Pion condensation in the two–flavor chiral quark model at finite baryochemical potential

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- The constituent quark model and its renormalization
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- Diagonalized propagators
- Parametrization
- Results at lowest order in ρ (arXiv:0811.2727 [hep-ph])
- Conclusion

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Where can pion condensation occur in nature?

- Quark matter can exist in neutron stars $\longrightarrow$ at very large bariochemical potential ($\mu_{\mathrm{B}} \approx 1$ GeV)

  If the isospin chemical potential is also different from zero $\longrightarrow$ possibility of pion condensation

- In heavy ion collisions μ_I is tunable with different isotopes of an element

Neutrino emission from pion condensed quark matter $\rightarrow$ direct Urca processes:

$$d \rightarrow u + e^- + \bar{\nu}$$
$$u + e^- \rightarrow d + \nu$$

$\Longrightarrow$ It might will be investigated experimentaly

In 2 flavoured NJL model (L. He *et al* Phys. Rev. **D74**, 036005 (2004)):

- if $\mu_I < 140$ MeV($= m_\pi$) $\rightarrow$ no pion condensation
- if 140 MeV$< \mu_I < 230$ MeV $\rightarrow$ BEC phase
- if $\mu_I > 230$ MeV $\rightarrow$ BCS phase

Interesting feature of pion condensation found in $SU(2)$ PNJL model:

At sufficiently high temperature the condesate evaporates above a certain μ_I. (e.g. Z. Zhang, Y. Liu: hep-ph/0610221v3)

Up to now:

- Investigations in $SU(2)$ NJL and PNJL model (BEC, BCS and CFL phases)
- Investigation in $O(4)$ model in the large N_c limit (leading order) (BEC phase)

The model and its renormalization

Our starting point is the renormalized $SU(2)_L \times SU(2)_R$ symmetric Lagrangian with explicit symmetry breaking term

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - \frac{\lambda}{4}\phi^4 + h\phi_0 + i\bar{\psi}\gamma_\mu\partial^\mu\psi - \frac{g_F}{2}\bar{\psi}T_i\phi_i\psi \\ &+ \frac{1}{2}\left(\delta Z\partial_\mu\phi\partial^\mu\phi - \delta m^2\phi^2\right) - \frac{\delta\lambda}{4}\phi^4\end{aligned}$$

$\psi = (u,\, d)^T \longrightarrow$ doublet quark fileds

$\phi = (\phi_0, \phi_1, \phi_2, \phi_3) \equiv (\sigma, \pi_1, \pi_2, \pi_3) \longrightarrow$ sigma and pion scalar fields

$h \longrightarrow$ symmetry breaking external field

$T_i = (\tau_0, i\tau_i\gamma_5) \longrightarrow$ quark–boson coupling matrix

The renormalized (finite) parameters of the Lagrangian: m^2, λ, g_F

$\delta z, \delta m^2, \delta\lambda$ are the usual (infinite) counterterms
(Fermions are treated at tree level $\rightarrow$ no wavefunction renormalization)

The genarating functional:

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\Pi \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left[\mathrm{i}\int_0^{-\mathrm{i}\beta} dt \int d^3x (\Pi\dot{\phi} + \mathrm{i}\bar{\psi}\gamma_0\dot{\psi} - \mathcal{H} + \mu_{\mathsf{B}} Q_{\mathrm{B}} + \mu_{\mathsf{I}} Q_{\mathrm{I}})\right],$$

where the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\left(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) + \frac{\lambda}{4}\phi^4 - h\phi_0 + \mathrm{i}\bar{\psi}\gamma_i\partial_i\psi + \frac{g_F}{2}\bar{\psi}T_\alpha\phi_\alpha\psi \\ &- \frac{1}{2}\delta Z\Pi^2 + \frac{1}{2}\delta Z(\nabla\phi)^2 + \frac{\delta\lambda}{4}\phi^4 + \frac{1}{2}\delta m^2\phi^2, \qquad (i = 1, 2, 3)\end{aligned}$$

and the canonical momenta of the scalar fields

$$\Pi = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = (1 + \delta Z)\dot{\phi}.$$

Q_B, Q_I are the conserved barion and isospin charges

$$\begin{aligned}Q_B &= \int \mathrm{d}^3x \frac{1}{3}(u^\dagger u + d^\dagger d) \\ Q_I &= \int \mathrm{d}^3x \left[(1 + \delta Z)(\pi_2\dot{\pi}_1 - \pi_1\dot{\pi}_2) + \frac{1}{2}\left(u^\dagger u - d^\dagger d\right)\right].\end{aligned}$$

Symmetry breaking:

At small T when either $h \neq 0$ or $h = 0$ and $m^2 < 0 \Rightarrow \langle\phi_0\rangle \equiv \langle\sigma\rangle \equiv v \neq 0$

At large $\mu_I \Rightarrow \langle\phi_1\rangle \equiv \langle\pi_1\rangle \equiv \rho \neq 0$ and $\langle\phi_i\rangle \equiv \langle\pi_i\rangle = 0$ for $i = 2, 3$

Shifting the corresponding fields

$$\mathcal{Z} = \int \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi \left[\mathrm{e}^{-\int_0^{-\mathrm{i}\beta} dt \int d^3x \bar{\psi} G^{f^{-1}} \psi} \mathrm{e}^{-\mathrm{i}\int_0^{-\mathrm{i}\beta} dt \int d^3x \tilde{\mathcal{L}}_I} \int \mathcal{D}\Pi \mathrm{e}^{i\int_0^{-\mathrm{i}\beta} dt \int d^3x (\Pi\dot{\phi} - \tilde{\mathcal{H}}_B)} \right].$$

where the Π dependent part of $\Pi\dot{\phi} - \tilde{\mathcal{H}}_B$:

$$\begin{aligned}
\Pi\dot{\phi} - \tilde{\mathcal{H}}_B = \; & - \; \frac{1}{2}(1-\delta Z)\Big(\Pi_0^2 - 2\Pi_0(1+\delta Z)\dot{\phi}_0\Big) \\
& - \; \frac{1}{2}(1-\delta Z)\Big(\Pi_3^2 - 2\Pi_3(1+\delta Z)\dot{\phi}_3\Big) \\
& - \; \frac{1}{2}(1-\delta Z)\Big(\Pi_1^2 - 2\Pi_1(1+\delta Z)(\dot{\phi}_1 - \mu_I\phi_2)\Big) \\
& - \; \frac{1}{2}(1-\delta Z)\Big(\Pi_2^2 - 2\Pi_2(1+\delta Z)(\dot{\phi}_2 + \mu_I(\phi_1+\rho))\Big).
\end{aligned}$$

Making whole squares in the above brakets

Then performing the Π integration $\rightarrow$ produce the inverse bosonic propagators and the tree-level EoS (linear terms).

Propagator matirices:

$$\mathrm{i}{G^{f}_{ij}}^{-1} = \begin{pmatrix} (-\mathrm{i}\omega_n + \frac{1}{3}\mu_{\mathsf{B}} + \frac{1}{2}\mu_{\mathsf{I}})\gamma_0 - \gamma_i p_i - \frac{g_F}{2}v & -\mathrm{i}\frac{g_F}{2}\gamma_5\rho \\ -\mathrm{i}\frac{g_F}{2}\gamma_5\rho & (-\mathrm{i}\omega_n + \frac{1}{3}\mu_{\mathsf{B}} - \frac{1}{2}\mu_{\mathsf{I}})\gamma_0 - \gamma_i p_i - \frac{g_F}{2}v \end{pmatrix},$$

$$\mathrm{i}{G^{b}_{44}}^{-1} = (-\mathrm{i}\omega_n)^2 - E^2_{\pi_3},$$

$$\mathrm{i}{G^{b}_{kl}}^{-1} = \begin{pmatrix} (-\mathrm{i}\omega_n - \mu_{\mathsf{I}})^2 - E^2_{\pi_3} - \lambda\rho^2 & -\lambda\rho^2 & -\sqrt{2}\lambda v\rho \\ -\lambda\rho^2 & (-\mathrm{i}\omega_n + \mu_{\mathsf{I}})^2 - E^2_{\pi_3} - \lambda\rho^2 & -\sqrt{2}\lambda v\rho \\ -\sqrt{2}\lambda v\rho & -\sqrt{2}\lambda v\rho & (-\mathrm{i}\omega_n)^2 - E^2_{\pi_3} - 2\lambda v^2 \end{pmatrix}$$

Tree-level EoS:

$$\begin{aligned} \mathrm{EoS}^{\mathsf{tree}}_{\sigma} &= v(m^2 + \lambda(v^2 + \rho^2)) - h = 0, \\ \mathrm{EoS}^{\mathsf{tree}}_{\pi_1} &= \rho(m^2 + \lambda(v^2 + \rho^2) - \mu_{\mathsf{I}}^2(1 + \delta Z)) = 0. \end{aligned}$$

Renormalizaiton conditions $\rightarrow$ finiteness of the perturbative N–point functions (the propagator and the four point boson vertex)

Practically it is easier to obtain the counterterms from the one–loop EoS

Note: there is some arbitrariness in choosing the finite parts

With cut-off regularization:

$$\begin{aligned}
\delta m^2 &= -6\lambda(\Lambda^2 - m^2 \ln\frac{\Lambda^2}{l_b^2}) + \frac{g_F^2}{4\pi^2} N_c \Lambda^2, \\
\delta\lambda &= 12\lambda^2 \ln\frac{\Lambda^2}{l_b^2} - \frac{g_F^2}{32\pi^2} N_c \ln\frac{\Lambda^2}{\mathrm{e} l_f^2}, \\
\delta Z &= -N_c \frac{g_F^2}{16\pi^2} \ln\frac{\Lambda^2}{\mathrm{e}^2 l_f^2},
\end{aligned}$$

$l_b, l_f \rightarrow$ bosonic and fermionic renormalization scales

Equations at one–loop level, the OPT

At finite temperature tree level mass squares can become negative $\rightarrow$ some sort of resummation is needed

Using the optimized perturbation theory (OPT):

- a temperature dependent mass term introduced in the Lagrangian
- the difference is treated as a higher order counterterm
- the new mass parameter determined by the FAC criterion ($m^{\text{1–loop}} = m^{\text{tree}}$) $\rightarrow$ can be transformed to an equation for a resummed particle mass
- conserves Ward–identities

Equation for the resummed π_3 mass:

$$m^2_{\pi_3}(T,\mu) = m^2 + \delta m^2 + (\lambda + \delta\lambda)(v^2 + \rho^2) + \Sigma_{\pi_3}(\omega = \mathbf{p} = 0, m^2_{\pi_3}, T, \mu)$$

One–loop level EoS for σ:

$$v\left(m^2 + \delta m^2 + (\lambda + \delta\lambda)(v^2 + \rho^2) + \lambda \sum\!\!\!\!\!\!\int_{\mathbf{p}} \mathrm{Tr}\{H^b G^b(\omega_n, \mathbf{p}, \mu_\mathsf{I})\} \right.$$
$$\left. + \, g_F \sum\!\!\!\!\!\!\int_{\mathbf{p}} \mathrm{Tr}\{H^f G^f(\omega_n, \mathbf{p}, \mu_\mathsf{I}, \mu_\mathsf{B})\}\right) = h$$

comparing the two equations → a Ward–identity is recognized

$$v m^2_{\pi_3} = h$$

One–loop level EoS for π_1:

$$\rho\left(m^2 + \delta m^2 + (\lambda + \delta\lambda)(v^2 + \rho^2) - \mu_\mathsf{I}^2(1 + \delta Z) + \lambda \sum\!\!\!\!\!\!\int_{\mathbf{p}} \mathrm{Tr}\{R^b G^b(\omega_n, \mathbf{p}, \mu_\mathsf{I})\} \right.$$
$$\left. + g_F \sum\!\!\!\!\!\!\int_{\mathbf{p}} \mathrm{Tr}\{R^f G^f(\omega_n, \mathbf{p}, \mu_\mathsf{I}, \mu_\mathsf{B})\}\right) = 0,$$

Calculation of the loop integrals requires the diagonalization of the propagators $\rightarrow$ Straightforward but the eigenvalues are non–rational functions of ω_n $\rightarrow$ diagonalization for small ρ $\rightarrow$ Landau–type analysis

Up to $\mathcal{O}(\rho^3)$

$$m_{\pi_3}^2 = m^2 + \lambda v^2 + t^{(0)}(m_{\pi_3}^2, v, T, \mu_{\mathsf{I,B}}) + (\lambda + t^{(2)}(m_{\pi_3}^2, v, T, \mu_{\mathsf{I,B}}))\rho^2$$

Due to the Ward identity the EoS for σ remains $v m_{\pi_3}^2 = h$

$$\rho \left[\mu_I^2 - m^2 - \lambda v^2 - r^{(0)}(m_{\pi_3}^2, v, T, \mu_{\mathsf{I,B}}) - (\lambda + r^{(2)}(m_{\pi_3}^2, v, T, \mu_{\mathsf{I,B}}))\rho^2 + \mathcal{O}(\rho^4) \right] = 0$$

$\Rightarrow$ Pion condensate non–zero only if the roots are real

If $\lambda + r^{(2)} > 0$ and $\mu_I^2 - m^2 - \lambda v^2 - r^{(0)} > 0$

$$\rho = \sqrt{\frac{\mu_I^2 - m^2 - \lambda v^2 - r^{(0)}(m_{\pi_3}^2, v, T, \mu_{\mathsf{I,B}})}{\lambda + r^{(2)}(m_{\pi_3}^2, v, T, \mu_{\mathsf{I,B}})}}$$

$\Rightarrow$ The transition is of second order

Diagonalized propagators

$$O_B = \begin{pmatrix} 1-|a|^2\rho^2 & b(1-2av)\rho^2 & -\sqrt{2}a\rho \\ b^*(1-2a^*v)\rho^2 & 1-|a|^2\rho^2 & -\sqrt{2}a^*\rho \\ \sqrt{2}a\rho & \sqrt{2}a^*\rho & 1-2|a|^2\rho^2 \end{pmatrix} + \mathcal{O}(\rho^3),$$

$a = a(\omega_n, \mu) = \lambda v/(\mu^2 + 2\lambda v^2 + 2\mathrm{i}\omega_n\mu)$ and $b = b(\omega_n, \mu) = \mathrm{i}\lambda/(4\mu\omega_n)$

$$\mathrm{i}\tilde{G}_{\pi^+} = \frac{1}{(\omega_n + \mathrm{i}\mu_\mathrm{I})^2 + E_\pi^2} - \rho^2 \frac{\lambda(2\mu_\mathrm{I}^2 + 2\lambda v^2 - 4\mathrm{i}\mu_\mathrm{I}\omega_n)}{((\omega_n + \mathrm{i}\mu_\mathrm{I})^2 + E_\pi^2)^2(\mu_\mathrm{I}^2 + 2\lambda v^2 - 2\mathrm{i}\mu_\mathrm{I}\omega_n)} + \mathcal{O}(\rho^4),$$

$$\mathrm{i}\tilde{G}_{\pi^-} = \frac{1}{(\omega_n - \mathrm{i}\mu_\mathrm{I})^2 + E_\pi^2} - \rho^2 \frac{\lambda(2\mu_\mathrm{I}^2 + 2\lambda v^2 + 4\mathrm{i}\mu_\mathrm{I}\omega_n)}{((\omega_n - \mathrm{i}\mu_\mathrm{I})^2 + E_\pi^2)^2(\mu_\mathrm{I}^2 + 2\lambda v^2 + 2\mathrm{i}\mu_\mathrm{I}\omega_n)} + \mathcal{O}(\rho^4),$$

$$\mathrm{i}\tilde{G}_{\sigma} = \frac{1}{\omega_n^2 + E_\sigma^2} - \rho^2 \frac{\lambda(\mu_\mathrm{I}^2 + 2\lambda v^2)(\mu_\mathrm{I}^2 + 6\lambda v^2 + 4\mu_\mathrm{I}^2\omega_n^2)}{(\omega_n^2 + E_\sigma^2)^2((\mu_\mathrm{I}^2 + 2\lambda v^2)^2 + 4\mu_\mathrm{I}^2\omega_n^2)} + \mathcal{O}(\rho^4),$$

while the π_3 propagator is

$$\mathrm{i}G_{\pi_3} = \frac{1}{\omega_n^2 + E_\pi^2} - \rho^2 \frac{\lambda}{(\omega_n^2 + E_\pi^2)^2} + \mathcal{O}(\rho^4).$$

Important to note $\longrightarrow$ the transformation matrix depends on ρ, ω_n, μ

$$O_F = \begin{pmatrix} 1 + \frac{g_F^2}{32k_0^2}\rho^2 & -\mathrm{i}\frac{g_F}{4k_0}\gamma_0\gamma_5\rho \\ -\mathrm{i}\frac{g_F}{4k_0}\gamma_0\gamma_5\rho & 1 + \frac{g_F^2}{32k_0^2}\rho^2 \end{pmatrix},$$

where $k_0 = (-\mathrm{i}\omega_n + \frac{1}{3}\mu_{\mathsf{B}})\gamma_0$ and the matrix is hermitian

$$\mathrm{i}\tilde{G}_{u/d} = -\frac{1}{\not{p}_{u/d} - m_f} - \rho^2 \frac{g_F^2}{8k_0} \frac{1}{\not{p}_{u/d} - m_f} \gamma_0 \frac{1}{\not{p}_{u/d} - m_f}$$

where $\not{p}_{u/d} = (-\mathrm{i}\omega_n + \mu_{u/d})\gamma_0 - \gamma_i p_i$ and $\mu_{u/d} = \mu_{\mathsf{B}}/3 \pm \mu_{\mathsf{I}}/2$.

Calculation of one–loop contributions (bosonic case):

$$\mathrm{Tr}\{B^b G^b\} = \mathrm{Tr}\{B^b O_B^{-1} O_B G^b O_B^{-1} O_B\} = \mathrm{Tr}\{O_B B^b O_B^{-1} \tilde{G}^b\} = \mathrm{Tr}\{\tilde{B}^b \tilde{G}^b\}$$

where $\tilde{G}^b = \mathrm{diag}(\tilde{G}_{\pi^+}^{-1}, \tilde{G}_{\pi^-}^{-1}, \tilde{G}_{\sigma}^{-1})$

Parameterization

For finite temperature calculations → necessary to fix the parameters of the model, namely: m^2, λ, g_F, h and v

Four physical quantity:

- one–loop pion mass: $M_\pi = 138\,\text{MeV}$
- one–loop sigma mass: $M_\sigma = 500\,\text{MeV}$
- tree level u–d fermon mass: $m_f = 938/3\,\text{MeV}$
- pion decay constant: $f_\pi = 93\,\text{MeV}$

and the EoS for σ is used.

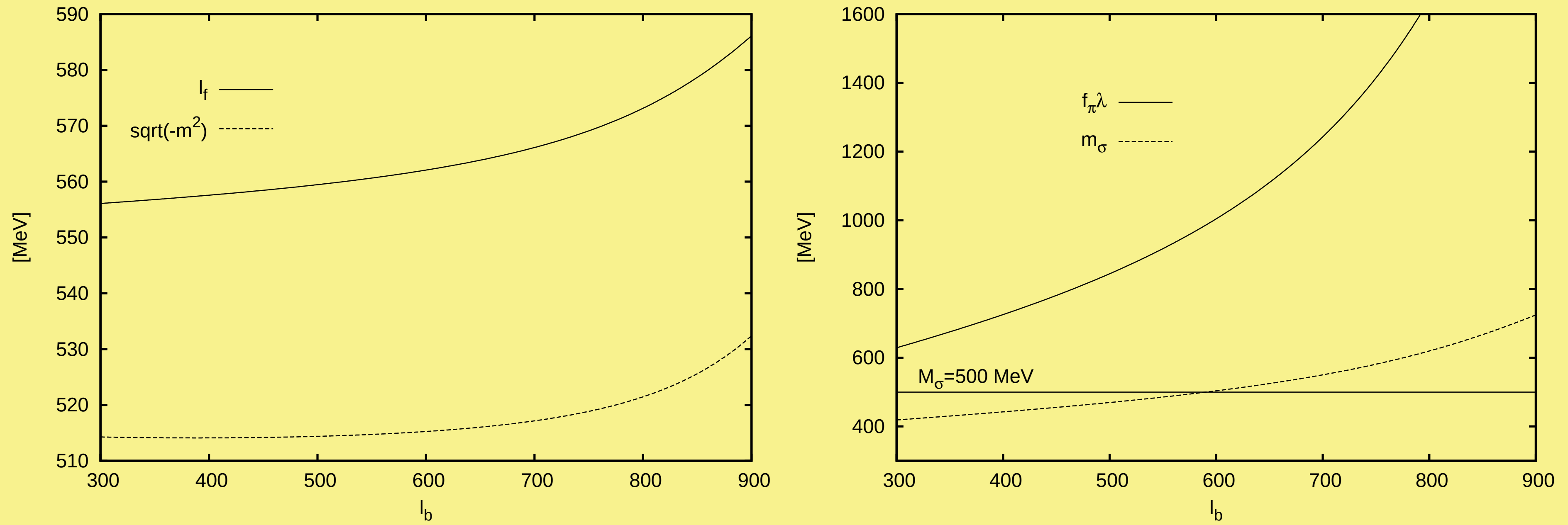
$$M_\pi^2 = m^2 + \lambda v^2 + 3\lambda(T_0^b(M_\pi, l_b) + T_0^b(m_\sigma, l_b)) + 2g_F^2 N_c T_0^{f,\pi}(m_f, l_f)$$

$$\begin{aligned} M_\sigma^2 &= m^2 + 3\lambda v^2 + 3\lambda(T_0^b(M_\pi, l_b) + T_0^b(m_\sigma, l_b)) + 18\lambda^2 v^2 B_0^b(m_\sigma, l_b) \\ &+ 6\lambda^2 v^2 B_0^b(M_\pi, l_b) + 6g_F^2 T_0^{f,\sigma}(m_f, l_f), \end{aligned}$$

$$v = f_\pi, \qquad g_F = 2\frac{m_f}{f_\pi}, \qquad h = f_\pi M_\pi^2,$$

note: fixing the finite part of $\delta Z \rightarrow$ fixes the fermionic renormalization scale l_f

The bosonic scale dependence of the parameters:



Left panel m^2 and l_f

Right panel the tree level σ mass and λ

At $l_b \approx 600$ MeV the tree level σ mass equals its one$-$loop level value $\rightarrow$ σ mass becomes selfconsistent

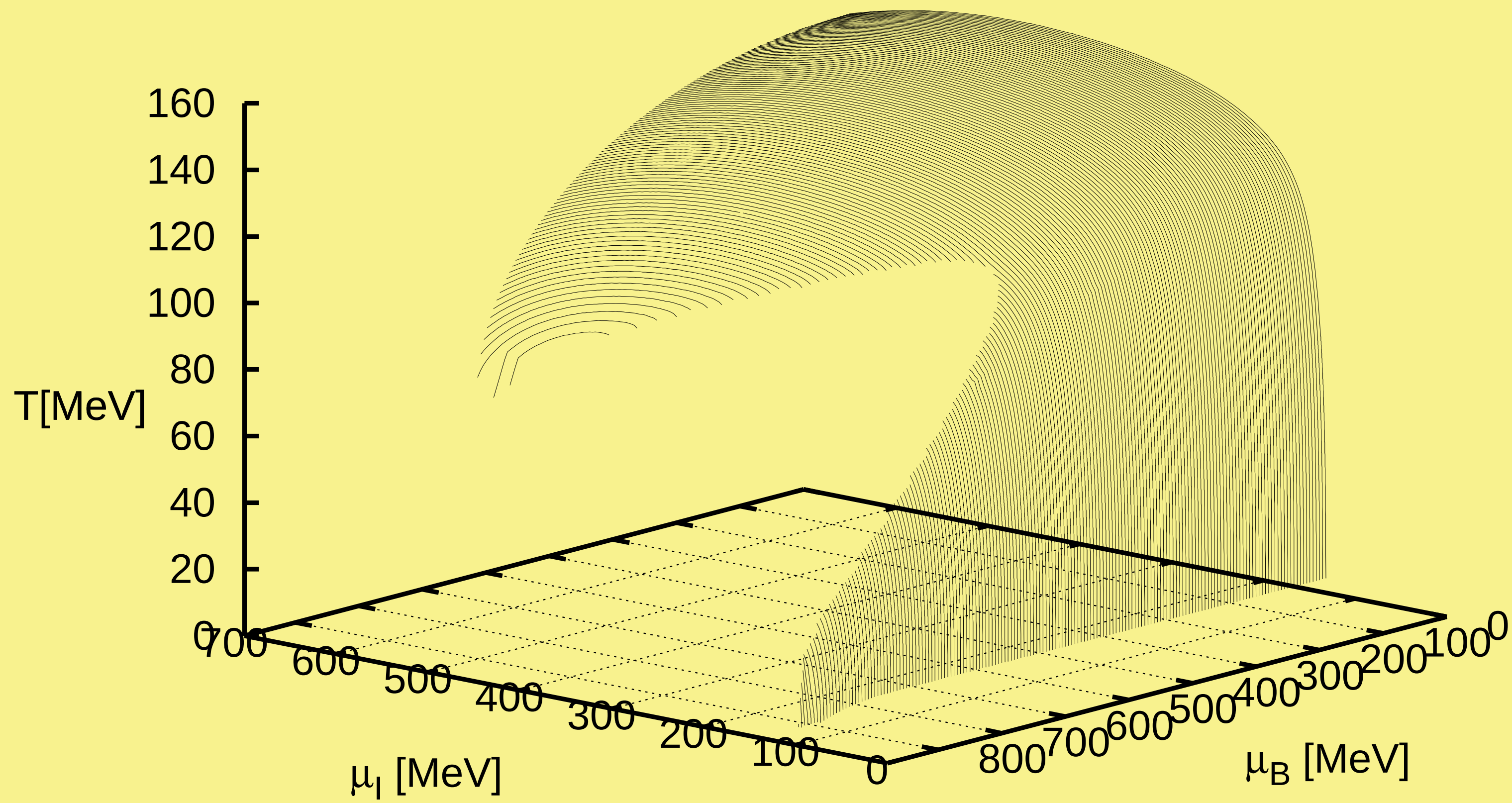
In addition m and l_f moderately depend on the renormalization scale

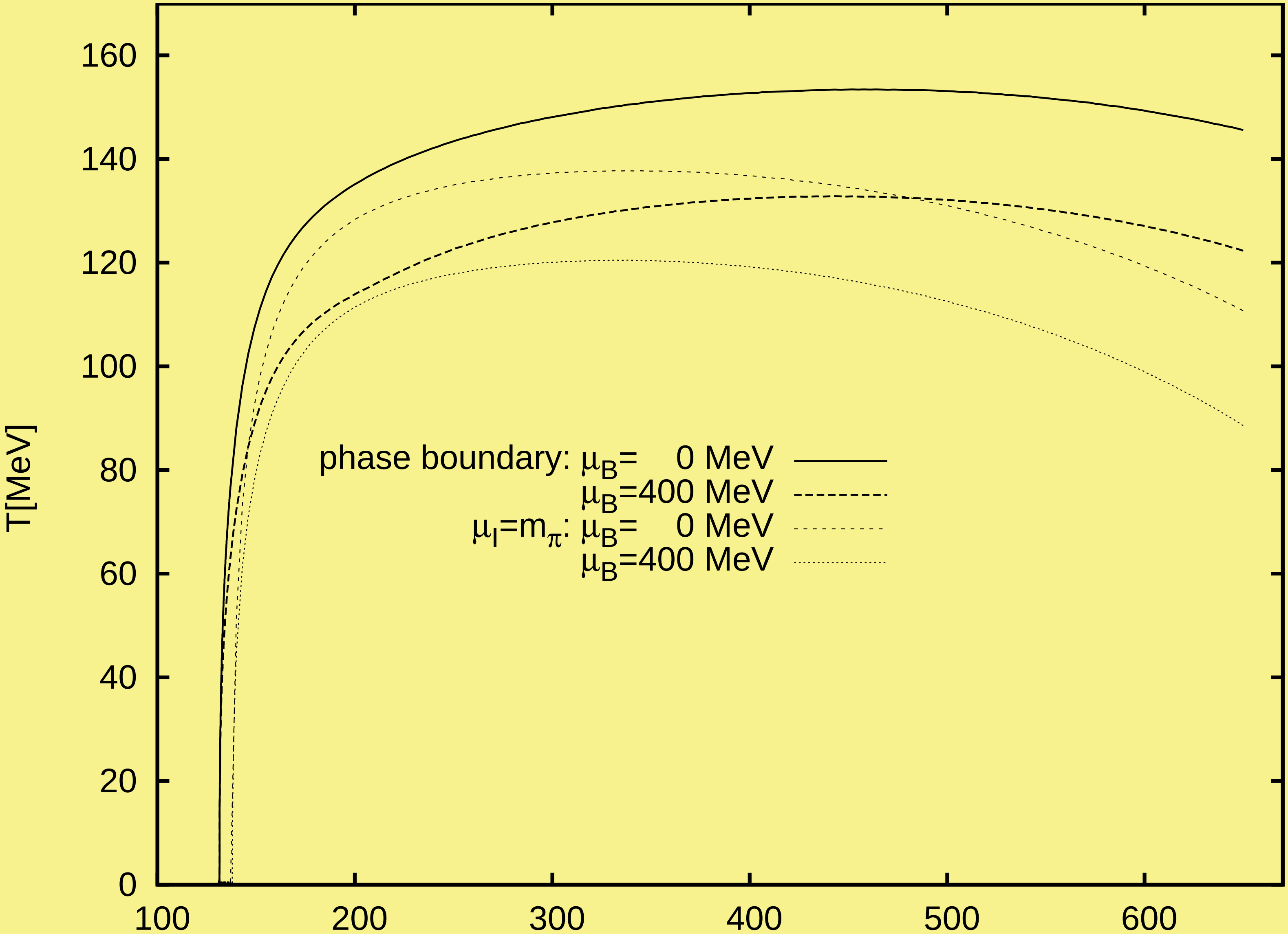
Choosing the following scale range: $l_b \in [400$ MeV, 800 MeV$]$

Results

The second order boundary for the occurrence of the pion condensation in the $\mu_{\mathrm{I}} - \mu_{\mathrm{B}} - T$ space:

$$\mu_I^2 - m_{\pi_3}^2(T, \mu_{\mathrm{I}}, \mu_{\mathrm{B}}) - R^{\text{1-loop}}(T, \mu_{\mathrm{I}}, \mu_{\mathrm{B}}) = 0$$



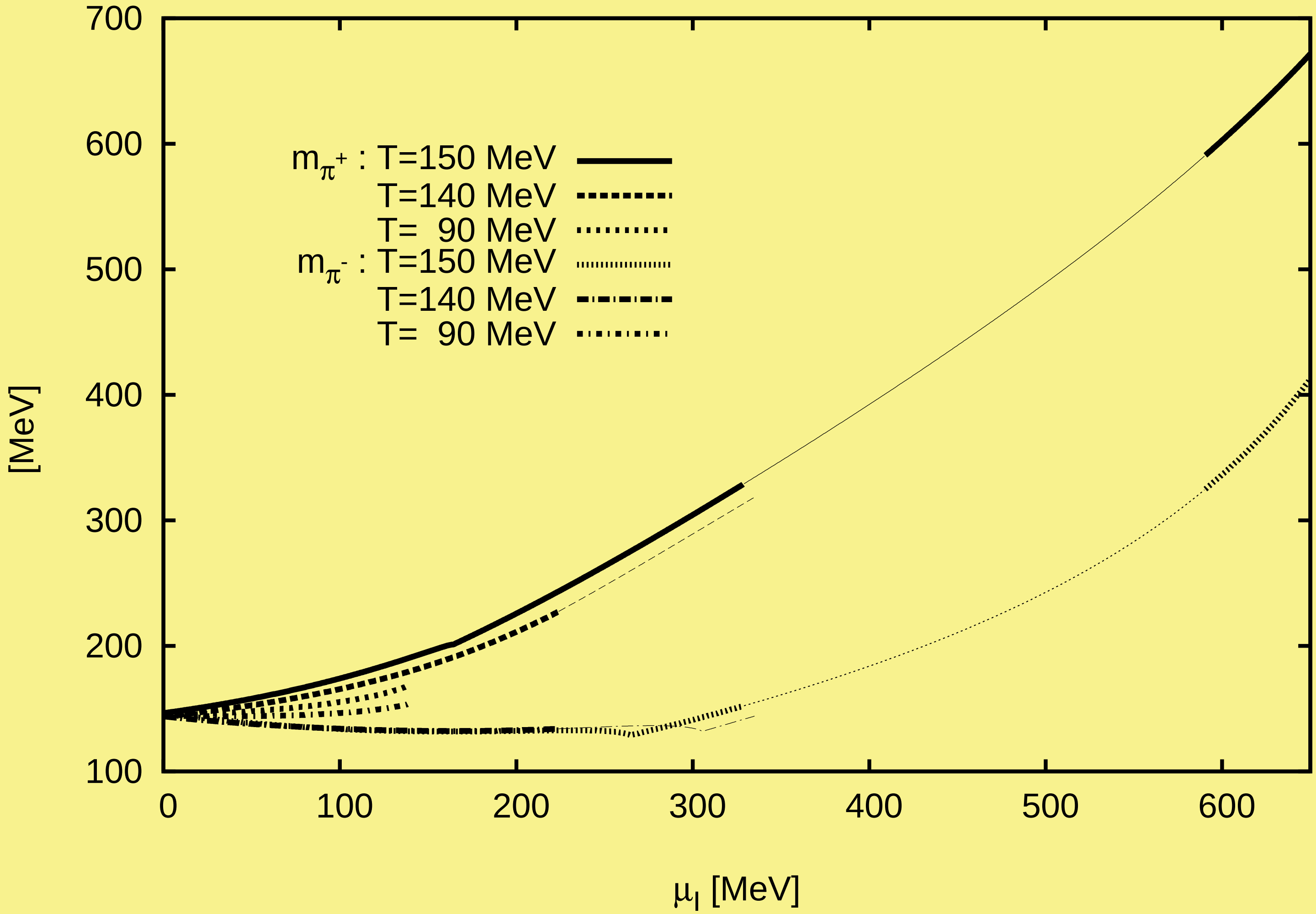


condensation starts at $\mu_I = 131$ MeV

The one–loop pole masses of the charged pions detrmined at $\rho = 0$:

$$\left(M_{\pi^\pm}^{\text{pole}}\right)^2 = \left(m_{\pi^\pm}^{\text{tree}}\right)^2 + \Sigma_{\pi^\pm}(\omega = M_{\pi^\pm}^{\text{pole}}, \mathbf{p} = 0, T, \mu_{\text{I,B}}),$$

The known solutions of $v(T, \mu_{\text{I,B}})$ and $m_{\pi_3}(T, \mu_{\text{I,B}})$ are used



Conclusion

- Bosonic scale dependence was investigated and a moderate dependence was found
- The 2^{nd} order boundary of pion condensation was determined in the $\mu_{\text{I}} - \mu_{\text{B}} - T$ space at one–loop level using a Landau-type of analysis.
- The surface starts steeply with increasing μ_{I} at fixed μ_{B} and towards large values of μ_{B} the pion condensed region shrinks and even disappears at around $\mu_{\text{B}} = 830$ MeV. (However, at such a high energy one should take into account the effects of the strange quark.)
- Investigating different sections of the surface at one–loop level the pion condensation curve slightly differ from the $\mu_{\text{I}} = m_{\pi_3}$ curve at small μ_{I} and this deviation increases with increasing μ_{I}.
- μ_I dependence of charged pion pole masses were obtained.
- Possible continuation: Next to leading order in ρ: scaling, BEC-BCS transition; Three flavor; Polyakov–loop