Dissipative relativistic hydrodynamics

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- Motivation
  - Problems with second order theories
  - Thermodynamics, fluids and stability
- Generic stability of relativistic dissipative fluids
- Temperature of moving bodies
- Summary

Internal energy:

$$\varepsilon = \sqrt{e^2 - \mathbf{q}^2}$$

Dissipative relativistic fluids

	Nonrelativistic	Relativistic
Local equilibrium (1st order)	Fourier+Navier-Stokes	~~Eckart~~ (1940), Tsumura-Kunihiro (2008)
Beyond local equilibrium (2^{nd} order)	Cattaneo-Vernotte, gen. Navier-Stokes	Israel-Stewart (1969-72), Pavón, Müller-Ruggieri, Geroch, Öttinger, Carter, conformal, etc.

Eckart:

$$S^a(T^{ab}, N^a) = s(e,n)u^a + \frac{q^a}{T}$$

Extended (Israel–Stewart – Pavón–Jou–Casas-Vázquez):

$$S^a(T^{ab}, N^a) = \left(\underline{s(e,\, n)} - \frac{\beta_0}{2T}\Pi^2 - \frac{\beta_1}{2T} q_b q^b - \frac{\beta_2}{2T}\pi^{bc}\pi_{bc} \right) u^a +$$

$$+ \underline{\frac{1}{T}} \left(q^a + \alpha_0 \Pi q^a + \alpha_1 \pi^{ab} q_b \quad \right)$$

(+ order estimates)

Remarks on causality and stability:

<u>Symmetric hyperbolic equations ~ causality</u>

$$\partial_t T - \kappa \partial_{xx} T = 0,$$

$$\gamma(\partial_{\tilde{t}} - v\partial_{\tilde{x}})T - \kappa\left(\partial_{\tilde{x}\tilde{x}} - 2\frac{v}{c^2}\partial_{\tilde{x}\tilde{t}} + \frac{v^2}{c^4}\partial_{\tilde{t}\tilde{t}}\right)T = 0$$

– The extended theories are *not* proved to be symmetric hyperbolic.

– In Israel-Stewart theory the symmetric hyperbolicity *conditions* of the *perturbation* equations follow from the stability conditions.

– Parabolic theories cannot be excluded – speed of the validity range can be small. Moreover, they can be extended later.

<u>Stability of the homogeneous equilibrium</u> (generic stability) is required.

– Fourier-Navier-Stokes limit. Relaxation to the (unstable) first order theory? (Geroch 1995, Lindblom 1995)

Fourier-Navier-Stokes

$$\dot{n} + n\partial_i v^i = 0,$$

$$\dot{\varepsilon} + \varepsilon\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j = 0,$$

$$\dot{k}^i + k^i\partial_j v^j + \partial_j P^{ij} = 0^i,$$

$$q^i = -\lambda\partial^i T,$$

$$\Pi^{ij} = -\xi\partial_k v^k \delta^{ij} - 2\eta\langle\partial_i v^j\rangle.$$

$$q^i\partial_i \frac{1}{T} - \frac{1}{T}\Big(P^{ij} - \underbrace{(Ts + \mu n - \varepsilon)}_{p}\delta^{ij}\Big)\partial_i v_j \geq 0$$

Isotropic linear constitutive relations,
<> is symmetric, traceless part

Equilibrium:

$$n(x_i,t) = const., \quad \varepsilon(x_i,t) = const., \quad v^i(x_i,t) = const.$$

Linearization, …, Routh-Hurwitz criteria:

$$\lambda > 0, \quad \eta > 0, \quad \xi > 0,$$

$$\partial_\varepsilon T > 0,$$

Thermodynamic stability (concave entropy)

$$\underbrace{(\varepsilon + p)\partial_\varepsilon p + n\partial_n p > 0}_{\text{Hydrodynamic stability}} \quad \Leftrightarrow \quad \underbrace{\partial_\varepsilon T\partial_n \frac{\mu}{T} - \partial_n T\partial_\varepsilon \frac{\mu}{T} > 0}_{-T^2 Det(\partial^2 s)}$$

Remarks on stability and Second Law:

Non-equilibrium thermodynamics:

basic variables
evolution equations (basic balances) } + Second Law

→ Stability of homogeneous equilibrium

Entropy ~ Lyapunov function

<u>Homogeneous systems</u> (equilibrium thermodynamics):

dynamic reinterpretation – ordinary differential equations

clear, mathematically strict

See e.g. Matolcsi, T.: Ordinary thermodynamics, Academic Publishers, 2005

<u>Continuum systems</u> (irreversible thermodynamics):

partial differential equations – Lyapunov theorem is more technical

→ Linear stability (of homogeneous equilibrium)

Stability conditions of the Israel-Stewart theory

(Hiscock-Lindblom 1985)

$$\Omega_1 = \frac{1}{e+p}\left.\frac{\partial e}{\partial p}\right|_{\frac{s}{n}} = \frac{T}{(e+p)^2\left.\frac{\partial T}{\partial e}\right|_n - n^2T^2\left.\frac{\partial}{\partial n}\frac{\mu}{T}\right|_e} \geq 0,$$

$$\Omega_2 = \frac{1}{e+p}\left.\frac{\partial e}{\partial(s/n)}\right|_p \left.\frac{\partial p}{\partial(s/n)}\right|_{\frac{\mu}{nT}} = \ldots \geq 0,$$

$$\Omega_5 = \beta_0 \geq 0, \qquad \Omega_8 = \beta_2 \geq 0, \qquad \Omega_7 = \beta_1 - \frac{\alpha_1^2}{2\beta_2} \geq 0,$$

$$\Omega_4 = e + p - \frac{2\beta_2 + \beta_1 + \alpha_1}{2\beta_1\beta_2 - \alpha_1^2} \geq 0, \qquad \Omega_6 = \beta_1 - \frac{\alpha_0^2}{\beta_0} - \frac{2\alpha_1^2}{3\beta_2} - \frac{1}{n^2T}\left.\frac{\partial T}{\partial e}\right|_n \geq 0,$$

$$\Omega_3 = (e+p)\left(1 - \left.\frac{\partial p}{\partial e}\right|_{\frac{s}{n}}\right) - \frac{1}{\beta_0} - \frac{2}{3\beta_2} - \frac{K^2}{\Omega_6} \geq 0, \qquad K = 1 + \frac{\alpha_0}{\beta_0} - \frac{2\alpha_1}{3\beta_2} - \left.\frac{\partial p}{\partial e}\right|_n \geq 0.$$

Special relativistic fluids (Eckart):

$$T^{ab} = eu^a u^b + q^a u^b + q^b u^a + P^{ab},$$ energy-momentum density

$$N^a = nu^a + j^a.$$ particle density vector

$$q^a u_a = j^a u_a = 0, \quad P^{ba} u_a = P^{ab} u_a = 0^b$$

q^a – momentum density or energy flux??

General representations by local rest frame quantities.

$$\partial_a S^a = \dot{s}(e,n) + s\partial_a u^a + \partial_a J^a \geq 0$$

$$\dot{e} = u^a \partial_a e$$

$$J^a = \frac{q^a}{T} - \frac{\mu}{T} j^a$$

$$u^a \partial_b T^{ab} = \dot{e} + e\partial_a u^a + \partial_a q^a + u_a \dot{q}^a$$

$$\partial_b N^b = \dot{n} + n\partial_a u^a + \partial_a j^a$$

$$-\frac{1}{T}\left(P^{ij} - p\delta^{ij}\right)\partial_i v_j + q^i \partial_i \frac{1}{T} \geq 0$$

$$\sigma_s = j^a \partial_a \frac{\mu}{T} - \frac{1}{T}(P^{ab} - p\delta^{ab})\partial_b u_a - \frac{q^a}{T^2}(\partial_a T - T\dot{u}_a) \geq 0$$

Eckart term

Second Law (Liu procedure) – first order weakly nonlocal:

Entropy inequality with the conditions of energy-momentum and particle number balances as constraints:

$$\partial_a S^a - \Lambda_a \partial_b T^{ab} - \lambda \partial_a N^a \geq 0$$

Consequences:

State space: (e, u^a, n)

1) $$s(e,u^a,n) = s(e, q^a(e,u^a), n)$$

2) $$e\frac{\partial s}{\partial q^a} = q_a \frac{\partial s}{\partial e} \Rightarrow$$

$$s(e,q^a) = \hat{s}(e^2 - \mathbf{q}^2) = \tilde{s}\left(\sqrt{e^2 - \mathbf{q}^2}\right)$$

3) $$J^a = \frac{q^a}{T} - \frac{\mu}{T} j^a$$

Ván: JMMS, 2008, **3**/6, 1161, **(**arXiv:07121437)

Modified relativistic irreversible thermodynamics:

Internal energy: $\varepsilon = \sqrt{e^2 - \mathbf{q}^2} = \sqrt{\varepsilon^a \varepsilon_a} = \sqrt{u_b T^{ba} T_{ac} u^c}$

$$\partial_a S^a = \dot{s}(\varepsilon, n) + s\partial_a u^a + \partial_a J^a \geq 0$$

$$J^a = \frac{q^a}{T} - \frac{\mu}{T} j^a$$

$$u^a \partial_b T^{ab} = \dot{e} + e\partial_a u^a + \partial_a q^a + u_a \dot{q}^a + u_a \partial_b P^{ab} = 0$$

$$\partial_b N^b = \dot{n} + n\partial_a u^a + \partial_a j^a$$

$$\sigma_s = j^a \partial_a \frac{\mu}{T} - \frac{1}{T}\left(P^{ab} - p\delta^{ab}\right)\partial_b u_a - \frac{q^a}{T^2}\left(\partial_a T + T\dot{u}_a + T\frac{\dot{q}_a}{e}\right) \geq 0$$

Eckart term

Ván and Bíró EPJ, (2008), **155**, 201. (arXiv:0704.2039v2)

Dissipative hydrodynamics

$$\partial_a N^a = \dot{n} + n\partial_a u^a + \partial_a j^a = 0,$$

$$u^a \partial_b T^{ab} = \dot{e} + (e+p)\partial_a u^a + \partial_a q^a + q^a \dot{u}_a - \Pi^{ab}\partial_b u_a = 0,$$

$$\Delta^a_c \partial_b T^{cb} = (e+p)\dot{u}^a + q^a \partial_b u^b + q^b \partial_b u^a + \Delta^a_c(\dot{q}^c + \partial_b \Pi^{cb}) = 0^a,$$

$$q^a = -\lambda \Delta^{ac}\left(\partial_c T + T\dot{u}_c + \boxed{T\frac{\dot{q}^a}{e}}\right),$$

$$\nu^a = -\varsigma \Delta^{ac}\partial_c \frac{\mu}{T},$$

$$\Pi^a_a = P^a_a - p = -\xi \partial_c u^c,$$

$$\Pi^a_b = -2\eta < \partial_b u^a > .$$

$< \; >$ symmetric traceless spacelike part

$\Rightarrow$ linear stability of homogeneous equilibrium

Conditions: thermodynamic stability, nothing more.

(Ván: arXiv:0811.0257)

Thermostatics:

$$de - \frac{q^a}{e} dq_a = Tds + \mu dn \;\Leftrightarrow\; s\left(\sqrt{e^2 + q^a q_a}, n\right) = \hat{s}\left(e, q^a, n\right)$$

Temperatures and other intensives are doubled:

$$\frac{\partial s}{\partial \boldsymbol{\varepsilon}} = \frac{1}{\Theta}; \quad \frac{\partial \hat{s}}{\partial e} = \frac{1}{T} \quad \Rightarrow \quad eT = \boldsymbol{\varepsilon}\Theta, \;\; e\boldsymbol{\mu} = \boldsymbol{\varepsilon}\mathrm{M}$$

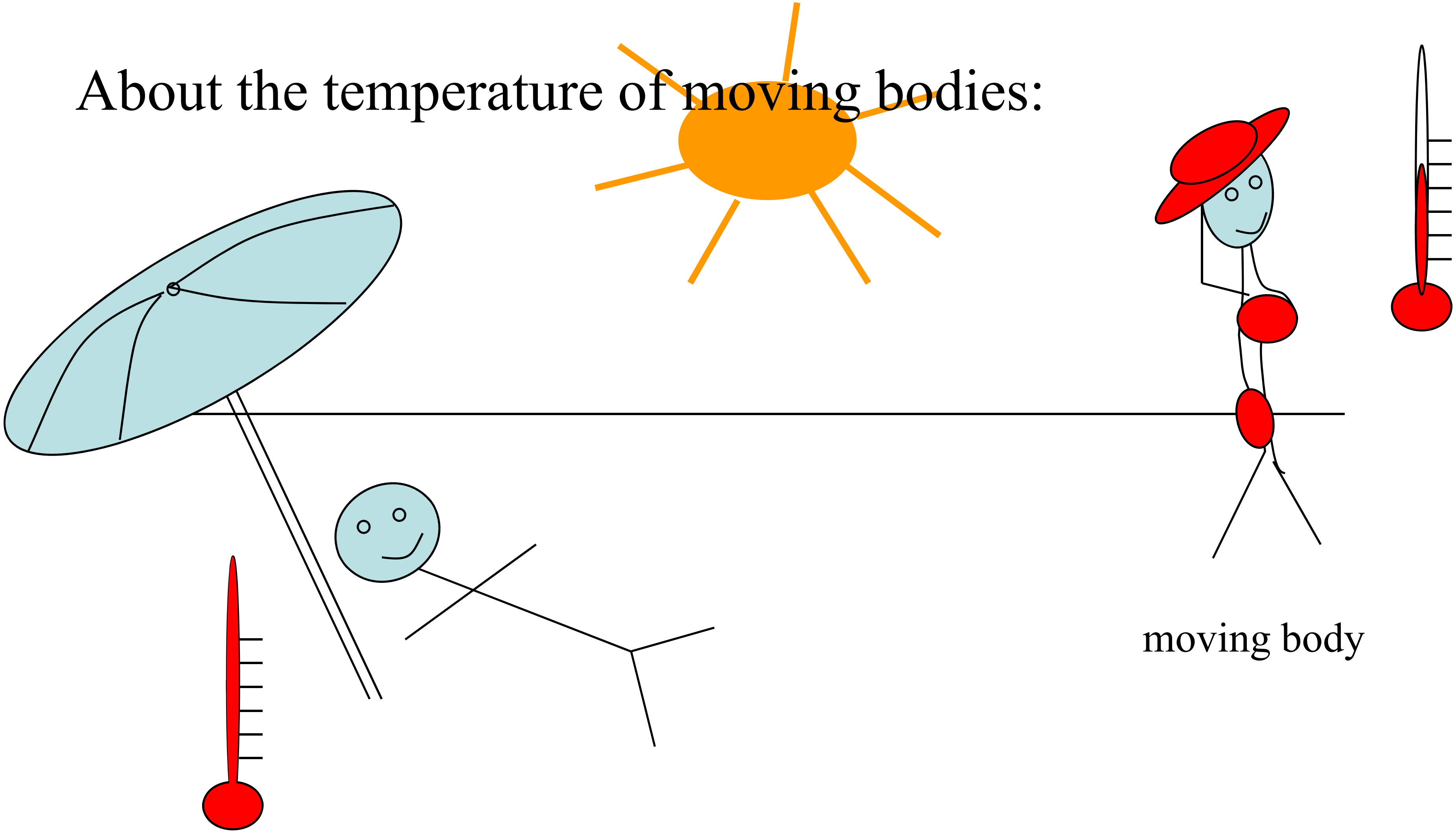
Different roles:

Equations of state: Θ, M

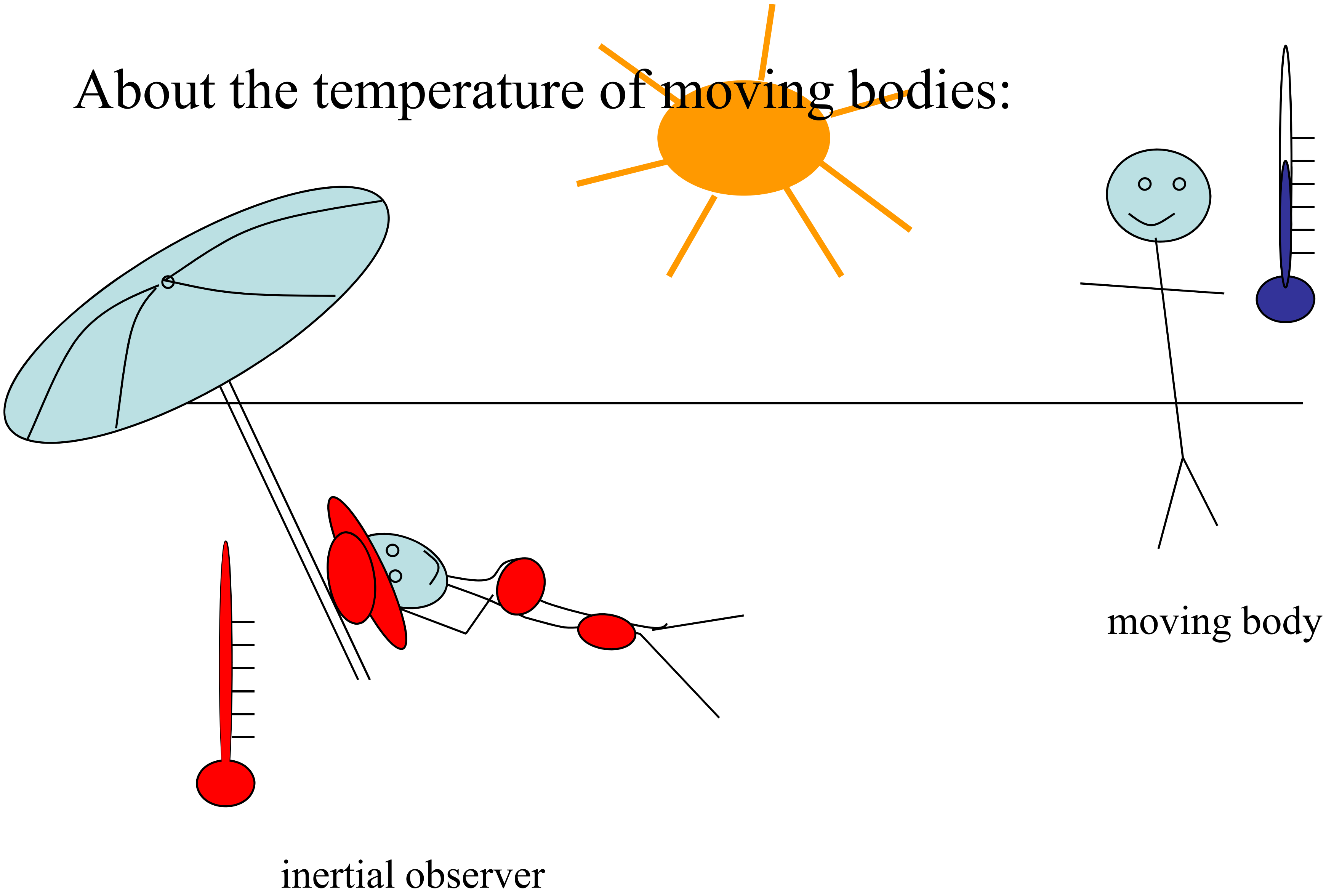
Constitutive functions: T, μ

$$q^a = -\lambda \Delta^{ac}\left(\partial_c T + T\dot{u}_c + T\frac{\dot{q}^a}{e}\right)$$

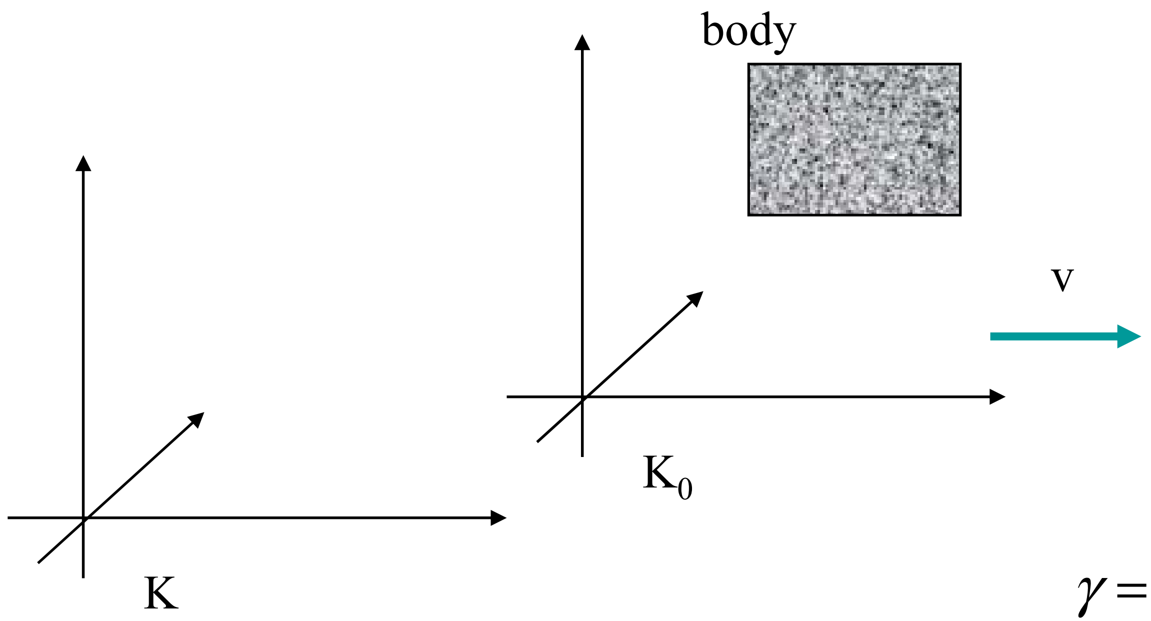
About the temperature of moving bodies:



About the temperature of moving bodies:



About the temperature of moving bodies:



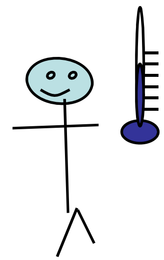
$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

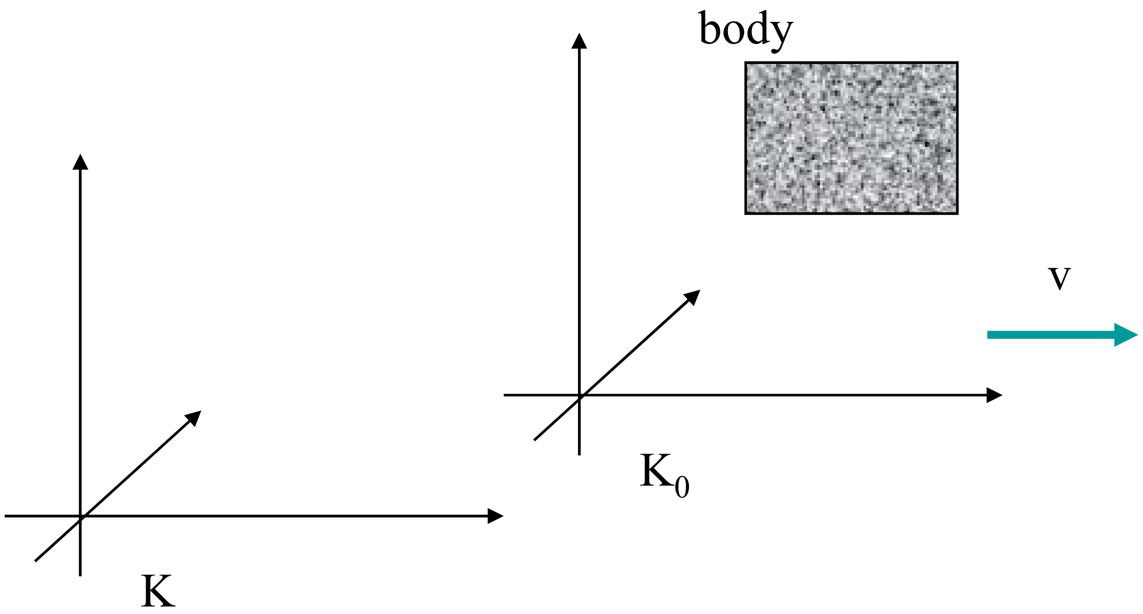
$$dE - \mathbf{v}d\mathbf{G} = TdS - pdV$$

translational work

Einstein-Planck: entropy is vector, energy + work is scalar

$$s = \gamma s_0, \quad e = e_0 \quad \Rightarrow \quad \boxed{T = \gamma^{-1}T_0}, \quad p = p_0$$

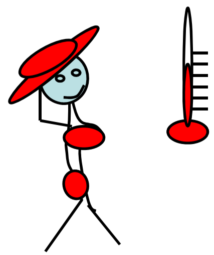




$$dE = TdS - pdV \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

Ott - hydro: entropy is vector, energy-pressure are from a tensor

$$s = \boldsymbol{\gamma} s_0, \quad e = \boldsymbol{\gamma}^2 e_0, \quad p = \boldsymbol{\gamma}^2 p_0 \quad \Rightarrow \quad \boxed{T = \boldsymbol{\gamma} T_0}$$



$$d\varepsilon = d\sqrt{e^2 - q^2} = \frac{e\,de - q^a dq_a}{\varepsilon} = \frac{e}{\varepsilon}\left(de - \frac{q^a}{\varepsilon} dq^a\right) = \theta ds + M dn$$

$\varepsilon = \sqrt{\varepsilon^a \varepsilon_a}$ $\qquad$ $\varepsilon^a = -u_b T^{ba}$ energy(-momentum) vector

$s = \gamma s_0, \quad \varepsilon = \gamma \varepsilon_0, \quad \Rightarrow \quad \theta = \theta_0$ $\qquad$ Landsberg

$T = \dfrac{\theta\varepsilon}{e}, \quad e = \gamma^2 e_0 \quad \Rightarrow \quad T = \gamma^{-1} T_0$ $\qquad$ Einstein-Planck

non-dissipative $\quad \Rightarrow \quad T = \gamma T_0$ $\qquad$ Ott

Simple transformation properties?

Equilibration:

Two bodies A and B have relative speed v. What must be the relation between their temperatures T_A and T_B, measured in their rest frames, if they are to be in thermal equilibrium?

Integration, homogeneity:

$$\varepsilon V = V\sqrt{\varepsilon_a \varepsilon^a} = \sqrt{E_a E^a} = \sqrt{E^2 - \mathbf{G}^2}, \quad sV = S, \quad nV = N.$$

$$\theta dS + M dN = \frac{E_a}{|E|} dE^a = d|E|$$

$$E_1^{\ a} + E_2^{\ a} = const.$$

$$N = const.$$

$$\theta_1 = \theta_2 = \theta$$

$$d(S_1 + S_2) = \frac{E_{1a}}{\theta_1 |E_1|} dE_1^a + \frac{E_{2a}}{\theta_2 |E_2|} dE_2^a = \left(\frac{E_{1a}}{\theta |E_1|} - \frac{E_{2a}}{\theta |E_2|} \right) dE_1^a \Rightarrow u_1^a = u_2^a$$

Thermal interaction requires uniform velocities.

Quasi-hyperbolic extension – relaxation of viscosity:

$$s(\varepsilon, n) = s\left(\sqrt{e^2 - \mathbf{q}^2 - \beta_0 \Pi^2 - \beta_2 \pi^{bc}\pi_{bc}}, n\right) = s\left(\sqrt{e^2 - D^2}, n\right)$$

Relaxation:

$$\beta_0 = 1, \quad \beta_2 = 1.$$

$$\frac{1}{e}\Delta^{ab}\dot{q}_b + \frac{1}{T}\nabla^a T + \Delta^{ab}\dot{u}_b + \frac{1}{\lambda T}q^a = 0,$$

$$\boxed{\frac{3}{e}\dot{\Pi}} + \partial_b u^b + \frac{1}{\zeta}\Pi = 0,$$

$$\boxed{\frac{1}{e}\dot{\pi}_{ab}} - \partial_{<a}u_{b>} + \frac{1}{2\eta}\pi_{ab} = 0.$$

Simpler than Israel-Stewart: there are no β derivatives.

Bíró, Molnár and Ván: PRC, (2008), **78**, 014909 (arXiv:0805.1061)

1) Generalized Bjorken flow - the role of q:

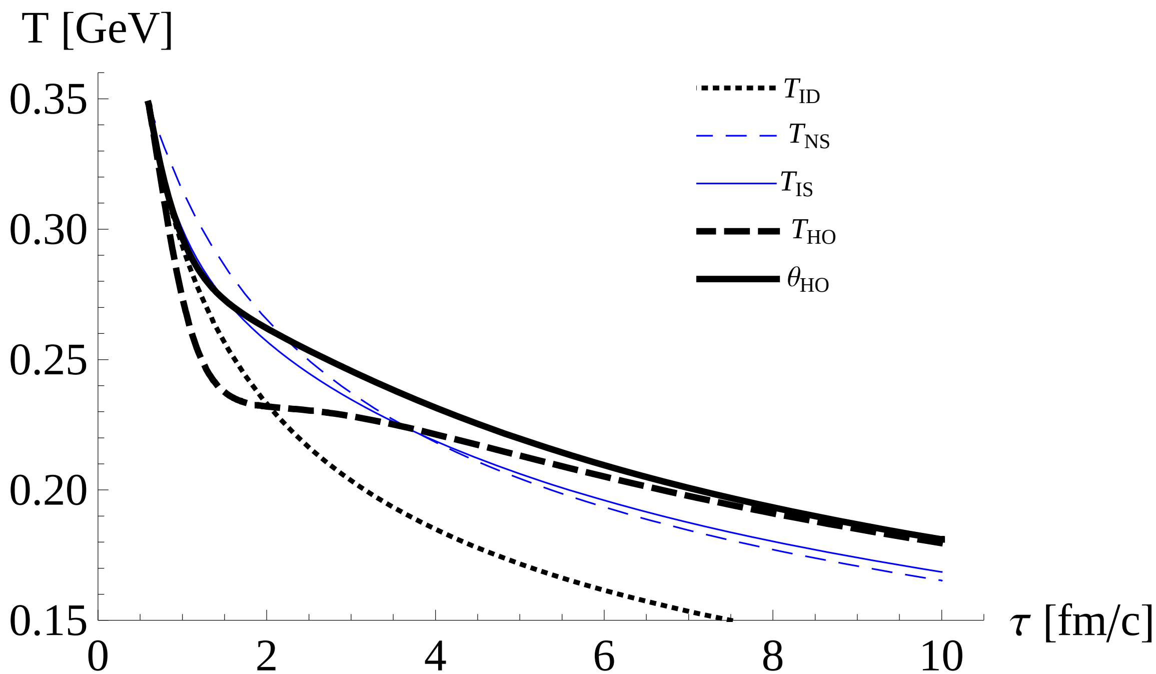
tetrad : $e_0^a, e_1^a, e_2^a, e_3^a$; axial symmetry $x^a = \tau e_0^a + r e_2^a$

$$u^a = \gamma(e_0^a + v e_1^a), \;\; q^a = \gamma q(v e_0^a + e_1^a).$$

Only for the q=0 solution remains the v=0 Bjorken-flow stationary.

2) Temperatures:

-qgp eos
- $\tau_0 = 0.6$fm/c,
- $e_0 = \varepsilon_0 = 30$GeV/fm^3
- $\eta/s = 0.4$,
- $\pi_0 = 0$.

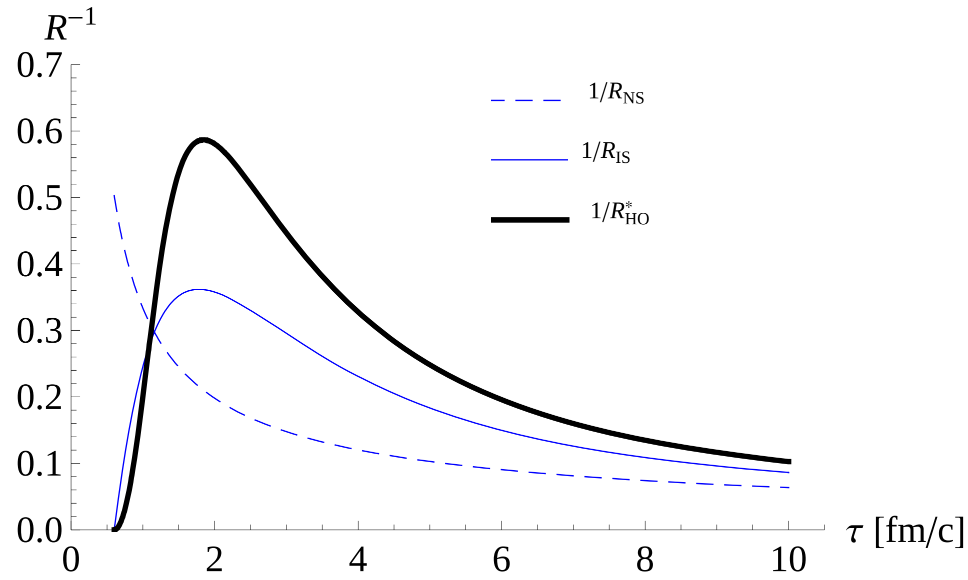


3) Reheating:

Eckart: $R^{-1}<1$ ($p<4\pi$) stability

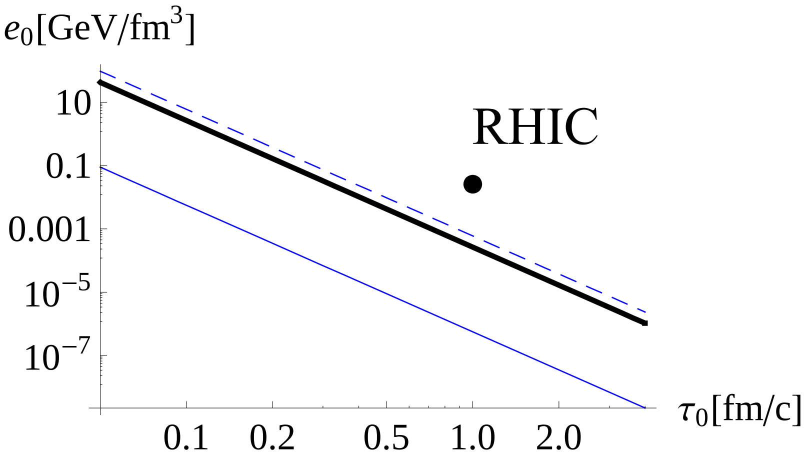
$$e_0 == \left(\frac{4\eta_0}{3\tau_0}\right)^4_{Eckart} = b\,\tau_0^{-4}$$

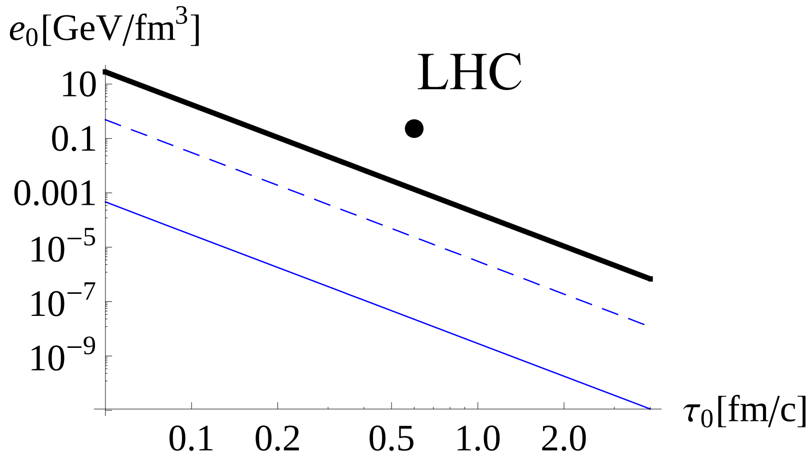
η_0	Eckart	IS	HO
0.3	$6\cdot10^{-4}$	$5.6\cdot10^{-7}$	$2.67\cdot10^{-4}$
0.08	$3\cdot10^{-6}$	$2.89\cdot10^{-9}$	$1.75\cdot10^{-4}$



(a)

(b)





Summary

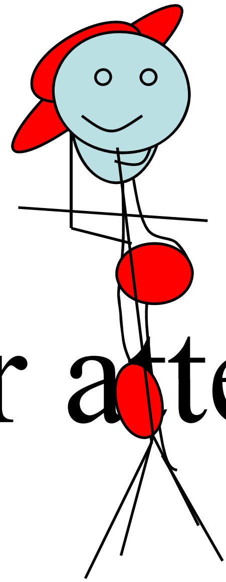
– Extended theories are not ultimate.

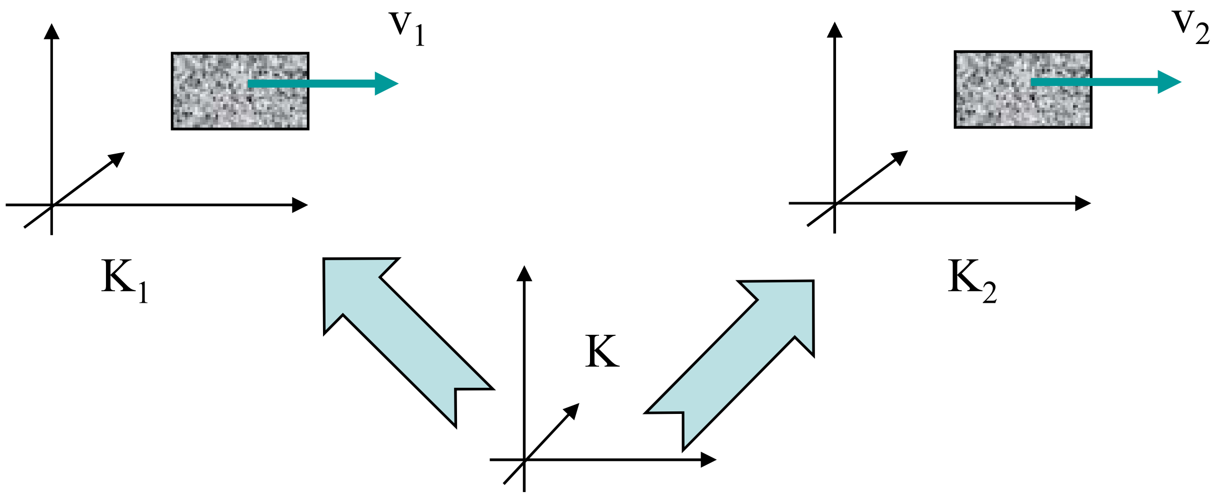
– energy $\neq$ internal energy
  → generic stability without extra conditions

– hyperbolic(-like) extensions, generalized Bjorken solutions, reheating conditions, etc…

– different temperatures in Fourier-law (equilibration) and in EOS out of local equilibrium
  → temperature of moving bodies - interpretation

Thank you for your attention!





$$E'^{a}_{1} + E'^{a}_{2} = const.$$
$$N = const.$$
$$dS_1 + dS_2 = 0$$

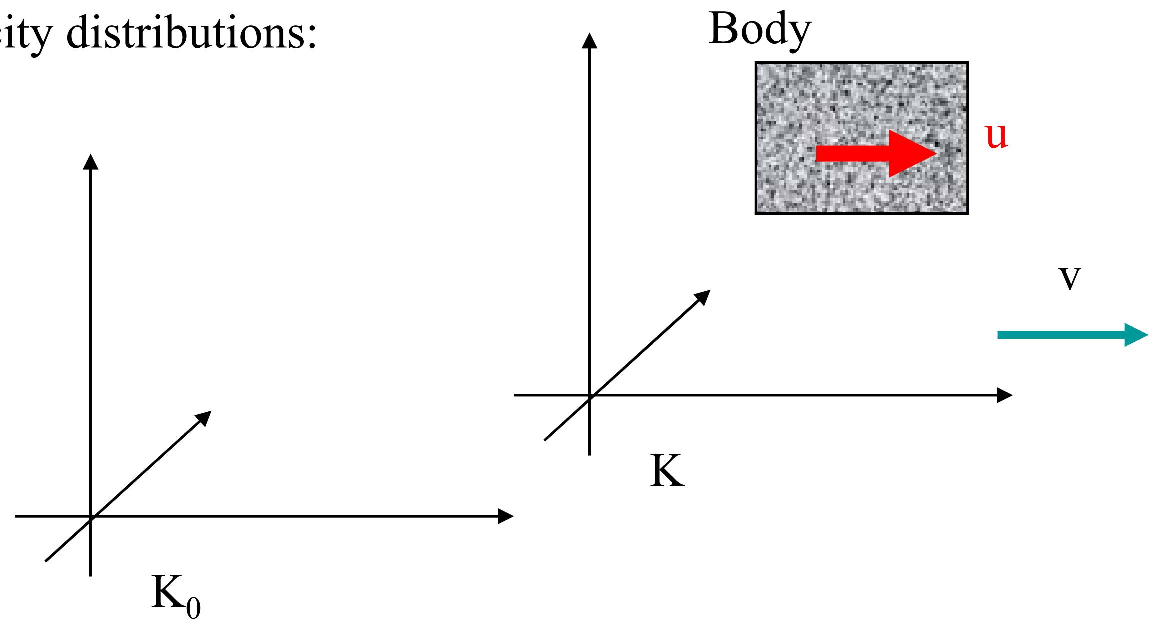
$$\frac{1}{T_2} = \frac{1}{T_1}\frac{E'_1}{E_1}$$
$$\frac{Q_2}{E_2} = \frac{Q'_1}{E_1}$$

$$Q_1 = 0 \quad \Rightarrow \quad \frac{1}{T_2} = \gamma\frac{1}{T_1}, \quad \frac{Q_2}{E_2} = -v$$ Einstein-Planck

$$Q'_1 = 0 \quad \Rightarrow \quad \frac{1}{T_2} = \frac{1}{\gamma T_1}, \quad Q_2 = 0$$ Ott

$$Q_1 = E_1 \quad \Rightarrow \quad \frac{1}{T_2} = \frac{\gamma(1-v)}{T_1}, \quad Q_2 = E_2$$ lightlike

Velocity distributions:



$$f(u) = \hat{f}\left(\frac{\varepsilon}{T}\right) \quad \Rightarrow \quad T = \gamma\left(1 - \frac{uv}{c^2}\right)T_0$$

Averages? (Cubero et. al. PRL 2007, **99** 170601)

Heavy-ion experiments, cosmology.

Liu procedure for relativistic fluids

$$\partial_b T^{ab} = \dot{e}u^a + eu^a\partial_b u^b + e\dot{u}^a + u^a\partial_b q^b + q^b\partial_b u^a + \dot{q}^a + q^a\partial_b u^b + \partial_b P^{ab} = 0$$

$$\partial_a S^a = \dot{s} + s\partial_a u^a + \partial_a J^a \geq 0.$$

Thermodynamics – local rest frame

– *basic state (fields):* (e, u^a)

– *constitutive state:* $(e, u^a, \partial_a e, \partial_b u^a)$

– *constitutive functions:* (q^a, P^{ab}, s, J^a)

$$\partial_a S^a - \Lambda_a \partial_b T^{ab} \geq 0 \qquad \Lambda_a = \Lambda u_a + l_a$$

4-vector (temperature ?)

Solution of Liu equations (Λ_a, A_a *are local):*

$$J^a = \Lambda q^a + l_b P^{ab} + a^a$$

$$s = \Lambda e + l_b q^b + A$$

Dissipation inequality $s(e,u^a)$??

$$\partial_a e\left[(\partial_1 s-\Lambda-l_b\partial_1 q^b)u^a+q^a\partial_1\Lambda+P^{ba}\partial_1 l_a+\partial_1 a^a\right]+$$
$$\partial_a u_b\left[-\Lambda P^{ab}+A\Delta^{ab}+q^a(\partial_2\Lambda)^b+P^{ca}(\partial_2 l_c)^a+(\partial_2 a^a)^b+\right.$$
$$\left.\left((\partial_2 s)^b-l_c(\partial_2 q^c)^b\right)u^a-\left(l^b e-\Lambda q^b\right)\dot{u}^a-l^b q^a\right]\geq 0$$

1) $s(e,u^a)=s(e,q^a(e,u^a))$

2) $e\dfrac{\partial s}{\partial q^a}=q_a\dfrac{\partial s}{\partial e}\;\Rightarrow\; s(e,q^a)=\hat{s}(e^2-\mathbf{q}^2)=\tilde{s}\left(\sqrt{e^2-\mathbf{q}^2}\right)$

$$\sqrt{e^2-\mathbf{q}^2}\approx e-\frac{\mathbf{q}^2}{2e}+....$$

Energy-momentum – momentum density and energy flux

$$T^{ab} = eu^a u^b + u^a q^b + \hat{q}^a u^b + P^{ab}, \quad u_a q^a = 0, \quad u_a P^{ab} = 0^\beta$$

$$T^{ab} = \begin{pmatrix} e & q^i \\ \hat{q}^j & P^{ij} \end{pmatrix}$$

$$\partial_b T^{ab} = 0^a,$$

$$u^a \partial_b T^{ab} = \dot{e} + e\partial_a u^a + \partial_a q^a + q^a \dot{u}_a - P^{ab}\partial_b u_a = 0,$$

$$\Delta^a_c \partial_b T^{cb} = e\dot{u}^a + q^a \partial_b u^b + q^b \partial_b u^a + \Delta^a_c (\dot{\hat{q}}^c + \partial_b P^{cb}) = 0^a.$$

Landau choice: $q^a = 0^a$

Linearization

$$A = A_0 + \delta A$$

$$
\begin{aligned}
&\delta\dot{n} + && && n\partial_a \delta u^a + && && \partial_a \delta j^a && = 0,\\
& && \delta\dot{e} + && (e+p)\partial_a \delta u^a + \partial_a \delta q^a && && && = 0,\\
&\partial_n p \partial_a \delta n + && \partial_e p \partial_a \delta e + && e\delta\dot{u}_a + && \delta\dot{q}_a + && \partial_a \delta\tau && = 0,\\
& && && \varsigma \partial_a \delta u^a + && && \delta\tau && = 0,\\
&\xi \partial_n \frac{\mu}{T} \partial_a \delta n + && \xi \partial_e \frac{\mu}{T} \partial_a \delta e + && && && \delta j_a && = 0,\\
&\frac{\lambda}{T^2} \partial_n T \partial_a \delta n + && \frac{\lambda}{T^2} \partial_e T \partial_a \delta e + && \frac{\lambda}{T^2} T\delta\dot{u}_a + && \delta q_a && && = 0.
\end{aligned}
$$

$\delta n \quad \delta e \quad \delta u_a \quad \delta q_a \quad \delta j_a \quad \delta\tau$

$$\delta A = A_0 \exp(\Gamma t + ikx)$$ exponential plane-waves

$$\mathbf{Q} = \begin{pmatrix} \Gamma & 0 & ikn & 0 & ik & 0 \\ 0 & \Gamma & ik(e+p) & ik & 0 & 0 \\ ik\partial_n p & ik\partial_e p & (e+p)\Gamma & \Gamma & 0 & ik \\ ik\partial_n T & ik\partial_e T & T\Gamma & \frac{1}{\lambda} + \frac{T}{e}\Gamma & 0 & 0 \\ ik\partial_n \frac{\mu}{T} & ik\partial_n \frac{\mu}{T} & 0 & 0 & \frac{1}{\xi} & 0 \\ 0 & 0 & ik & 0 & 0 & \frac{1}{\tilde{\eta}} \end{pmatrix} \begin{pmatrix} \delta n \\ \delta e \\ \delta u^x \\ \delta q^x \\ \delta j^x \\ \delta \tau \end{pmatrix}$$

$$\tilde{\eta} = \frac{\eta}{3/4 + \eta/\varsigma}$$

$$\mathbf{R} = \begin{pmatrix} (e+p)\Gamma & \Gamma & ik & 0 \\ \lambda T\Gamma & 1+\lambda\Gamma\frac{T}{e} & 0 & 0 \\ ik\eta & 0 & 1 & 0 \\ ik\tilde{\eta}_v & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta u^y \\ \delta q^y \\ \delta \Pi^{xy} \\ \delta \Pi^{yy} \end{pmatrix}$$

$$
\begin{aligned}
&\lambda\chi\eta\, Det(Q) = \\
&\Gamma^3\, T^2 (e+p) + \\
&\Gamma^2\, k^2 (T^2\eta + (p+e))\left(\lambda\partial_e T + \eta T^2 \partial_n \frac{\mu}{T}\right) + \\
&\Gamma k^2 \left(T^2(\partial_e p(e+p) + n\partial_n p) + k^2\eta\left(\lambda\partial_e T + \eta T^2\partial_n \frac{\mu}{T}\right) + \right. \\
&\qquad \left. \lambda\chi k^2 (e+p)\left(\partial_e T\partial_n \frac{\mu}{T} - \partial_n T\partial_e \frac{\mu}{T}\right)\right) + \\
&k^4 \left(T^2\chi(e+p)\left(\partial_e p\partial_n \frac{\mu}{T} - \partial_n p\partial_e \frac{\mu}{T}\right) + \lambda n\left(\partial_e T\partial_n p - \partial_n T\partial_e p\right) + \right. \\
&\qquad \left. \lambda\chi\eta\, k^2\left(\partial_e T\partial_n \frac{\mu}{T} - \partial_n T\partial_e \frac{\mu}{T}\right)\right)
\end{aligned}
$$

Routh-Hurwitz:

$$
\frac{\partial T}{\partial e} > 0, \quad \frac{\partial}{\partial n}\frac{\mu}{T} > 0
$$

$$
\frac{\partial T}{\partial e}\frac{\partial}{\partial n}\frac{\mu}{T} - \frac{\partial T}{\partial n}\frac{\partial}{\partial e}\frac{\mu}{T} \geq 0
$$

thermodynamic stability

Causality hyperbolic or parabolic?

→ Well posedness
→ Speed of signal propagation

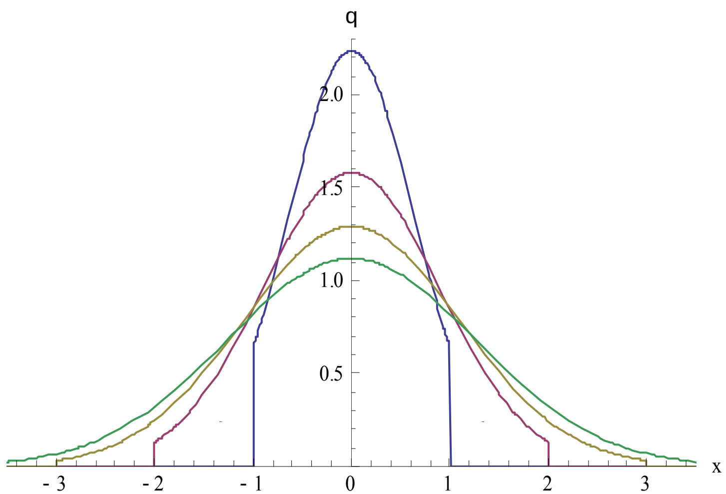
Hydrodynamic range of validity:
ξ – mean free path
τ – collision time

$$\partial_x T << \frac{\Delta T}{\xi}, \quad \partial_t T << \frac{\Delta T}{\tau}$$

$$T(x,t) = \frac{A}{\sqrt{2\kappa t}} e^{-\frac{x^2}{4\kappa t}} \longrightarrow \boxed{v_{\max} \approx \frac{\kappa}{\xi} = \frac{\lambda}{\xi c_V \rho}}$$

Water at room temperature:

$$v_{\max} \approx \frac{\lambda}{\xi c_V \rho} \cong 14 \frac{m}{s}$$



More complicated equations,
more spacetime dimensions, ….

Remarks on hyperbolicity

1) Hyperbolicity does not result in automatic causality, because the propagation speed of small perturbations can be large.

$$\text{hyperbolic} \not\Rightarrow \text{causal}$$

2) Parabolic equations and first order theories are not automatically excluded. The validity range of the theory could prevent large speeds if the perturbations were relaxing fast.

$$\text{parabolic+stable} \Rightarrow \text{causal}$$

3) Instability in first order theories is not acceptable.

$\Rightarrow$ Second order dissipative theories are corrections to first order stable theories.

Causality hyperbolic or parabolic?

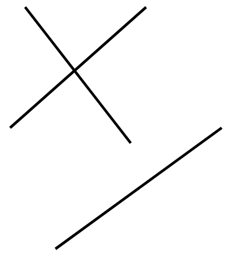
→ Well posedness
→ Speed of signal propagation

Second order linear partial differential equation:

$$A\partial_{xx}T + 2B\partial_{xt}T + C\partial_{tt}T + F(x,t,T,\partial_x T,\partial_t T) = 0$$

Corresponding equation of characteristics:

$$(*) \qquad A(\partial_x \boldsymbol{\theta})^2 + 2B\partial_x \boldsymbol{\theta}\,\partial_t \boldsymbol{\theta} + C(\partial_t \boldsymbol{\theta})^2 = 0$$



i) Hyperbolic equation: two distinct families of real characteristics
Parabolic equation: one distinct families of real characteristics
Elliptic equation: no real characteristics

Well posedness: existence, unicity, continuous dependence on initial data.

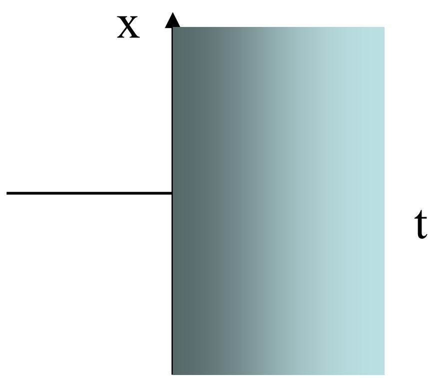
A characteristic Cauchy problem of (1) is well posed. (initial data on the characteristic surface: $T(x,0) = f(x)$)

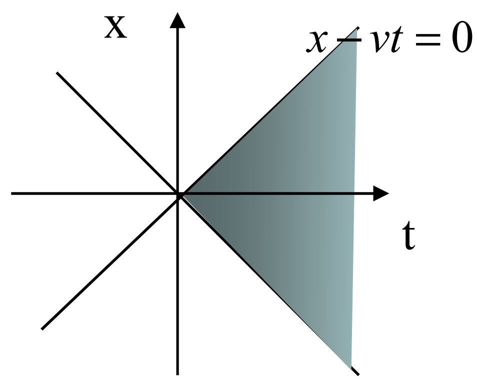
ii) (*) is transformation invariant $\tilde{x} = \tilde{x}(x,t), \quad \tilde{t} = \tilde{t}(x,t)$

(1) $\partial_t T - \kappa \partial_{xx} T = 0,$

$$\gamma(\partial_{\tilde{t}} - v\partial_{\tilde{x}})T - \kappa\left(\partial_{\tilde{x}\tilde{x}} - 2\frac{v}{c^2}\partial_{\tilde{x}\tilde{t}} + \frac{v^2}{c^4}\partial_{\tilde{t}\tilde{t}}\right)T = 0$$

iii) The outer real characteristics that pass through a given point (x_0, t_0) give its *domain of influence* Ω_0 .





E.g.

$$\partial_t T - \kappa \partial_{xx} T = 0,$$
$$T(x,0) = \delta(x)$$

$$T(x,t) = \frac{A}{\sqrt{2\kappa t}} e^{-\frac{x^2}{4\kappa t}}$$

Infinite speed of signal propagation?
physics - mathematics

Hydrodynamic range of validity:
ξ – mean free path
τ – collision time

$$\partial_x T << \frac{\Delta T}{\xi}, \quad \partial_t T << \frac{\Delta T}{\tau}$$

$$T(x,t) = \frac{A}{\sqrt{2\kappa t}} e^{-\frac{x^2}{4\kappa t}} \longrightarrow \boxed{v_{\max} \approx \frac{\kappa}{\xi} = \frac{\lambda}{\xi c_V \rho}}$$

Water at room temperature: $v_{\max} \approx \frac{\lambda}{\xi c_V \rho} \cong 14 \, \frac{m}{s}$

Fermi gas of light quarks
at T_c: $v_{\max} \approx \frac{\lambda}{\xi c_V \rho} \approx \frac{m c_V \xi \bar{v}}{\xi c_V \rho} = \frac{\sqrt{mT}}{\rho} \cong 10^{-31} \, \frac{m}{s}$

More complicated equations, more spacetime dimensions, ….

Non-relativistic fluid mechanics

local equilibrium, Fourier-Navier-Stokes

$$\dot{n} + n\partial_i v^i = 0,$$
$$\dot{\varepsilon} + \varepsilon\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j = 0,$$
$$\dot{k}^i + k^i\partial_j v^j + \partial_j P^{ij} = 0^i.$$

n	*particle number density*
v^i	*relative (3-)velocity*
e	*internal energy density*
q^i	*internal energy (heat) flux*
P^{ij}	*pressure*
k^i	*momentum density*

Thermodynamics

$$d\varepsilon = Tds + \mu dn, \qquad p = Ts + \mu n - \varepsilon, \quad \varepsilon = e - \frac{mnv^2}{2}.$$

$$\dot{s}(\varepsilon, n) + s\partial_i v^i + \partial_i J^i = \frac{1}{T}\dot{\varepsilon} - \frac{\mu}{T}\dot{n} + s\partial_i v^i + \partial_i \frac{q^i}{T} = \ldots =$$

$$\boxed{q^i \partial_i \frac{1}{T} - \frac{1}{T}\left(P^{ij} - \underbrace{(Ts + \mu n - \varepsilon)}_{p}\delta^{ij}\right)\partial_i v_j \geq 0}$$

About the temperature of moving bodies: