s)\vec{\nabla}\right]\phi(\vec{x},s) = -\rho_{ext}(\vec{x},s)$$

$$\varepsilon_{eff}(\vec{x},s) = \varepsilon(\vec{x},s) + \frac{1}{s}\sigma(\vec{x},s)$$

Extended version of Green's 2nd theorem

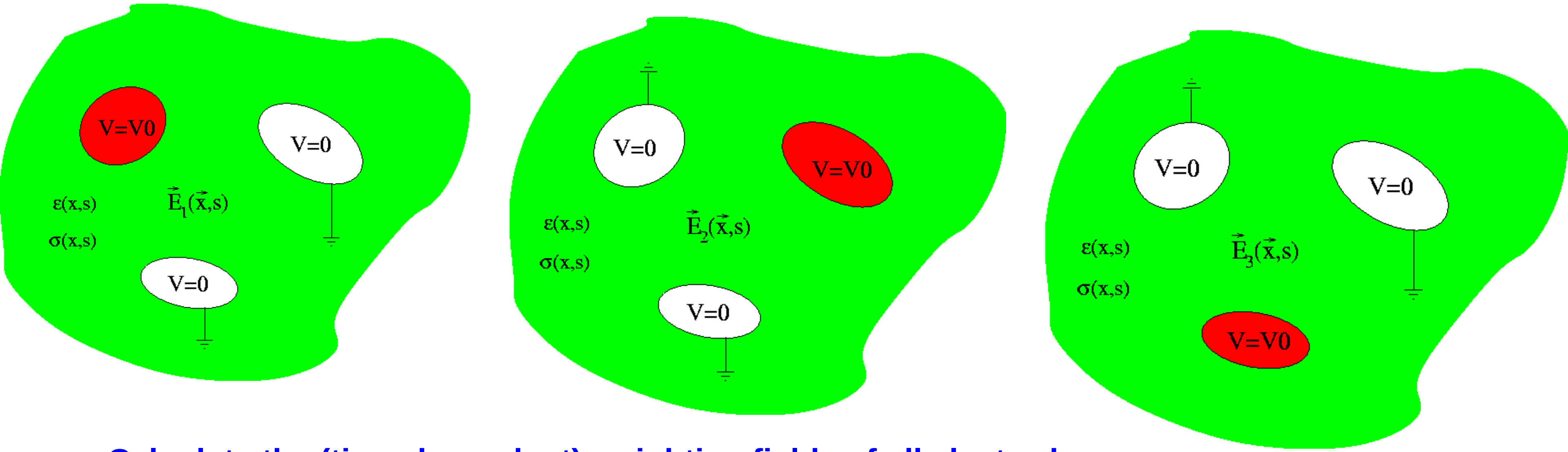
$$\int_A \left[\psi(\vec{x})f(\vec{x})\vec{\nabla}\phi(\vec{x}) - \phi(\vec{x})f(\vec{x})\vec{\nabla}\psi(\vec{x})\right] d\vec{A}$$

$$= \int_V \left[\psi(\vec{x})\vec{\nabla}[f(\vec{x})\vec{\nabla}]\phi(\vec{x}) - \phi(\vec{x})\vec{\nabla}[f(\vec{x})\vec{\nabla}]\psi(\vec{x})\right] d^3x$$

Theorem (1,4)

Remove the charges and the discrete elements and calculate the weighting fields of all electrodes by putting a voltage $V_0\delta(t)$ on the electrode in question and grounding all others.

In the Laplace domain this corresponds to a constant voltage V_0 on the electrode.

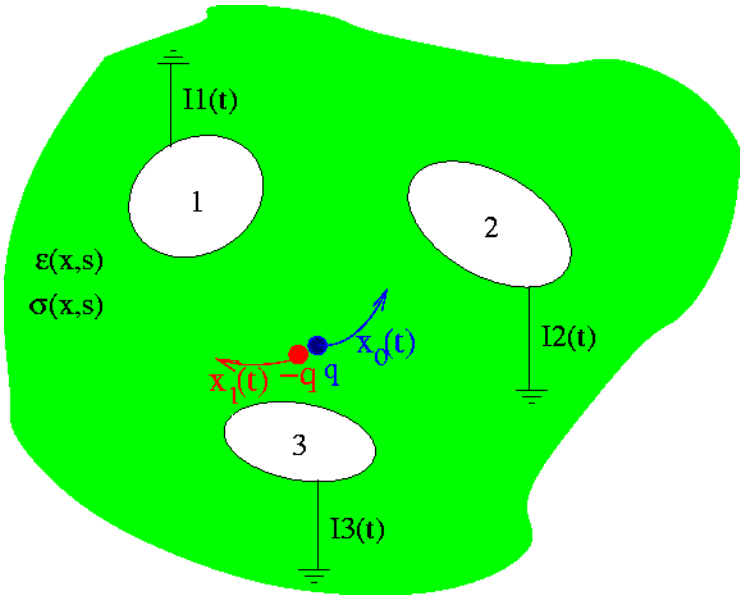


Calculate the (time dependent) weighting fields of all electrodes

$$\vec{\nabla}\left[\varepsilon_{eff}(\vec{x},s)\vec{\nabla}\right]\phi(\vec{x},s)=0 \qquad \phi_n(\vec{x},s)|_{\vec{x}=\vec{A}_m}=V_0\delta_{nm}$$

$$\vec{E}_n(\vec{x},s)=-\vec{\nabla}\phi_n(\vec{x},s) \qquad \vec{E}_n(\vec{x},t)=\mathcal{L}^{-1}\left[\vec{E}_n(\vec{x},s)\right]$$

Theorem (2,4)



Calculate induced currents in case the electrodes are grounded

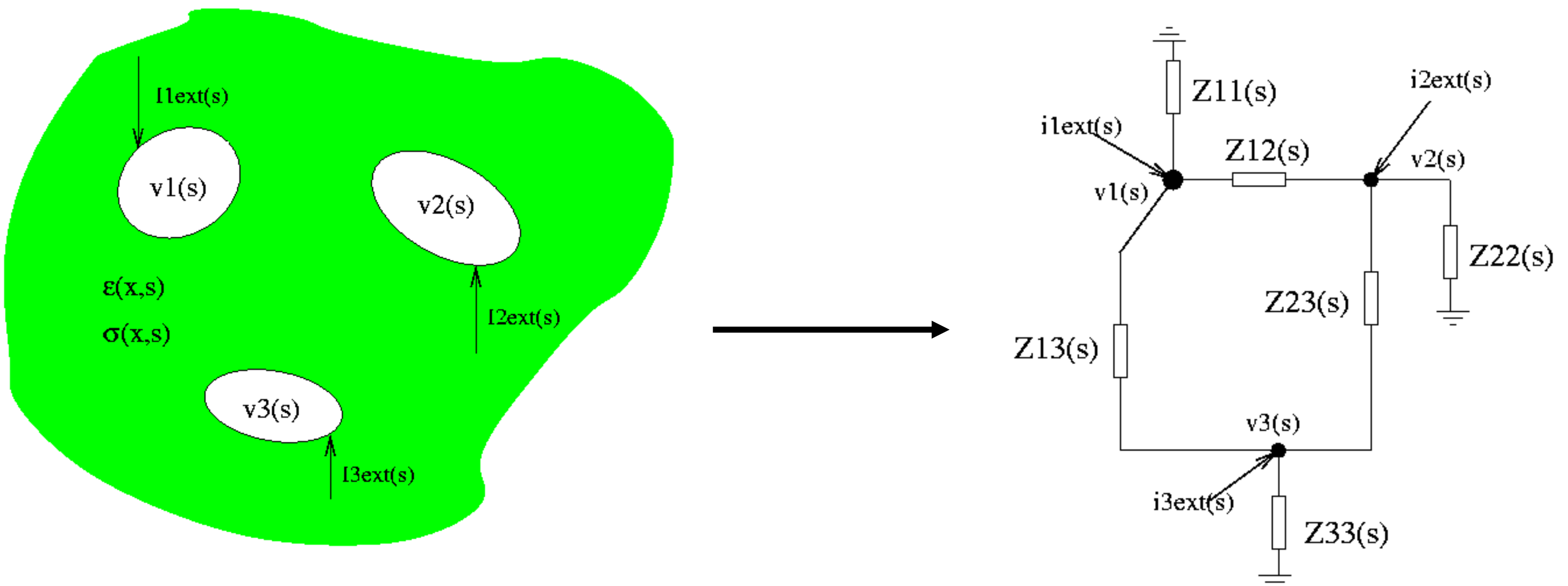
$$I_n(t) = \frac{q}{V_0}\int_0^t \vec{E}_n\left[\vec{x}_0(t'), t-t'\right]\dot{\vec{x}}_0(t')dt' - \frac{q}{V_0}\int_0^t \vec{E}_n\left[\vec{x}_1(t'), t-t'\right]\dot{\vec{x}}_1(t')dt'$$

Theorem (3,4)

Calculate the admittance matrix and equivalent impedance elements from the weighting fields.

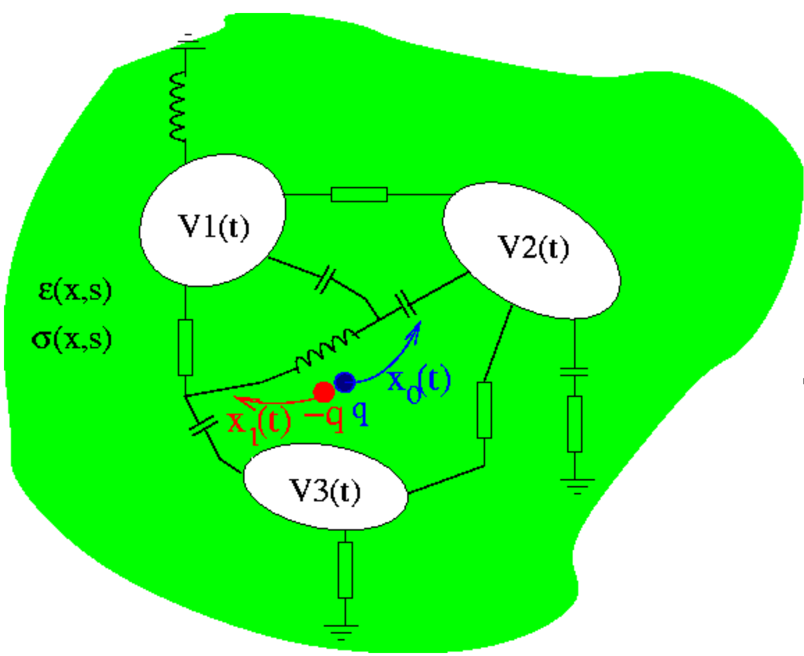
$$y_{nm}(s) = \int_{A_n} \varepsilon_{eff}(\vec{x}, s)\vec{E}_m(\vec{x}, s) d\vec{A} \qquad y_{nm}(s) = y_{mn}(s)$$

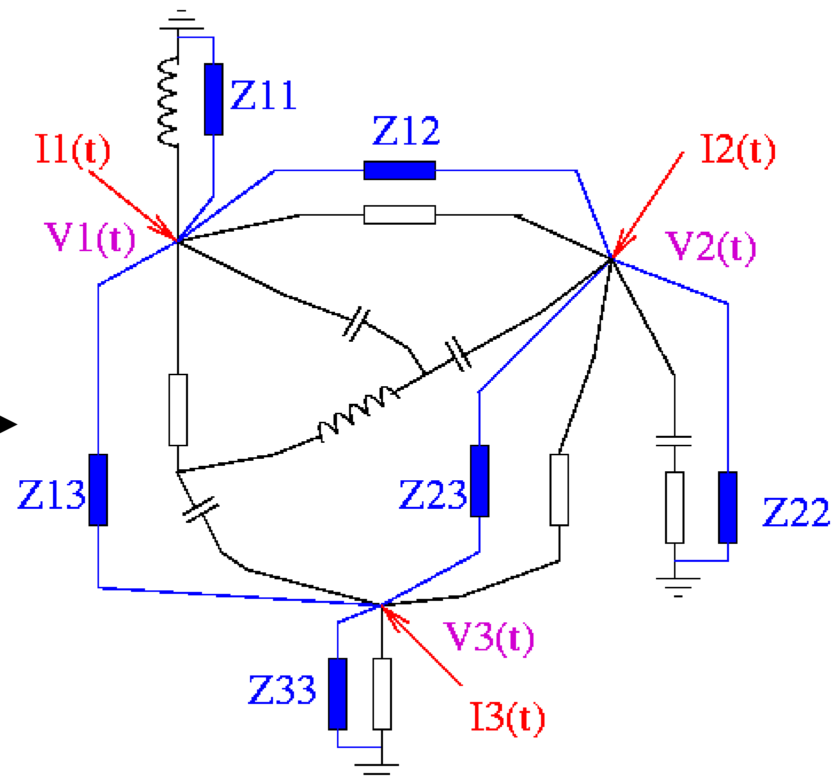
$$Z_{nm}(s) = -\frac{1}{y_{nm}(s)} \qquad Z_{nn}(s) = \frac{1}{\sum_{m=1}^{N} y_{nm}(s)}$$



Theorem (4,4)

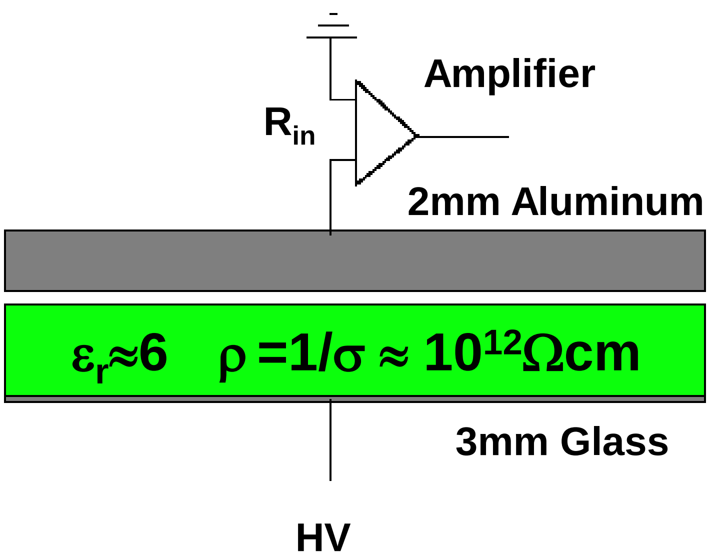
Add the impedance elements to the original circuit and put the calculated currents On the nodes 1,2,3. This gives the induced voltages.





Examples

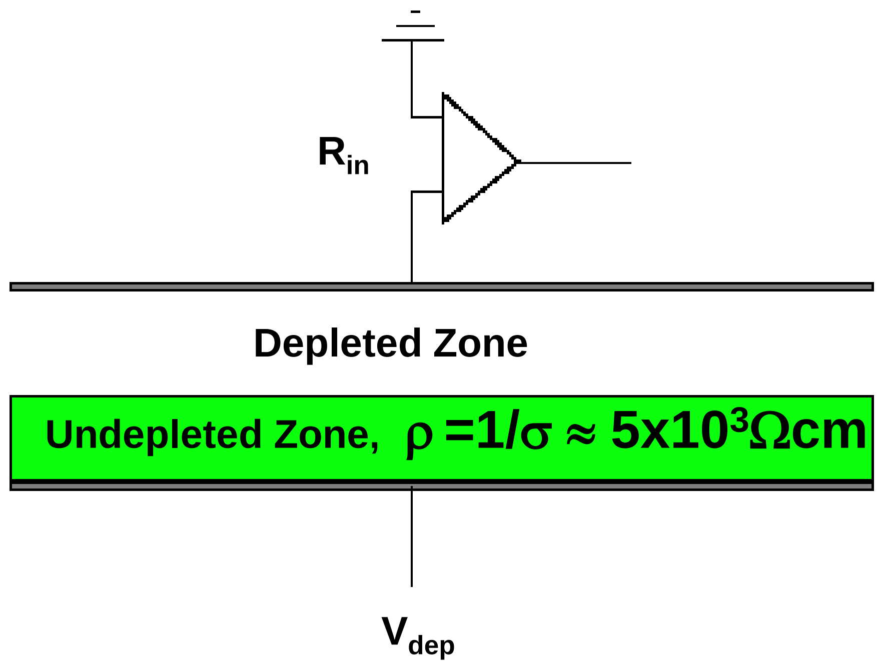
RPC



300µm Gas Gap

$\tau \approx \varepsilon_0 / \sigma \approx$ 100msec

Silicon Detector



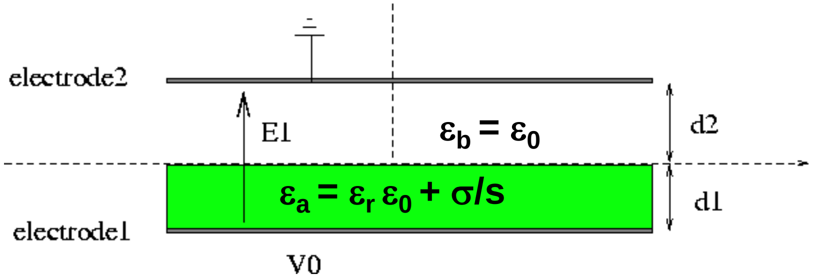
$\tau \approx \varepsilon_0 / \sigma \approx$ 1ns

heavily irradiated silicon has larger resistivity

that can give time constants of a few hundreds of ns,

Example, Weighting Fields (1,4)

Weighting Field of Electrode 1

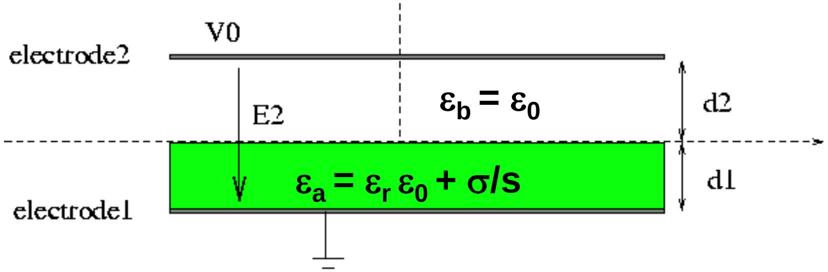


$$E_{1z}(s) = \frac{\varepsilon_a V_0}{\varepsilon_a d_2 + \varepsilon_b d_1} = \frac{V_0 \varepsilon_r}{(d_1 + d_2 \varepsilon_r)} \frac{s + \frac{1}{\tau_1}}{s + \frac{1}{\tau_2}} \qquad z > 0$$

$$= \frac{\varepsilon_b V_0}{\varepsilon_a d_2 + \varepsilon_b d_1} = \frac{V_0}{(d_1 + d_2 \varepsilon_r)} \frac{s}{s + \frac{1}{\tau_2}} \qquad z < 0$$

$$\tau_1 = \frac{\varepsilon_r \varepsilon_0}{\sigma} \qquad \tau_2 = \frac{\varepsilon_0}{\sigma} \left(\frac{d_1 + d_2 \varepsilon_r}{d_2} \right)$$

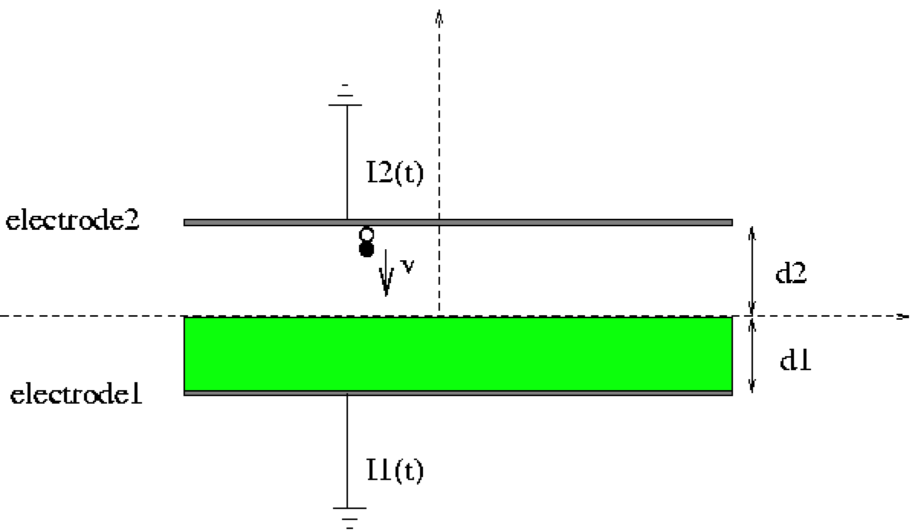
Weighting Field of Electrode 2



$$E_{2z}(s) = -E_{1z}(s)$$

Example, Induced Currents (2,4)

At t=0 a pair of charges q, -q is created at z=d_2.
One charge is moving with velocity v to z=0
Until it hits the resistive layer at T=d_2/v.



$$x_0(t) = \begin{cases} d_2 - vt & t < T \\ 0 & t > T \end{cases}$$

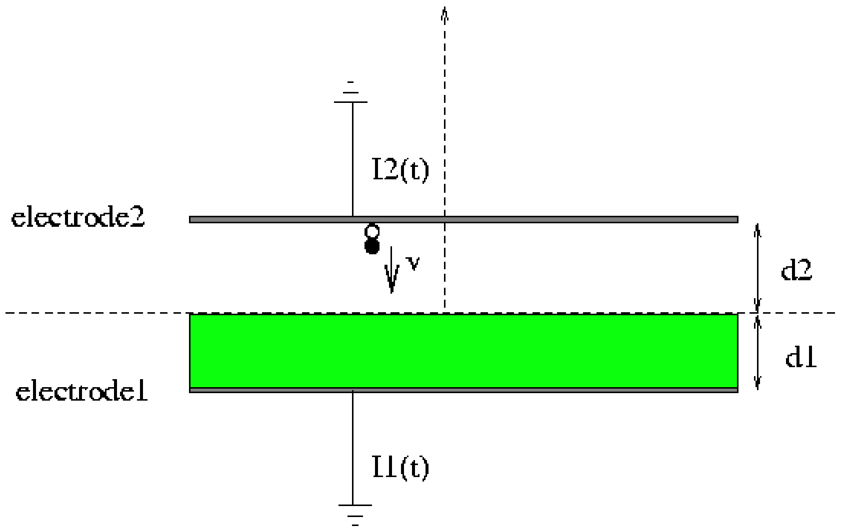
$$\dot{x}_0(t) = \begin{cases} -v & t < T \\ 0 & t > T \end{cases}$$

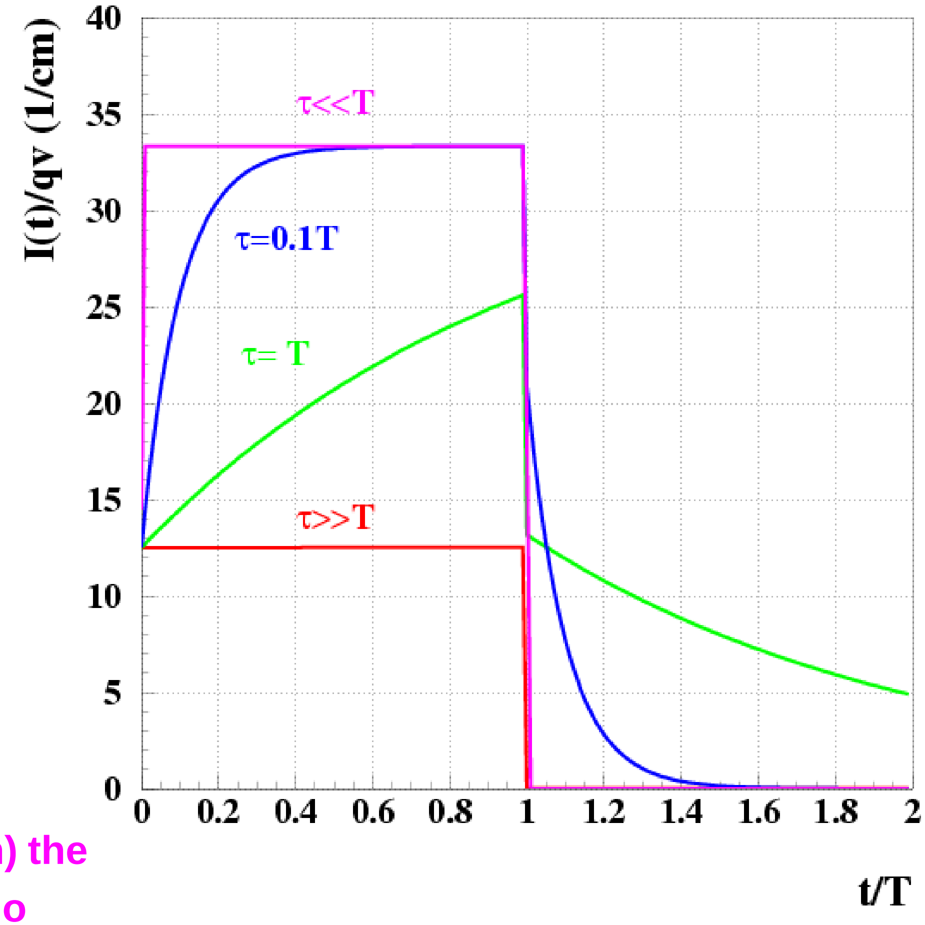
$$E_{1z}(\vec{x}, t) = \frac{\varepsilon_r V_0}{d_1 + \varepsilon_r d_2} \left[\delta(t) + \frac{\tau_2 - \tau_1}{\tau_1 \tau_2} e^{-\frac{t}{\tau_2}} \right] \qquad z > 0$$

$$I_1(t) = qv \frac{\varepsilon_r}{d_1 + \varepsilon_r d_2} \left[1 + \frac{d_1}{d_2 \varepsilon_r} (1 - e^{-\frac{t}{\tau_2}}) \right] \qquad t < T$$

$$= qv \frac{1}{d_1 + \varepsilon_r d_2} \frac{d_1}{d_2} \left(e^{\frac{T}{\tau_2}} - 1 \right) e^{-\frac{t}{\tau_2}} \qquad t > T$$

Example, Induced Currents (2,4)

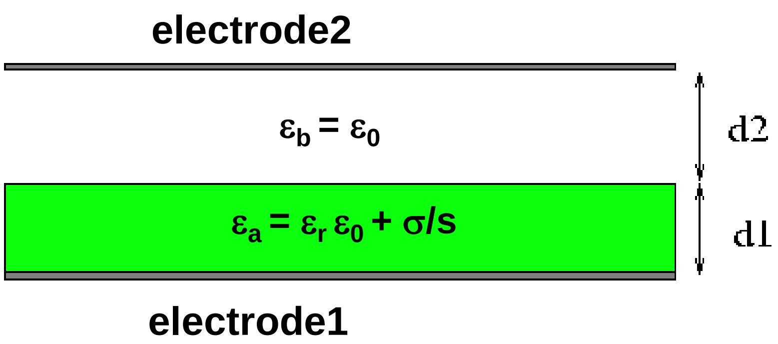


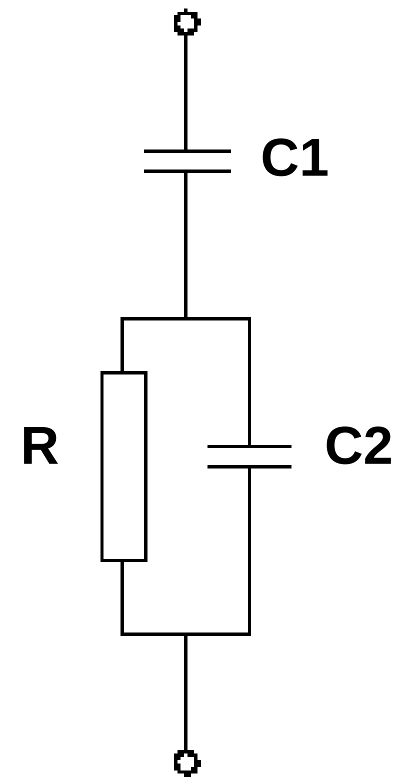


In case of high resistivity (τ>>T, RPCs, irradiated silicon) the layer is an insulator.

In case of very low resistivity (τ <<T, silicon) the layer acts like a metal plate and the scenario is equal to a parallel plate geometry with plate separation d_2.

Example, Admittance Matrix (3,4)





$$y_{nm}(s) = \frac{A\varepsilon_0 s(\sigma + \varepsilon_0 s)}{\sigma d_2 + (d1_1 + d_2)\varepsilon_0 s} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

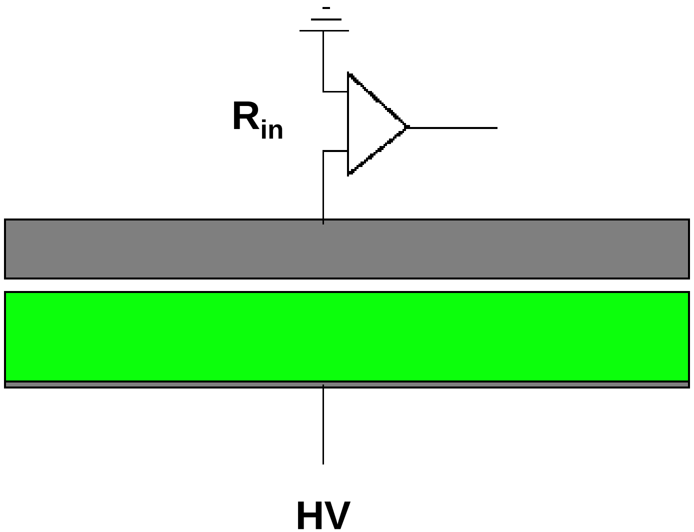
$$Z_{11}(s) = \infty$$

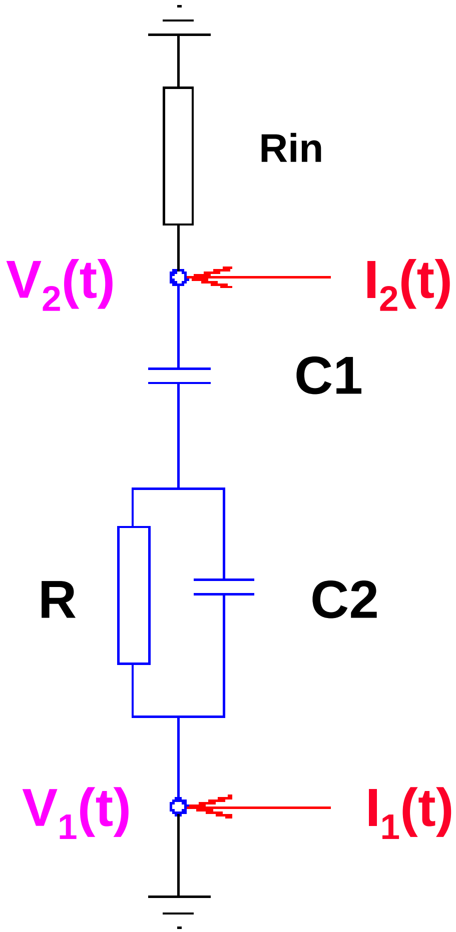
$$Z_{22}(s) = \infty$$

$$Z_{12}(s) = \frac{1}{sC_1} + \frac{R/sC_2}{R + 1/sC_2}$$

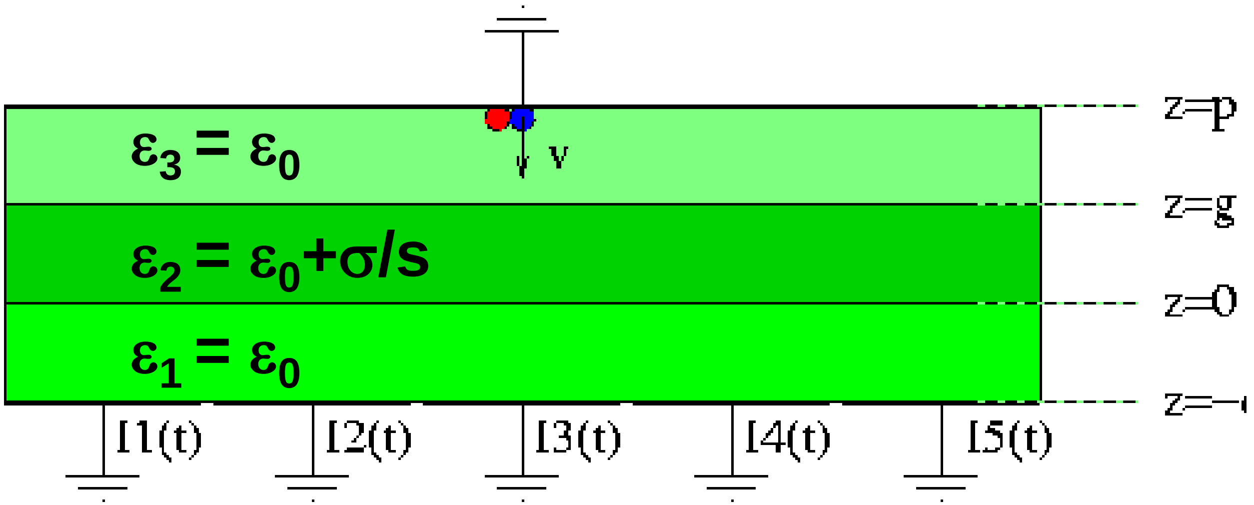
$$C_1 = \varepsilon_0 \frac{A}{d_2} \qquad C_2 = \varepsilon_r \varepsilon_0 \frac{A}{d_1} \qquad R = \frac{1}{\sigma}\frac{d_1}{A}$$

Example, Voltage (4,4)



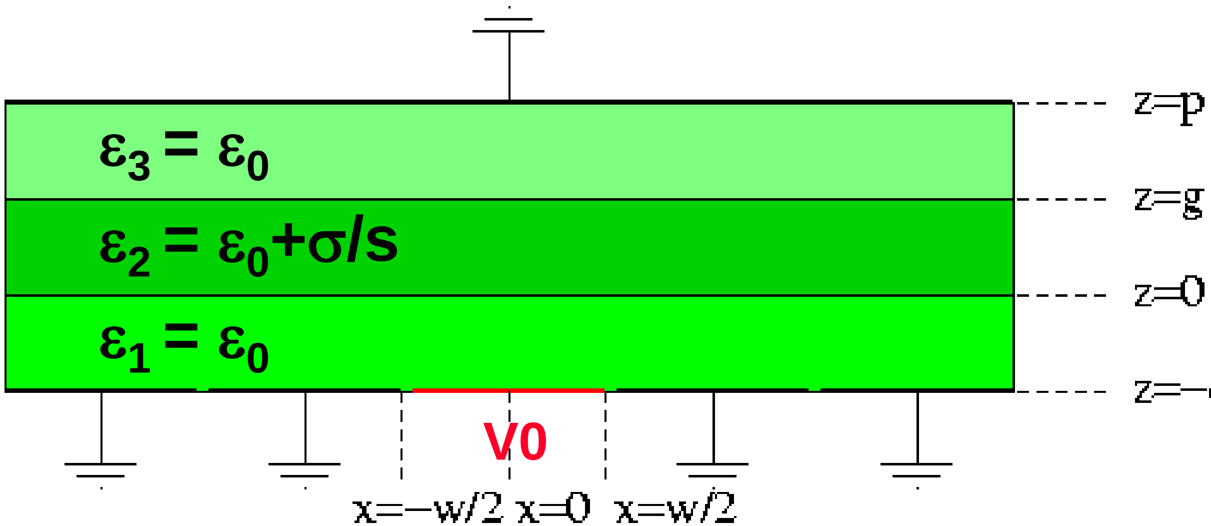


Strip Example



What is the effect of a conductive layer between the readout strips and the place where a charge is moving ?

Strip Example



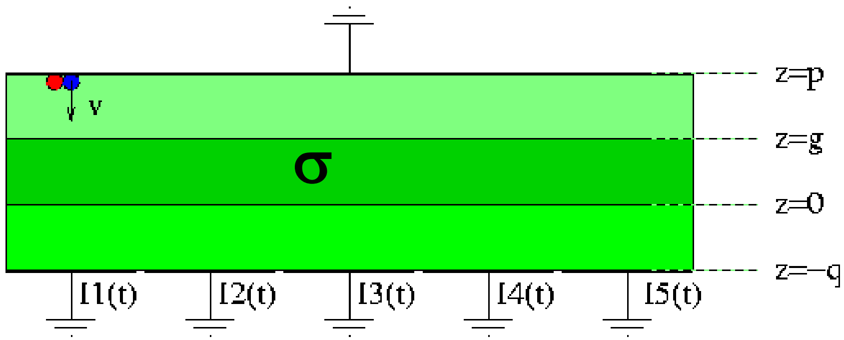
Electrostatic Weighting field (derived from B. Schnizer et. al, CERN-OPEN-2001-074):

$$E_z(x,z) = \frac{4V_0}{\pi}\int_0^\infty d\kappa \cos(\kappa x)\sin\left(\kappa\frac{w}{2}\right)\frac{2\varepsilon_1\varepsilon_2\cosh[\kappa(p-z)]}{(\varepsilon_1+\varepsilon_2)(\varepsilon_2+\varepsilon_3)\sinh[\kappa(p+q)]-(\varepsilon_1-\varepsilon_2)(\varepsilon_2+\varepsilon_3)\sinh[\kappa(q-p)]-(\varepsilon_1+\varepsilon_2)(\varepsilon_2-\varepsilon_3)\sinh[\kappa(2g+q-p)]+(\varepsilon_1-\varepsilon_2)(\varepsilon_2-\varepsilon_3)\sinh[\kappa(p+q-2g)]}$$

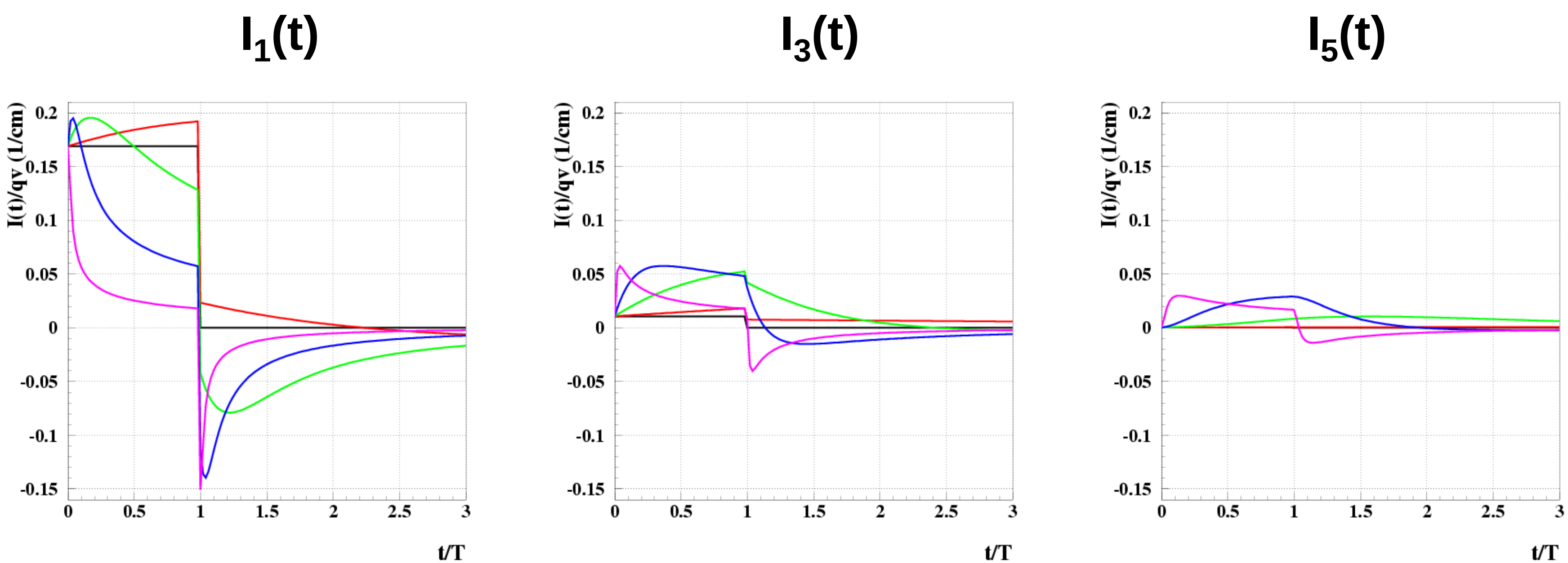
Replace $\varepsilon_1 \to \varepsilon_0$, $\varepsilon_2 \to \varepsilon_0+\sigma/s$, $\varepsilon_3 \to \varepsilon_0$ and perform inverse Laplace Transform $\to E_z(x,z,t)$. Evaluation with MATHEMATICA:

Strip Example

$T \ll \tau$
$T=\tau$
$T=10\tau$
$T=50\tau$
$T=500\tau$



$\tau = \varepsilon_0/\sigma$



The conductive layer 'spreads' the signals across the strips.

Examples for different geometries with thin resistive layers

Infinitely extended resistive layer

First we investigate an infinitely extended layer as shown in Fig. 12a. The charge Q will cause

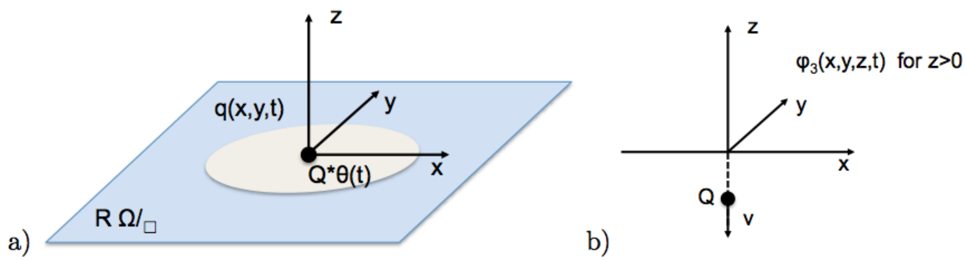


Figure 12: a) A point charge placed at an infinitely extended resistive layer at $t = 0$. b) The solution for the time dependent potential is equal to a point charge moving with velocity v along the z-axis.

$$\phi_1(r,z,t) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+(-z+vt)^2}} \qquad \phi_3(r,z,t) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+(z+vt)^2}} \tag{111}$$

We therefore conclude that the field due to a point charge placed on an infinite resistive layer at $t = 0$ is equal to the field of a charge Q that is moving with a velocity $v = 1/2\varepsilon_0 R$ away from the layer along the $z-$axis. As an example for a surface resistivity of $R = 1\,\mathrm{M}\Omega$/square the velocity is $5.6\,\mathrm{cm}/\mu\mathrm{s}$.

The time dependent surface charge density on the resistive surface is given by

$$q(r,t) = \varepsilon_0\frac{\partial\phi_1}{\partial z}|_{z=0} - \varepsilon_0\frac{\partial\phi_3}{\partial z}|_{z=0} \tag{112}$$

which evaluates to

$$\boxed{q(r,t) = \frac{Q}{2\pi}\frac{vt}{\sqrt{(r^2+v^2t^2)^3}}} \tag{113}$$

The total charge on the resistive surface $Q_{tot} = \int_0^\infty 2r\pi q(r,t)dr$ is equal to Q at any time. The peak and the FWHM of the charge density are given by

$$q_{max} = \frac{Q}{2\pi}\frac{1}{v^2t^2} \qquad FWHM = 2(4^{1/3}-1)^{1/2} \approx 1.53vt \tag{114}$$

The charge is therefore 'diffusing' with a velocity v, and does not assume a gaussian shape as expected from a diffusion effect but has $1/r^3$ tails for large values of r. The radial current $I(r)$ at distance r are given by

$$I(r) = \frac{2r\pi}{R}E(r) = -\frac{2r\pi}{R}\frac{\partial\phi_1}{\partial r}|_{z=0} = \frac{Qvr^2}{(r^2+v^2t^2)^{3/2}} \tag{115}$$

It is easily verified that the rate of change of the total charge inside a radius r i.e. $dQ_r(t)/dt = d/dt\int_0^r 2r'\pi q(r',t),dr'$ is equal the the current $I(r)$.

A point charge Q is placed on an infinitely extended resistive layer with surface resistivity of R Ohms/square at t=0.

What is the charge distribution at time t>0 ?

Note that this is not governed by any diffusion equation.

The solution is far from a Gaussian.

The timescale is giverned by the velocity $v=1/(2\varepsilon_0 R)$

Resistive layer grounded on a circle

If we now assume the geometry to be grounded at a radius $r = c$ as shown in Fig, 13a, we use Eq. 41 with $r_0 = 0$ and have the solution

$$\phi_1(r,z,t) = \frac{Q}{2\pi\varepsilon_0 c}\sum_{l=1}^{\infty}\frac{J_0(j_{0l}\frac{r}{c})}{j_{0l}J_1^2(j_{0l})}e^{-j_{0l}(t/T - z/c)} \qquad T = c/v \tag{116}$$

and $\phi_3(r,z,t) = \phi_1(r,-z,t)$. The charge inside the radius c is not a constant but it will disappear with a characteristic time constant $T = c/v$ by currents flowing into the 'grounded' ring at $r = c$. As before we can calculate the surface charge density and charge inside the radius r, which evaluate to

$$q(r,t) = \frac{Q}{c^2\pi}\sum_{l=1}^{\infty}\frac{J_0(j_{0l}r/c)}{J_1^2(j_{0l})}e^{-j_{0l}t/T} \quad Q_{tot}(t) = 2Q\sum_{l=1}^{\infty}\frac{1}{j_{0l}J_1(j_{0l})}e^{-j_{0l}t/T} \tag{117}$$

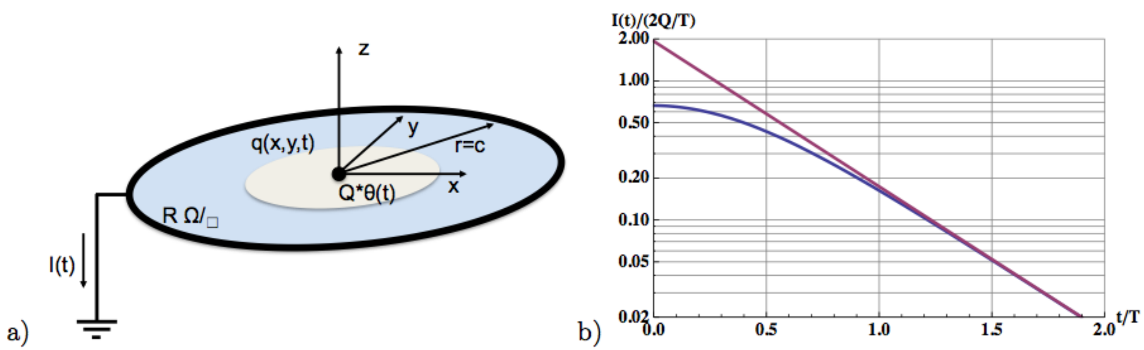


Figure 13: a) A point charge placed in the center of a resistive layer that is grounded at $r = c$. b) Current flowing to ground, where the straight line corresponds to the approximation from Eq. 119.

The current flowing into the 'grounded' ring is then again

$$I(t) = -\frac{dQ_{tot}}{dt} = \frac{2r\pi}{R}E_r(r,t) = \frac{2Q}{T}\sum_{l=1}^{\infty}\frac{1}{J_1(j_{0l})}e^{-j_{0l}t/T} \tag{118}$$

One can verify that the total amount of charge flowing to ground $\int_0^\infty I(t)dt$ is again Q. The current can be pictured to decay with an infinite number of time constants $\tau_l = T/j_{0l}$, so for large times the longest one i.e. $T/j_{01} \approx 0.42T$ will dominate and the current decays as $I(t)$. The current is plotted in Fig. 13b.

$$I(t) \approx \frac{2Q}{TJ_1(j_1)}e^{-j_{01}t/T} \qquad t \gg T \tag{119}$$

A point charge Q is placed on a resistive layer with surface resistivity of R Ohms/square that is grounded on a circle

What is the charge distribution at time t>0 ?

Note that this is not governed by any diffusion equation.

The solution is far from a Gaussian.

The charge disappears 'exponentially' with a time constant of T=c/v (c is the radius of the ring)

Resistive layer grounded on a rectangle

Next we assume a rectangular grounded boundar
Q at position x_0, y_0 at $t = 0$ as indicated in Fig. 14a

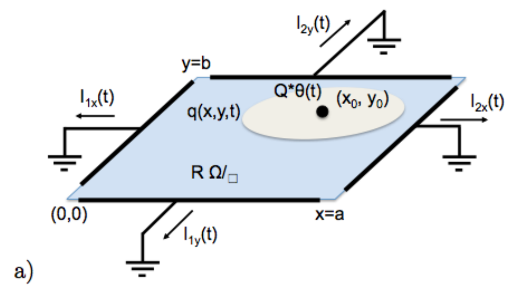


Figure 14: a) A point charge placed on a resistive layer tha
resistive layer that is grounded on at $x = 0$ and $x = a$ but in

expression Eq. 42. Assuming the currents pointing to the outside of the boundary, the currents flowing through the 4 boundaries are

$$I_{1x} = -\frac{1}{R}\int_0^b -\frac{\partial \phi_1}{\partial x}|_{x=0}dy \qquad I_{2x} = \frac{1}{R}\int_0^b -\frac{\partial \phi_1}{\partial x}|_{x=a}dy \tag{120}$$

$$I_{1y} = -\frac{1}{R}\int_0^a -\frac{\partial \phi_1}{\partial x}|_{y=0}dx \qquad I_{2y} = \frac{1}{R}\int_0^a -\frac{\partial \phi_1}{\partial x}|_{y=b}dx \tag{121}$$

which evaluates to

$$I_{1x}(t) = \frac{4Qv}{a^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{l}{m}\frac{1}{k_{lm}}\left[1-(-1)^m\right]\sin\frac{l\pi x_0}{a}\sin\frac{l\pi y_0}{b}e^{-k_{lm}vt} \tag{122}$$

$$I_{2x}(t) = \frac{4Qv}{a^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{l}{m}\frac{1}{k_{lm}}(-1)^l\left[(-1)^m-1\right]\sin\frac{l\pi x_0}{a}\sin\frac{l\pi y_0}{b}e^{-k_{lm}vt} \tag{123}$$

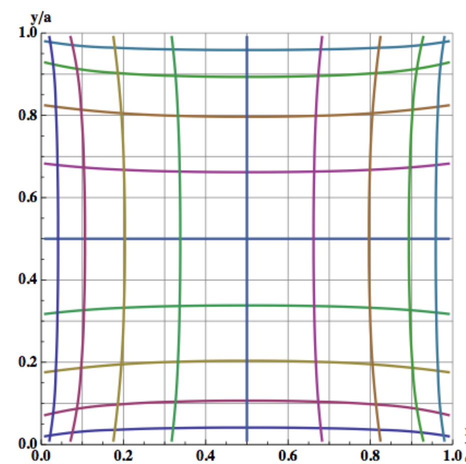
$$I_{1y}(t) = \frac{4Qv}{b^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{l}\frac{1}{k_{lm}}\left[1-(-1)^l\right]\sin\frac{l\pi x_0}{a}\sin\frac{l\pi y_0}{b}e^{-k_{lm}vt} \tag{124}$$

$$I_{2y}(t) = \frac{4Qv}{b^2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{l}\frac{1}{k_{lm}}(-1)^m\left[(-1)^l-1\right]\sin\frac{l\pi x_0}{a}\sin\frac{l\pi y_0}{b}e^{-k_{lm}vt} \tag{125}$$

In case we want to know the total charge flowing through the grounded sides we have to integrate the above expressions from $t = 0$ to ∞ which results in the same expressions and just $e^{-k_{lm}vt}$ replaced by $1/(k_{lm}v)$. These measured currents can be used to find the position of the charge, a principle that is applied in the MicroCat detector. As an example, Fig. 15 shows the correction map that has to be applied in case one just uses linear interpolation of the measured charges.

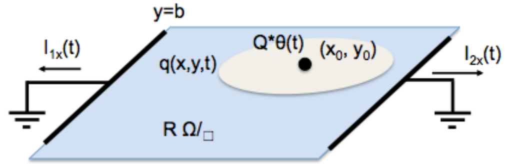
A point charge Q is placed on a resistive layer with surface resistivity of R Ohms/square that is grounded on 4 edges

What are the currents induced on these grounded edges for time t>0 ?



for the case where the position of the charge is determined by linear i
boundaries of the geometry in Fig. 14a.

Resistive layer grounded on two sides and i



b)

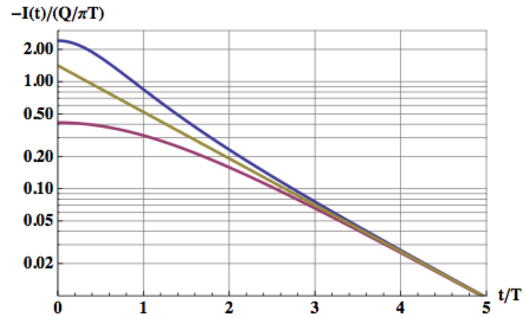


Figure 16: Currents for the geometry of Fig. 14b for $x_0 = a/4$.

5.4. Resistive layer grounded at $\pm a$ and insulated at $\pm b$.

In case the resistive layer is grounded at $x = 0, x = a$ and insulated at $y = 0, y = b$, as shown in Fig. 14, the currents are only flowing into the grounded elements at $x = 0$ and $x = a$. We use Eq. 43 and with some effort the summation can be achieved and evaluates to

$$I_{1x}(t) = -\frac{1}{R}\int_0^b -\frac{\partial \phi_1}{\partial x}|_{x=0}dy = -\frac{Q}{\pi T}\frac{\sin(\pi\frac{x_0}{a})}{\cosh(\frac{t}{T}) - \cos(\pi\frac{x_0}{a})} \tag{126}$$

$$I_{2x}(t) = \frac{1}{R}\int_0^b -\frac{\partial \phi_1}{\partial x}|_{x=a}dy = -\frac{Q}{\pi T}\frac{\sin(\pi\frac{x_0}{a})}{\cosh(\frac{t}{T}) + \cos(\pi\frac{x_0}{a})} \tag{127}$$

with $T = \frac{2a\varepsilon_0 R}{\pi} = \frac{a}{\pi v}$. For large times both expressions tend to

$$I_{1x}(t) = I_{2x}(t) \approx -\frac{2Q}{\pi T}\cos\left(\pi\frac{x_0}{a}\right) e^{-t/T} \tag{128}$$

Fig. 16 shows the two currents for a charge deposit at position $x_0 = a/4$ together with the asymptotic expression from Eq. 128. The total charge that is flowing through the grounded ends is given by

$$q_1 = \int_0^\infty I_{1x}(t)dt = Q\frac{a - x_0}{a} \qquad q_2 = \int_0^\infty I_{2x}(t)dt = Q\frac{x_0}{a} \tag{129}$$

so we learn that the charges are just shared in proportion to the distance from the grounded boundary, equal to the resistive charge division.

A point charge Q is placed on a resistive layer with surface resistivity of R Ohms/square that is grounded on 2 edges and insulated on the other two.

What are the currents induced on these grounded edges for time t>0 ?

The currents are monotonic.

Both of the currents approach exponential shape with a time constant T.

The measured total charges satisfy the simple resistive charge division formulas.

Infinitely extended resistive layer with parallel ground plane

Assuming an infinitely extended geometry, the time dependent charge density evaluates to

$$q(r,t) = \frac{Q}{b^2\pi}\frac{1}{2}\int_0^\infty \kappa J_0(\kappa\frac{r}{b})\,\exp\left[-\kappa(1-e^{-2\kappa})\frac{t}{T}\right]d\kappa \qquad T=\frac{b}{v}=2b\varepsilon_0 R \tag{134}$$

It can be verified that $\int_0^\infty 2r\pi q(r,t)dr = Q$ at any time. For long times i.e. large values of t/T we can approximate the exponent of the above expression by

$$-\kappa(1-e^{-2\kappa})\frac{t}{T} \approx -2\kappa^2\frac{t}{T} \tag{135}$$

and the integral evaluates to

$$q(r,t) = \frac{Q}{b^2\pi}\frac{1}{8t/T}e^{-\frac{r^2}{8b^2t/T}} \tag{136}$$

In analogy to the one dimensional transmission line, the discussed geometry is often assumed to be defined by the two dimensional diffusion equation

$$\frac{\partial q}{\partial t} = h\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}\right) \qquad h=1/RC \quad C=\frac{\varepsilon_0}{b} \tag{137}$$

where C is the capacitance per unit area between the resistive layer and the grounded plate. The solution of this equation for a point charge Q put at $r=0, t=0$ evaluates exactly to the above Gaussian expression In Fig. 19 the charge distribution from Eq. 134 is compared to the above Gaussian as well as Eq. 113 for the geometry without a ground plane. Although the order of magnitude is similar, the solution of the diffusion equation does not work very well. The reason for the discrepancy can be understood when investigating how Eq. 135 is derived: the current $\vec{j}(x,y,t)$ flowing inside the resistive layer is related to the electric field $\vec{E}(x,y,t)$ in the resistive layer by $\vec{j}=\vec{E}/R$. The relation between the current and the charge density $q(x,y,t)$ is $\vec{\nabla}\vec{j} = -\partial q/\partial t$. With $\vec{E}=-\vec{\nabla}\phi$ we then get

$$\frac{\partial q}{\partial t} = \frac{1}{R}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right) \tag{138}$$

If we set $q=C\phi$ we have the diffusion equation Eq. 135. This relation between voltage and charge($Q=CU$) is however only a good approximation if the charge distribution does not have a significant gradient over distances of the order of b. For small times when the charge distribution is very peaked around zero this is certainly not a good approximation. It means that for long times when the distribution if very broad when compared to the distance b the two solutions should approach each other. Indeed this can be seen if we calculate the current that is induced on the grounded plate, which we do next. The presence of the charge on the resistive layer induces a charge on the grounded metal plane. If we assume that the metal plane is segmented into strips, as shown in Fig. 20b, we can calculate the induced charge through the electric field on the surface of the plane. Assuming a strip centred at $x=x_p$ with a width of w and infinite extension in y direction, we find the induced charge to

$$Q_{ind}(t) = \int_{x_p-w/2}^{x_p+w/2}\int_\infty^\infty -\varepsilon_0\frac{\partial\phi_1}{\partial z}|_{z=-b}dydx \tag{139}$$

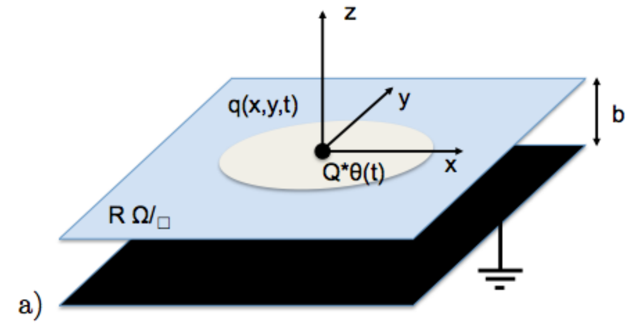


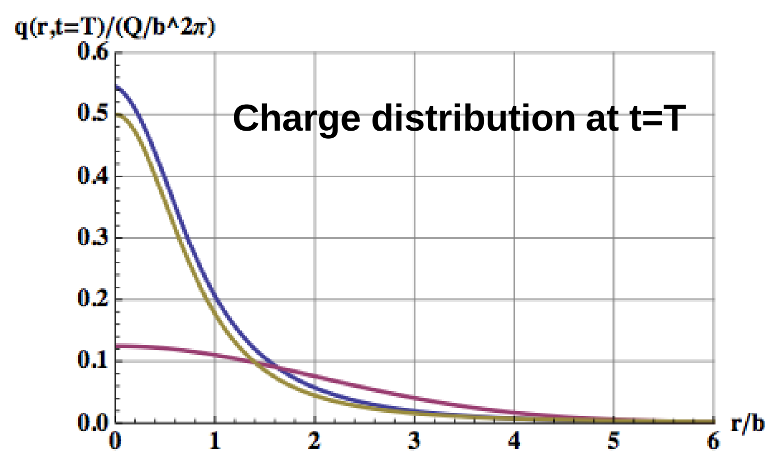
Figure 18: a) An infinitely extended resistive layer in presenc radius $r = c$.

A point charge Q is placed on an infinitely extended resistive layer with surface resistivity of R Ohms/square and a parallel ground plane at t=0.

What is the charge distribution at time t>0 ?

This process is in principle NOT governed by the diffusion equation.

In practice is is governed by the diffusion equation for long times.



Infinitely extended resistive layer with parallel ground plane

In analogy to the one dimensional transmission line, the discussed geometry is often assumed to be defined by the two dimensional diffusion equation

$$\frac{\partial q}{\partial t} = h\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}\right) \qquad h = 1/RC \quad C = \frac{\varepsilon_0}{b} \tag{137}$$

where C is the capacitance per unit area between the resistive layer and the grounded plate. The solution of this equation for a point charge Q put at $r = 0, t = 0$ evaluates exactly to the above Gaussian expression In Fig. 19 the charge distribution from Eq. 134 is compared to the above Gaussian as well as Eq. 113 for the geometry without a ground plane. Although the order of magnitude is similar, the solution of the diffusion equation does not work very well. The reason for the discrepancy can be understood when investigating how Eq. 135 is derived: the current $\vec{j}(x,y,t)$ flowing inside the resistive layer is related to the electric field $\vec{E}(x,y,t)$ in the resistive layer by $\vec{j} = \vec{E}/R$. The relation between the current and the charge density $q(x,y,t)$ is $\vec{\nabla}\vec{j} = -\partial q/\partial t$. With $\vec{E} = -\vec{\nabla}\phi$ we then get

$$\frac{\partial q}{\partial t} = \frac{1}{R}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) \tag{138}$$

If we set $q = C\phi$ we have the diffusion equation Eq. 135. This relation between voltage and charge($Q = CU$) is however only a good approximation if the charge distribution does not have a significant gradient over distances of the order of b. For small times when the charge distribution is very peaked around zero this is certainly not a good approximation. It means that for long times when the distribution if very broad when compared to the distance b the two solutions should approach each other. Indeed this can be seen if we calculate the current that is induced on the grounded plate, which we do next. The presence of the charge on the resistive layer induces a charge on the grounded metal plane. If we assume that the metal plane is segmented into strips, as shown in Fig. 20b, we can calculate the induced charge through the electric field on the surface of the plane. Assuming a strip centred at $x = x_p$ with a width of w and infinite extension in y direction, we find the induced charge to

$$Q_{ind}(t) = \int_{x_p-w/2}^{x_p+w/2}\int_{\infty}^{\infty} -\varepsilon_0 \frac{\partial \phi_1}{\partial z}|_{z=-b}\, dy dx \tag{139}$$

which evaluates to

$$Q_{ind}(t) = \frac{2Q}{\pi}\int_0^\infty \frac{1}{\kappa}\cos(\kappa\frac{x_p}{b})\sin(\kappa\frac{w}{2b})\exp\left[-\kappa - \kappa(1-e^{-2\kappa})\frac{t}{T}\right]d\kappa \tag{140}$$

The solution of the diffusion equation assumes the relation of a capacitor where the ground plate should just carry the charge density $-q(x,y,t)$, so the total charge on the strip is

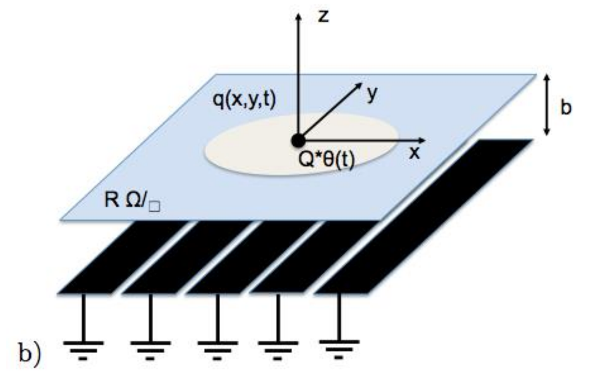
$$Q^g_{ind}(t) = \int_{x_p-w/2}^{x_p+w/2}\int_{\infty}^{\infty} q_g(x,y)dxdy = \frac{Q}{2}\left[\text{erf}\left(\frac{2x_p+w}{4b\sqrt{2t/T}}\right) - \text{erf}\left(\frac{2x_p-w}{4b\sqrt{2t/T}}\right)\right] \tag{141}$$

Both expression are shown in Fig. 19b. Although there are significant differences at small times the curves approach each other for longer times when the charge distribution becomes broad. Indeed, if take Eq. 139 we see that for large values of t/T only small values of κ contribute to the integral, so if we expand the exponent as

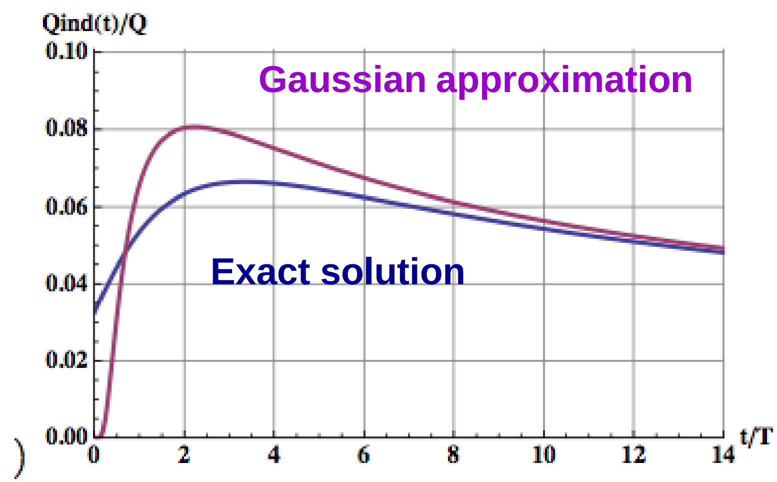
$$-\kappa - \kappa(1-e^{-2\kappa})\frac{t}{T} \approx -2\kappa^2\frac{t}{T} \tag{142}$$

the integral evaluates precisely to expression Eq. 140.

What are the charges induced metallic readout electrodes by this charge distribution?



ıce of a grounded layer. b) The same geometry grounded at a



Resistive layer grounded on a circle with parallel ground plane

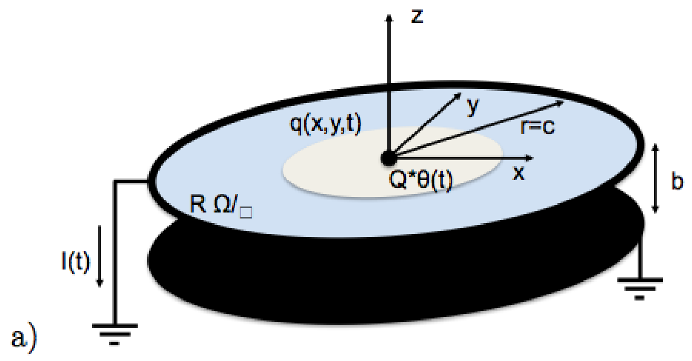


Figure 20: b) The same geometry grounded at a radius $r = c$.

To conclude this geometry we assume the geometry to be grounded at $r = 0$ as shown in Fig. 20b. We proceed as above and the charge Q_{tot} inside the radius c is given by

$$Q_{tot}(t) = 2Q \sum_{l=1}^{\infty} \frac{1}{j_{0l} J_1(j_{0l})} \exp\left[-j_{0l}(1 - e^{-2j_{0l}b/c})\frac{t}{T}\right] \tag{142}$$

The charge disappears with and infinite number of time constants

$$\tau_l = \frac{T}{j_{0l}(1 - e^{-2j_{0l}b/c})} \tag{143}$$

If the radius of the circle c is much larger than the distance b the longest time constant approximates as above to $\tau_1 \approx T/j_{01}$. In case $b \ll c$ we have $\tau_1 \approx 0.2c/b$, which tells us that the closer the resistive layer is to the grounded plane the slower the charge will disappear.

Uniform currents on resistive layers

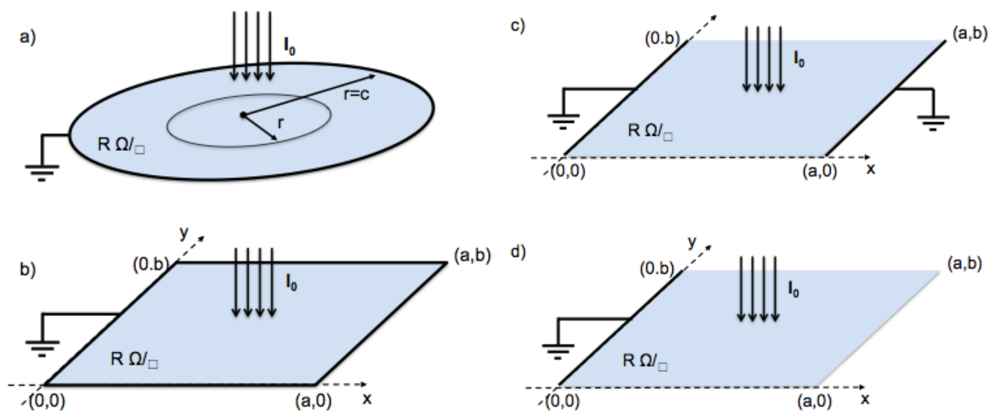


Figure 21: A uniform current 'impressed' on the resistive layer will result in a potential distribution that depends strongly on the boundary conditions. The 4 geometries shown in this figure are discussed.

Uniform illumination of the resistive layers results in 'chargeup' and related potentials.

In this section we want to discuss the potentials that are created on thin resistive layers for uniform charge deposition. In detectors like RPCs and Resistive Micromegas such resistive layers are used for application of the high voltage and for spark protection. The resistivity must be chosen small enough to ensure that potentials that are established on these layers are not influencing the applied electric fields responsible for the proper detector operation. If such detectors are in an environment of uniform particle irradiation the situation can be formulated by placing a uniform 'externally impressed' current per unit area $I_0\,[\mathrm{A/cm^2}]$ on the resistive layer. First we want to investigate the geometry shown in Fig. 21a) where the layer is grounded on a circle at $r = c$. The charge dq placed on an infinitesimal area at position r_0, ϕ_0 after time t is given by $dq(t) = I_0 r_0 dr_0 d\phi_0 t$, or in the Laplace domain $dq(s) = I_0 r_0 dr_0 d\phi_0 / s^2$. We therefore have to replace Q/s in Eq. ?? by $q(s)$, which results in

$$f_1(k,z,s) = \frac{I_0}{s}\frac{R r_0 dr_0 d\phi_0}{k + 2\varepsilon_0 R s}\, e^{kz} \qquad f_2(k,z,s) = \frac{I_0}{s}\frac{R r_0 dr_0 d\phi_0}{k + 2\varepsilon_0 R s}\, e^{-kz} \tag{144}$$

Since we want to know the steady situation for long times i.e. for $t \to \infty$ we $f(k,z,t\to\infty) = \lim_{s\to 0} s f(k,z,s)$ and have

$$f_1(k,z) = \frac{R I_0 r_0 dr_0 d\phi_0}{k}\, e^{kz} \qquad f_2(k,z) = \frac{R I_0 r_0 dr_0 d\phi_0}{k}\, e^{-kz} \tag{145}$$

Inserting this into Eq. ?? and integration over the area of the disk $\int_0^c dr_0 \int_0^{2\pi} d\phi_0$ we find the expression

$$\phi_1(r,z) = \phi_3(r,-z) = 2c^2 R I_0 \sum_{l=1}^{\infty} \frac{J_0(j_{ml} r/c)}{j_{0l}^3 J_1(j_{ml})}\, e^{j_{0l} z/c} \tag{146}$$

For $z = 0$ i.e. on the surface of the resistive layer, the expression can be summed and we have

$$\phi_1(r,z=0) = \phi_3(r,z=0) = \frac{1}{4} R I_0 (c^2 - r^2) \tag{147}$$

This expression can also be derived in an elementary way: the total current on a disc of radius r i.e. $r^2\pi I_0$, is equal to the total correct flowing at radius r i.e. $2r\pi E_r/R$. This defines the radial field inside the layer to $E_r = RI_0 r/2$. With the boundary condition $\phi(r) = \int_0^c E_r(r)dr = 0$ we find back the above expression. The maximum potential is therefore in the centre of the disc and is equal to

$$\phi(r=0) = \frac{c^2\pi RI_0}{4\pi} = \frac{1}{4\pi}RI_{tot} \approx 0.08\,RI_{tot} \tag{148}$$

For the potentials in the rectangular geometry of Fig. 21b we again have f_1, f_2 from Eq. 145 we just have to replace $r_0 dr_0 d\phi_0$ by $dx_0 dy_0$ and perform the integration $\int_0^a dx_0 \int_0^b dy_0$ of Eq. ??, which results in

$$\phi_1(x,y,z) = \phi_3(x,y,-z) = abRI_0 \frac{4}{\pi^4}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{[1-(-1)^l][1-(-1)^m]\sin(l\pi x/a)\sin(m\pi y/b)}{l^3 mb/a + m^3 la/b}\,e^{k_{lm}z} \tag{149}$$

The expression cannot be written in closed form but converges quickly, so numerical evaluation is straight forward. The peak of the potential can be found by setting $d\phi_1/dx = 0, d\phi_1/dy = 0$ and is found at $x = a/2, y = b/2$, which is also evident by the symmetry of the geometry. The maximum potential on the resistive layer is then

$$\phi_{max} = \phi(a/2, b/2, z=0) = \frac{1}{8}RI_0 a^2 \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{128}{\pi^4}\frac{(-1)^{l+m}}{(2l-1)^3(2m-1)+(2m-1)^3(2l-1)a^2/b^2} \tag{150}$$

For a square geometry ($b = a$) the sum evaluates to ≈ 0.59 so the peak voltage in the center is

$$\phi_{max} \approx 0.074 RI_0 a^2 = 0.074\,RI_{tot} \tag{151}$$

We see that the value is only less than 10 % different from the peak voltage for the circular boundary in Eq. 148. For uniform illumination of the geometry Fig. 21c that is grounded at $x = 0, a$ and insulated at $y = 0, b$ we use expression Eq. ?? and proceed as before and find

$$\phi_1(x,z) = \phi_3(x,-z) = 2RI_0 a^2 \sum_{l=1}^{\infty}\frac{(1-(-1)^l)\sin(l\pi x/a)}{l^3\pi^3}\,e^{l\pi z/a} \tag{152}$$

The potential is is independent of y and for $z = 0$ the sum can be written inclosed form

$$\phi_1(x, z=0) = \frac{1}{2}RI_0(ax - x^2) \qquad \phi_{max} = \frac{1}{8}a^2 RI_0 \tag{153}$$

Again this expression can be found in an elementary way by the fact that due to symmetry the currents can only flow in x-direction and the current at $x = a/2$ must be zero. The total current arriving on the area of $x = a/2 \pm s$ i.e. $2sbI_0$ is equal to the total current flowing at distance s i.e. $2E(s)/Rb$. With $x = a/2 + s$ we find back the above expression. The potential is therefore independent of b. For large values of b/a the expression Eq. 151 must therefore approach the same value. Indeed for $a/b = 0$ the sum evaluates to unity and the expression agree. From Fig. ?? we see that for aspect ratios of 4:1 the expressions is agree already within 10 % of the

Finally, in case the layer is only grounded at $x = 0$ and all other boundaries are insulated, the maximum potential is at $x = a$ and the results are

$$\phi_1(x) = \frac{1}{2}RI_0(2ax - x^2) \qquad \phi_{max} = \frac{1}{2}RI_0 a^2 \tag{154}$$

7. Uniform currents on thin resistive layers

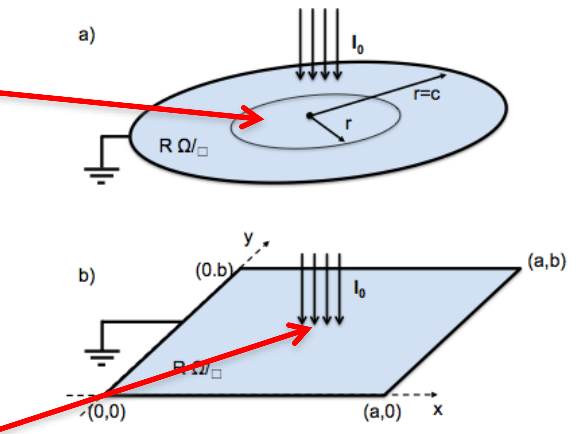
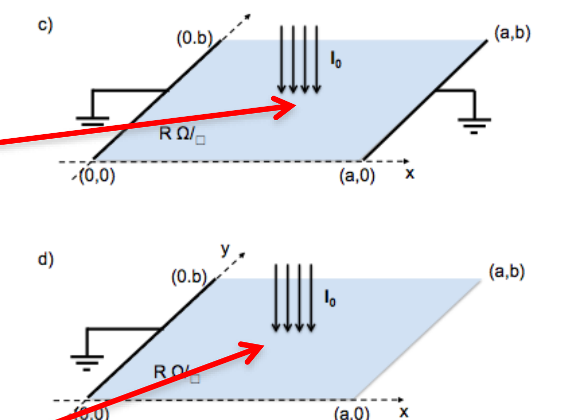


Figure 21: A uniform current 'impressed' on the resistive laye
on the boundary conditions. The 4 geometries shown in this f



er will result in a potential distribution that depends strong
figure are discussed.

Summary

Theorems for calculating fields and signals in detectors with resistive elements exist.

Exact solutions for a few basic geometries were given.

The diffusion equation is only an approximate description of charge diffusion on thin resistive layers.

Under well defined conditions,

specifically when the gradient of the charge distributions over distances on the order of the ground plane distance are small ($t \gg T$)

the diffusion equation which leads to Gaussian charge distributions can be applied.