# Conformal viscous hydrodynamics -
## *AdS/CFT correspondence at finite temperature*

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# CONTENT

- motivation: viscous effects in RHIC data

viscosity $\eta$ - relaxation time $\tau_\pi$, ...

- conformal invariance,

  relativistic hydrodynamics -

  second-order gradients

- gauge/gravity duality -

  (AdS/CFT) correspondence

A fluid which has no shear stresses, viscosity or heat conduction is called a

# PERFECT FLUID

i.e. it looks isotropic in its rest frame

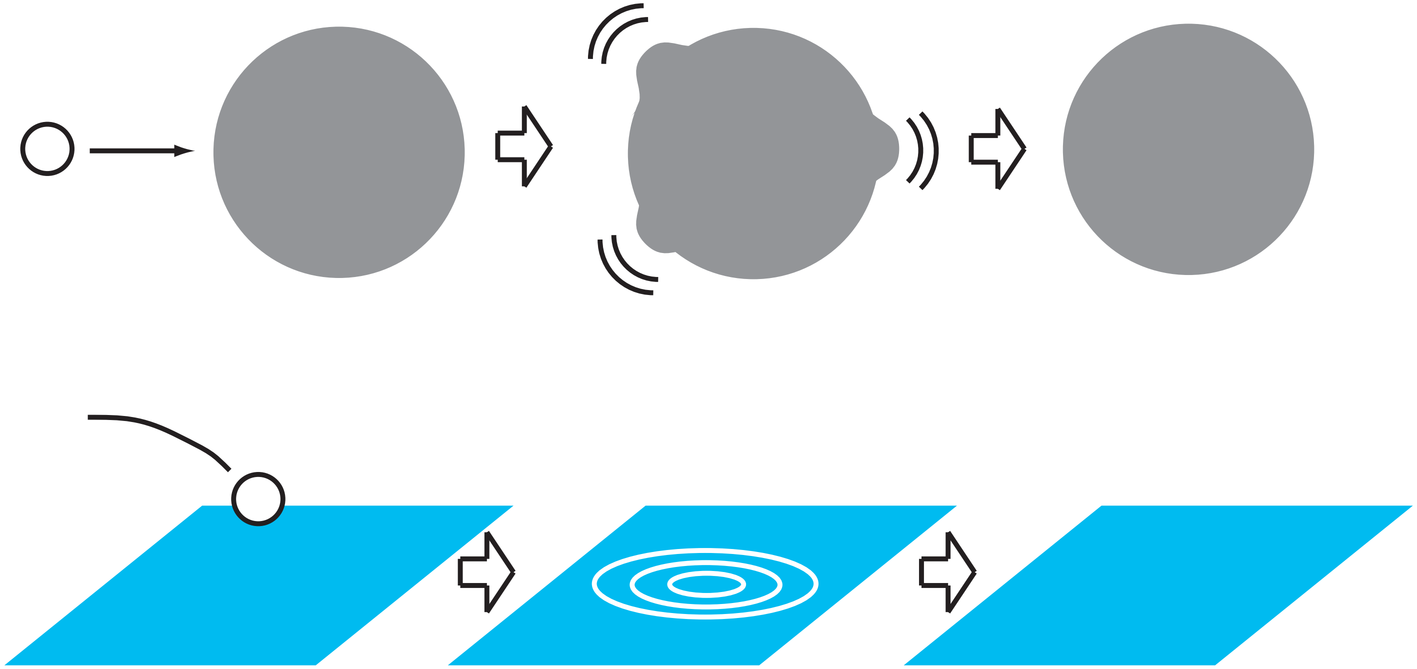
Top physics story of 2005 is the RHIC discovery of the strongly interacting quark-gluon plasma (called sQGP), which behaves almost like a perfect fluid, with very low viscosity

[T. D. Lee 06 ]

## black hole theory

is now used to explain properties of colliding nuclei

[L. Susskind 08]



[from M. Natsuume]

quasinormal modes:

gravitational perturbation to a black hole

and to a hydrodynamic system

# hydrodynamics

energy momentum tensor: $\epsilon$ energy density and $p$ pressure

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} + \Pi^{\mu\nu}$$

$u^\mu$ fluid velocity, $u_\mu u^\mu = -1$, and $\Pi^{\mu\nu}$ shear tensor

with

$$u_\mu \Pi^{\mu\nu} = 0 \;\; (u_\mu T^{\mu\nu} = -\epsilon u^\nu),$$

momentum density is due to energy flow, and

$$\Pi^\mu_\mu = 0$$

required from conformal invariance

local conservation law (covariant derivative $\nabla_\mu$): $\nabla_\mu T^{\mu\nu} = 0$

(assume: no net charge in the system)

# a few definitions

conformal fluid in $d = 4$ (curved) dimensions -
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

metric $g_{\mu\nu}$ with signature $(- + ++)$,
$\mu, \nu$.. space-time indices $0, 1, 2, 3$

(geometric) covariant derivative: $\nabla_\mu$

$$\nabla_\mu \Phi(x) = \partial_\mu \Phi(x) \ ,$$

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^{\nu} V^\rho \ \text{ etc., e.g. } \nabla_\mu g^{\nu\rho} = 0$$

Christoffel symbols:

$$\Gamma_{\mu\rho}{}^{\nu} = \frac{1}{2} g^{\nu\sigma} \left( \partial_\mu g_{\rho\sigma} + \partial_\rho g_{\mu\sigma} - \partial_\sigma g_{\mu\rho} \right)$$

Riemann tensor: $[\nabla_\mu, \nabla_\nu] V^\rho = R_{\mu\nu\sigma}{}^{\rho} V^\sigma$

# approximation

only retaining shear viscosity terms

- Navier - Stokes $=$ first-order theory in gradients with constitutive relation:

$$\Pi^{\mu\nu} = -2\eta < \nabla^{\mu} u^{\nu} > \equiv -2\eta\sigma^{\mu\nu}$$

with the shear strain tensor

$$\sigma^{\mu\nu} \equiv \frac{1}{2}\left(\Delta^{\mu\rho}\nabla_{\rho}u^{\nu} + \Delta^{\nu\rho}\nabla_{\rho}u^{\mu}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\rho\sigma}\nabla_{\rho}u_{\sigma}$$

and projection

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu}\ , \quad \Delta^{\mu\nu}u_{\nu} = 0$$

# Navier-Stokes

projection $\Delta^{\mu}_{\alpha}\nabla_{\beta}T^{\alpha\beta}=0 \quad \rightarrow$
relativistic Navier-Stokes equation in first-order theory

$$(\epsilon+p)u^{\alpha}\nabla_{\alpha}u^{\mu}+\Delta^{\mu\alpha}\nabla_{\alpha}p+\Delta^{\mu}_{\alpha}\nabla_{\beta}\left[-2\eta < \nabla^{\alpha}u^{\beta} >\right]=0$$

i.e. parabolic equation:

time derivative is of first order ($u^{\alpha}\nabla_{\alpha}\rightarrow\partial/\partial t$)

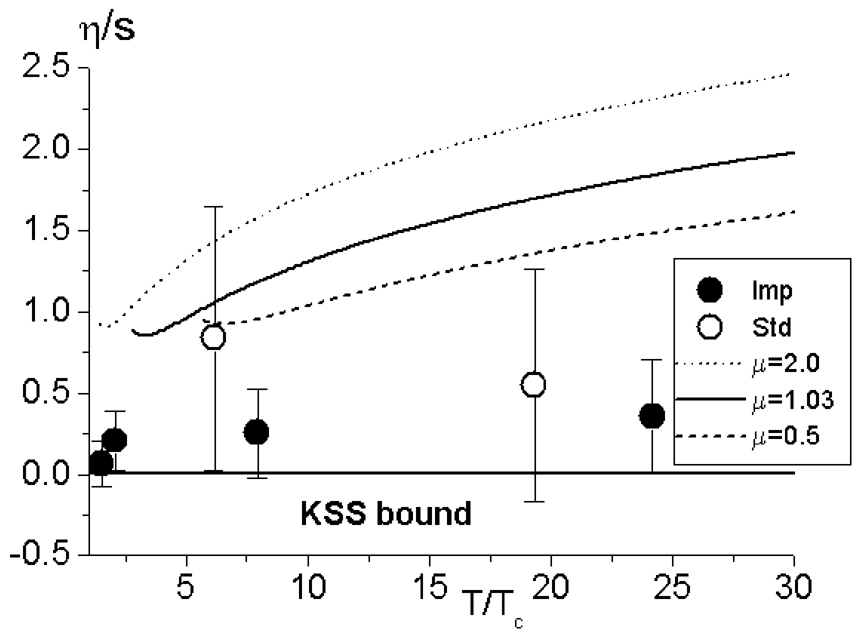
while

space derivative is of second order ($\nabla^2$)

“relativistic first-order dissipative theory is highly pathological, and therefore should be discarded in favor of the second-order one”

[Hiscock and Lindblom 1983-1985]

# viscosity



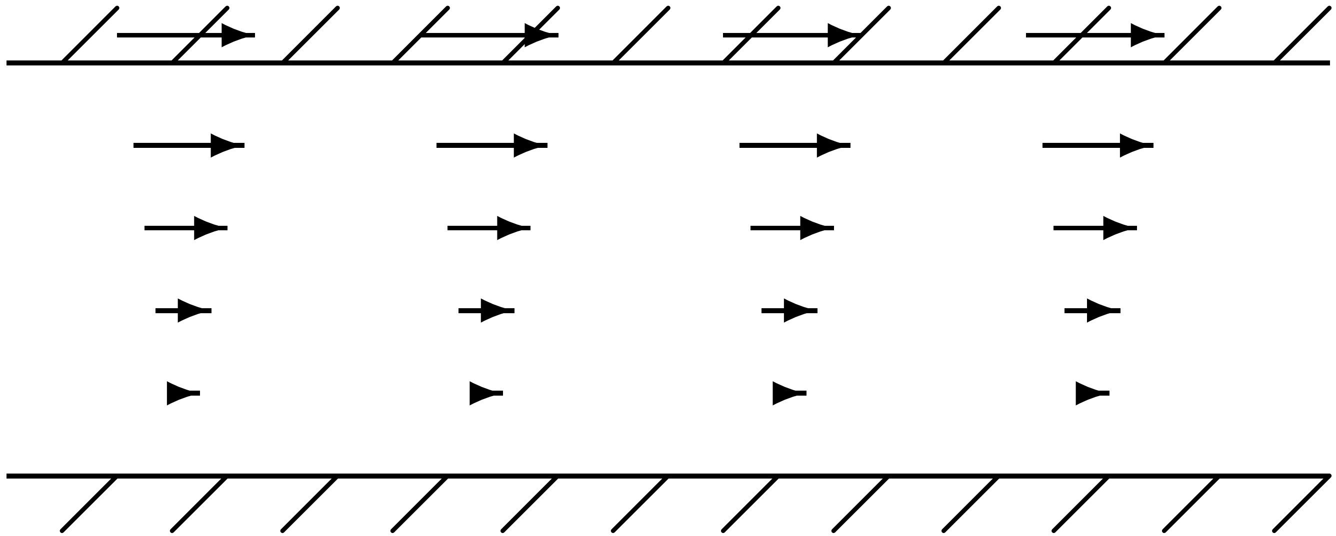

$\eta/s$ from quenched QCD lattice simulations [Nakamura and Sakai 05]

compared with the perturbative result

for strong coupling $\lambda = g_{YM}^2 N_c$ [Kovtun, Son and Starinets 05]
AdS/CFT universal (?) lower bound

$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$

# shear viscosity $\eta$




shear flow in $x-$ direction $\rightarrow$ force per unit area/momentum transfer

$$\frac{F_x}{A} = -\Pi_{xz} = \eta \frac{\partial u_x}{\partial z}$$

measure how quickly a system, perturbed from thermal equilibrium, goes back: it takes longer for a weakly coupled system, because particles interact less than for a strongly coupled system

# transverse mode

small linear perturbation and first-order $\rightarrow$
diffusion equation in shear channel:

$$\delta u_\perp = \sqrt{\frac{(\epsilon+p)}{4\pi\eta t}} \exp\left[-\frac{(\epsilon+p)x^2}{4\eta t}\right]$$

i.e. propagates outside the light-cone (Gaussian)
starting from $\delta u_\perp(x, t=0) = \delta(x)$ !!

minimal modification $\rightarrow$ second-order:
relaxation time $\tau_\pi > 0$ $\rightarrow$ hyperbolic equation

$$\left[\tau_\pi \partial_t^2 + \partial_t - \frac{\eta}{(\epsilon+p)}\partial_x^2\right] \delta u_\perp = 0$$

# conformal hydrodynamics

mainly based on more recent work by:

R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov

“Relativistic viscous hydrodynamics, conformal invariance, and holography”

JHEP 0804 (2008) 100 (arXiv: 0712.2451 [hep-th])

and

R. Loganayagam

“Entropy current in conformal hydrodynamics”

JHEP 0805 (2008) 087 (arXiv: 0801.3701 [hep-th])

and references therein

# a new result

*all* second order terms classified by conformal symmetry,
Weyl transformation: $g^{\mu\nu} = e^{-2\phi(x)}\,\tilde{g}^{\mu\nu}$,
$T^{\mu\nu} = e^{-6\phi}\,\tilde{T}^{\mu\nu},\;\; T^{\mu}_{\mu} = 0,\; \dots$

constitutive relation (d= 4):

$$\begin{aligned}\Pi^{\mu\nu} \;\; = \;\; & -2\eta\sigma^{\mu\nu} + 2\eta\tau_\pi u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} \\ & +4\lambda_1 \sigma^{<\mu}{}_\lambda \sigma^{\nu>\lambda} + 2\lambda_2 \sigma^{<\mu}{}_\lambda \Omega^{\nu>\lambda} + \lambda_3 \Omega^{<\mu}{}_\lambda \Omega^{\nu>\lambda} \\ & \quad +2\kappa u_\alpha C^{\alpha\mu\nu\beta} u_\beta\end{aligned}$$

$C^{\alpha\beta\gamma\delta}$... Weyl tensor, $\Omega^{\alpha\beta}$... antisymmetric vorticity tensor

[Baier, Romatschke, Son, Starinets, Stephanov 07]

[Bhattacharyya, Hubeny, Minwalla, Rangamani 07, Loganayagam 08]

# Weyl transformation

$$g_{\mu\nu} = \mathrm{e}^{2\phi}\tilde{g}_{\mu\nu}; \qquad g^{\mu\nu} = \mathrm{e}^{-2\phi}\tilde{g}^{\mu\nu}\ ,\ \phi = \phi(x)$$

leads to:

$$\Gamma_{\lambda\mu}{}^{\nu} = \widetilde{\Gamma}_{\lambda\mu}{}^{\nu} + \delta^{\nu}_{\lambda}\partial_{\mu}\phi + \delta^{\nu}_{\mu}\partial_{\lambda}\phi - \tilde{g}_{\lambda\mu}\tilde{g}^{\nu\sigma}\partial_{\sigma}\phi$$

$$u^{\mu} = \mathrm{e}^{-\phi}\tilde{u}^{\mu}, \quad \Delta^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu} = \mathrm{e}^{-2\phi}\widetilde{\Delta}^{\mu\nu}$$

usefull quantities (d dimensions):

$$\vartheta \equiv \nabla_{\mu}u^{\mu} = \mathrm{e}^{-\phi}\left[\tilde{\vartheta} + (d-1)\tilde{u}^{\sigma}\partial_{\sigma}\phi\right],$$

$$a^{\nu} \equiv u^{\mu}\nabla_{\mu}u^{\nu} = \mathrm{e}^{-2\phi}\left[\tilde{a}^{\nu} + \widetilde{\Delta}^{\nu\sigma}\partial_{\sigma}\phi\right],$$

$$\mathcal{A}_{\nu} \equiv a_{\nu} - \frac{\vartheta}{d-1}u_{\nu} = \tilde{\mathcal{A}}_{\nu} + \partial_{\nu}\phi$$

# Weyl derivative

cf. (abelian) gauge transform:

$$\tilde{\psi} = \mathrm{e}^{i\Lambda(x)}\psi\ ,\ \tilde{A}^\mu = A^\mu - \partial^\mu\Lambda\ ,$$

$$\tilde{D}^\mu\tilde{\psi} = \mathrm{e}^{i\Lambda}(\partial^\mu + iA^\mu)\psi = \mathrm{e}^{i\Lambda}D^\mu\psi$$

Define covariant Weyl derivative for:

$$Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{Q}^{\mu\ldots}_{\nu\ldots}\ \ ,\mathcal{D}_\lambda\, Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{\mathcal{D}}_\lambda\widetilde{Q}^{\mu\ldots}_{\nu\ldots}$$

where

$$\begin{aligned}\mathcal{D}_\lambda\, Q^{\mu\ldots}_{\nu\ldots} \equiv{}& \nabla_\lambda\, Q^{\mu\ldots}_{\nu\ldots} + w\,\mathcal{A}_\lambda Q^{\mu\ldots}_{\nu\ldots}\\ &+ \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda\mathcal{A}_\alpha - \delta^\mu_\alpha\mathcal{A}_\lambda\right] Q^{\alpha\ldots}_{\nu\ldots} + \ldots\\ &- \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda\mathcal{A}_\nu - \delta^\alpha_\nu\mathcal{A}_\lambda\right] Q^{\mu\ldots}_{\alpha\ldots} - \ldots\end{aligned}$$

## examples

$$\mathcal{D}_\lambda g_{\mu\nu} = 0\ ,\ \mathcal{D}_\mu u^\mu = 0\ ,\ u^\mu \mathcal{D}_\mu u^\nu = 0$$

and

$$\mathcal{D}_\mu u^\nu = \nabla_\mu u^\nu + u_\mu a^\nu - \frac{\vartheta}{d-1}\Delta_\mu{}^\nu = \sigma_\mu{}^\nu + \Omega_\mu{}^\nu = \mathrm{e}^{-\phi}\widetilde{\mathcal{D}}_\mu \tilde{u}^\nu,$$

$$\sigma^{\mu\nu} \equiv \frac{1}{2}\left(\Delta^{\mu\lambda}\nabla_\lambda u^\nu + \Delta^{\nu\lambda}\nabla_\lambda u^\mu\right) - \frac{1}{d-1}\vartheta\Delta^{\mu\nu} = \frac{1}{2}\left(\mathcal{D}^\mu u^\nu + \mathcal{D}^\nu u^\mu\right)$$
$$= \mathrm{e}^{-3\phi}\tilde{\sigma}^{\mu\nu},$$

$$\Omega^{\mu\nu} \equiv \frac{1}{2}\left(\Delta^{\mu\lambda}\nabla_\lambda u^\nu - \Delta^{\nu\lambda}\nabla_\lambda u^\mu\right) = \frac{1}{2}\left(\mathcal{D}^\mu u^\nu - \mathcal{D}^\nu u^\mu\right) = \mathrm{e}^{-3\phi}\widetilde{\Omega}^{\mu\nu}\ ,$$

temperature $T$:

$$T = \mathrm{e}^{-\phi}\ \tilde{T}\ ,\ \mathcal{D}_\mu T = \mathrm{e}^{-\phi}\widetilde{\mathcal{D}}_\mu \tilde{T}$$

# Weyl tensor

$$(d-2)C_{\mu\alpha\nu\beta}u^\alpha u^\beta =$$
$$\Delta^{\mu\lambda}\Delta^{\nu\sigma}R_{\lambda\sigma} + (d-2)\Delta^{\mu\lambda}\Delta^{\nu\sigma}R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta$$
$$-\frac{\Delta^{\mu\nu}}{d-1}(\Delta^{\lambda\sigma}R_{\lambda\sigma}+(d-2)\Delta^{\lambda\sigma}R_{\lambda\alpha\sigma\beta}u^\alpha u^\beta)$$

$R_{\mu\nu}\ldots$ Ricci tensor

symmetry properties (same as for the Riemann tensor):

$$C_{\mu\nu\lambda\sigma} = -C_{\nu\mu\lambda\sigma} = -C_{\mu\nu\sigma\lambda} = C_{\lambda\sigma\mu\nu}$$
and $\quad C_{\mu\alpha\lambda}{}^{\alpha} = 0$

# dispersion relations

quasi-normal modes in the limit of vanishing momentum $k \to 0$:

- sound mode/pole

$$\omega_{1,2} = \pm c_s k - i\Gamma k^2 \pm \frac{\Gamma}{c_s}(c_s^2 \tau_\pi - \frac{\Gamma}{2})k^3 + \mathcal{O}(k^3) \ ,$$

with $\Gamma = \frac{2}{3}\frac{\eta}{\epsilon+p}$

- shear mode/pole

$$\omega = -i\frac{\eta}{\epsilon + p}k^2 + \mathcal{O}(k^4)$$

# references

NOT a COMPLETE LIST!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- R. Loganayagam, *Entropy current in conformal hydrodynamics*
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- M. Luzum and P. Romatschke, *Conformal relativistic viscous hydrodynamics: Applications to RHIC at* $\sqrt{s_{NN}} = 200$ *GeV*
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# AdS/CFT correspondence

**duality $\sim$ equality**

is between

Quantum Field Theory (a special one) in $d = 4$

and

Classical Gravity in $d = 5$ (for $N_c >> 1,\ g^2_{YM} N_c >> 1$)

(at finite temperature)

# some classical gravity

Einstein-Hilbert with nonzero cosmological constant ($d$ dimensions)

- $$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\Lambda_d g_{\mu\nu} \ , \quad \mu, \nu = 0, 1, ...., d-1$$
- ansatz:

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + r^2 d\Omega^2_{(d-2)} \ , \quad g_{tt} = -\frac{1}{g_{rr}} = -f(r)$$

- $$R(1 - \frac{d}{2}) = -\Lambda_d d, \quad R_{\mu\nu} = \frac{2\Lambda_d}{d-2} g_{\mu\nu}$$

# AdS gravity

- with

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + p g_{\mu\nu} = -\frac{\Lambda_d}{8\pi G_d} g_{\mu\nu}$$

follows

$$\epsilon_{vac} = -p_{vac} = \frac{\Lambda_d}{8\pi G_d} < 0$$

for $\Lambda_d < 0$ ...

Anti-deSitter

[Hawking and Page 83]

# solution

- $$R_{tt} = \frac{f}{2}\frac{[r^{d-2}f']'}{r^{d-2}} = -\frac{2\Lambda_d}{d-2}f \ ,$$

$$f' = -\frac{4\Lambda_d}{(d-1)(d-2)}r \quad (+const) \ ,$$

$$f = 1 - \frac{2\Lambda_d}{(d-1)(d-2)}r^2 \equiv 1 + \frac{r^2}{L^2}$$

$r >> L \equiv R$ (radius of AdS space = distance scale)

$$ds^2 = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2 + \dots$$

- $AdS_5$

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) \ , \quad z = \frac{R^2}{r}$$

# exercise

consider $AdS_5$ metric, but start from coordinates $y^\mu, \mu = 0, 1, ..., 5$ :

$$(y^0)^2 - (y^1)^2 - .. - (y^4)^2 + (y^5)^2 = R^2$$

and change to

$$x^\mu = \frac{R}{r}(y^0, y^1, y^2, y^3) \quad , \quad r = y^4 + y^5 \quad ,$$

$$y^4 - y^5 = \frac{R^2}{r} + \frac{r}{R^2}((x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2)$$

leads to

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2$$

$AdS_5$ has $SO(2,4)$ symmetry

# Hawking temperature

black hole metric:

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + .. \, , \; g_{tt} = -\frac{1}{g_{rr}} = -[1 - (\frac{r_0}{r})^{(d-3)}]$$

expand around the horizon $r \approx r_0$:

$$g_{tt} \approx -\frac{d-3}{r_0}(r - r_0) = -\gamma_t(r - r_0) \; , \; g_{rr} \approx \frac{\gamma_r}{r - r_0}, \; \gamma_r = \frac{r_0}{d-3}$$

claim:

$$T_H = \frac{1}{4\pi}\sqrt{\frac{\gamma_t}{\gamma_r}} = \frac{d-3}{4\pi r_0}$$

and $AdS_5$: $\gamma_t = 1/\gamma_r = 4r_0/R^2$

$$T_H = \frac{r_0}{\pi R^2}$$

proof: with $\rho^2 = \frac{4r_0(r-r_0)}{d-3}$

$$ds^2 = \rho^2 (\frac{d-3}{2r_0})^2 d\tau^2 + d\rho^2 + ..., \;\; euclidean \; d\tau^2 = -dt^2$$

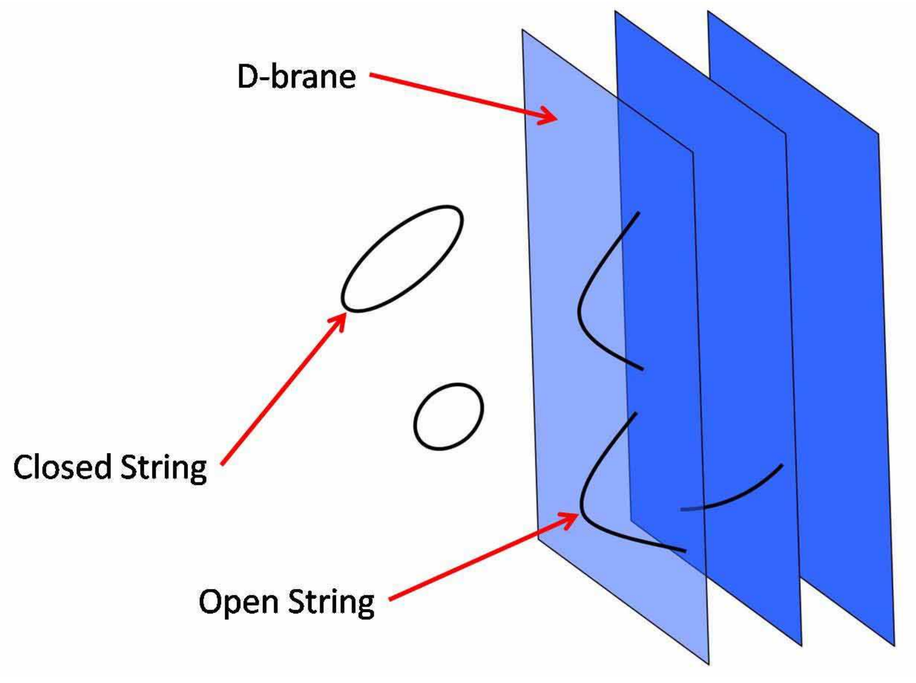
$$\rightarrow$$

$$ds^2 = \rho^2 d\phi^2 + d\rho^2 \; , \;\; \phi = \frac{d-3}{2r_0}\tau$$

requirement: periodicity $\phi \rightarrow \phi + 2\pi,$ i.e. NO conical singularity

$$2\pi = \frac{(d-3)}{2r_0}\beta \; , \;\; \beta = \frac{1}{T_H}$$

# $D-$**branes**




[from Myers and Vazquez 08]

dynamical walls on which strings can end:

theory of open strings living on $N_c - D3-$branes ($\mathcal{N} = 4$ SYM, $d = 4$) $\Longleftrightarrow$
gravity theory of fields living in the space curved by the branes ($AdS_5,\ d = 5$)

## $AdS_5$ metric at finite $T$

generalization of the Einstein-Hilbert-Maxwell equation
- strong analogy to Reissner-Nordström black hole

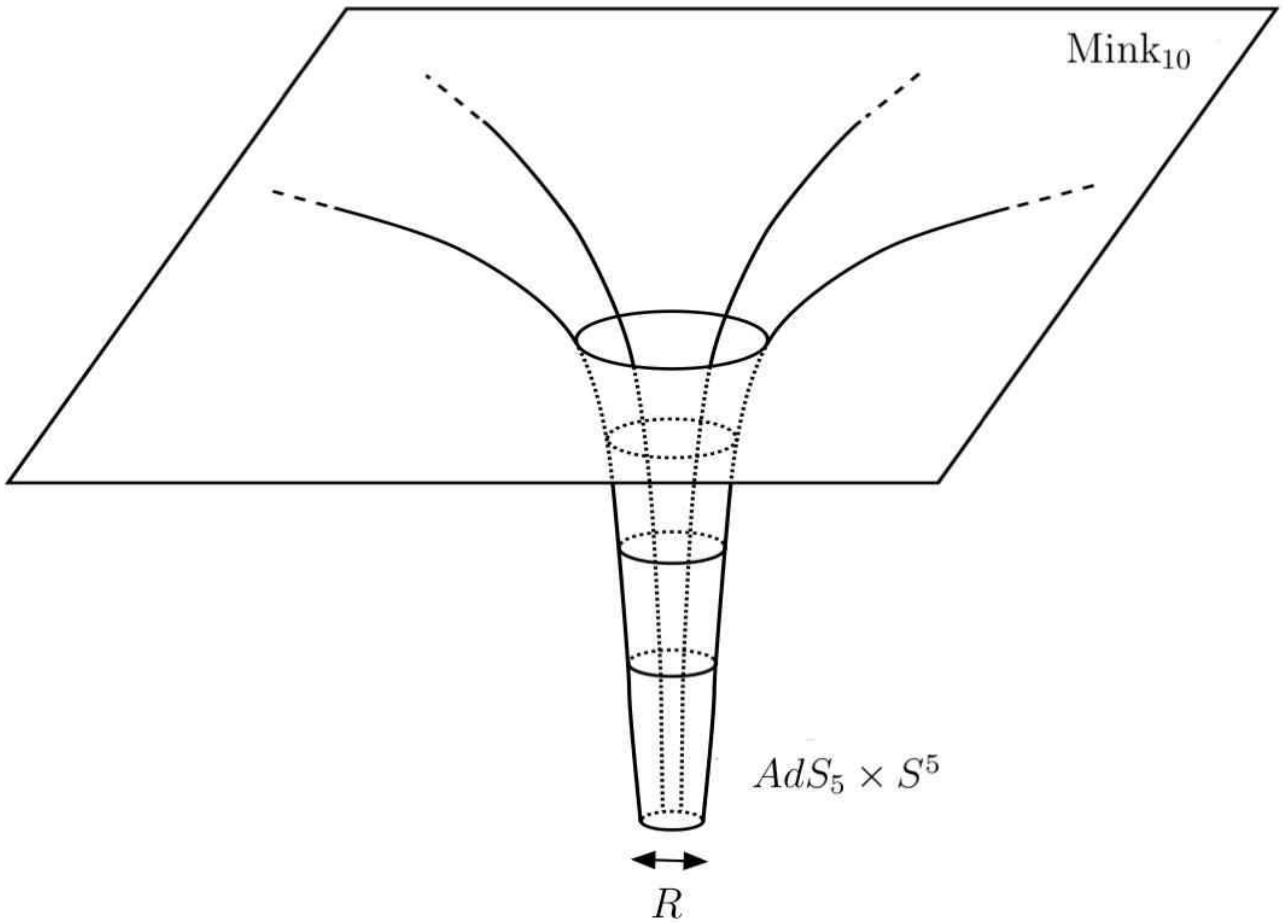
$$S = \frac{1}{16\pi G_{(10)}} \int d^{10}x \sqrt{-g} \left[R - \frac{F_5^2}{2 \cdot 5!} + \ldots\right] ,$$

$$\int_{s^5} *F_5 \propto N_c \text{ , } G_{(10)} = \frac{\pi^4 L^8}{2N_c^2}, \; H(r) = 1 + \frac{L^4}{r^4} \; , \quad f(r) = 1 - \frac{r_0^4}{r^4}$$

$$ds^2 = -\frac{f(r)}{\sqrt{H(r)}} dt^2 + \frac{1}{\sqrt{H(r)}} (dx_1^2 + dx_2^2 + dx_3^2) + \frac{\sqrt{H(r)}}{f(r)} dr^2 + r^2 \sqrt{H} d\Omega_5^2$$

near extremal black $D3-$ brane metric with horizon $r = r_0$
for $r_0 < r << L = R$, i.e. factorized metric for $AdS_5 \otimes S^5$

$$ds^2 = \frac{r^2}{R^2}(-f(r)dt^2 + d\vec{x}^2) + \frac{R^2}{r^2 f(r)} dr^2 + R^2 d\Omega_5^2$$




[from D. Mateos]

space-time around $D3-$branes

# AdS/CFT [Maldacena 98]

strongly coupled quantized conformal gauge theory
in $d = 4$ dimensions ($\mathcal{N} = 4$ SYM with $8N_c^2$ (1 gauge and 6 scalar) bosons and ($4N_c^2$ ) Weyl fermions) [[ NOT QCD !]]

$$\Longleftrightarrow$$

weakly coupled classical supergravity (type IIB)
in $d = 10$ dimensions (on $AdS_5 \times S^5$)
via holographic property: radial coordinate $r_0 \leq r < \infty$ with gauge theory on the boundary at $\infty$
in the limit:
't Hooft coupling $\lambda = g_{YM}^2 N_c$ is large, $N_c \to \infty, \quad g_{YM}^2 << 1$
i.e. string coupling $g_s = \frac{g_{YM}^2}{4\pi} << 1$ – NO LOOPS

and

small curvature $\frac{l_s^4}{R^4} = \frac{1}{\lambda} << 1$ – RADIUS $R$ of CURVATURE is LARGE compared to the STRING SCALE $l_s = \sqrt{\alpha'}$

# symmetries

duality maps the operators of QFT to the boundary values of the SUGRA fields

- QFT ($\mathcal{N} = 4$, $d = 4$ CFT)

$$S = -\frac{1}{g_{YM}^2} \int d^4x \, [\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} D_\mu \Phi^a_i D^\mu \Phi^a_i + ....]$$

  .... massless Weyl fermions + interactions , $i = 1, .., 6, a = 1, ..., N_c^2$
  has Poincaré and conformal symmetry $SO(2,4)$, e.g. $x^\mu \to \lambda x^\mu$, $\Phi^a_i(x) \to \Phi^a_i(\lambda x)$, etc, $\beta(\mu) = \mathbf{0}$, and $SU(\mathcal{N} = 4)$ symmetry and $SU(N_c)$

- $AdS_5 \otimes S^5$:

$$SO(2,4) \otimes SO(6) = SO(2,4) \otimes SU(4)$$

  besides $SU(N_c)$ due to $N_c$ parallel $D-3$ branes

# relaxation phenomena/strong coupling limit

hydrodynamic transport coefficients by gauge/gravity duality:
compare with quite involved AdS/CFT-gravity calculations at Hawking temperature $T_H = \frac{r_0}{\pi R^2}$,
and for momentum $\omega, k << T_H$,
e.g. from sound channel dispersion up to $O(k^3)$, etc :

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad \tau_\pi = \frac{2 - \ln 2}{2\pi T}, \quad \kappa = \frac{\eta}{\pi T} = 2\lambda_1 \ ,$$

$$\lambda_2 = -\frac{\ln 2}{2\pi T}\eta \ , \quad \lambda_3 = 0$$

# conformal $\Pi^{\mu\nu}$

*all* $5$ second order terms classified by conformal symmetry

constitutive relation (d= 4):

$$\begin{aligned}\Pi^{\mu\nu} &= -2\eta\sigma^{\mu\nu} + 2\eta\tau_\pi u^\lambda \mathcal{D}_\lambda \sigma^{\mu\nu} \\ &\quad +4\lambda_1 \sigma^{<\mu}{}_\lambda \sigma^{\nu>\lambda} + 2\lambda_2 \sigma^{<\mu}{}_\lambda \Omega^{\nu>\lambda} + \lambda_3 \Omega^{<\mu}{}_\lambda \Omega^{\nu>\lambda} \\ &\quad +2\kappa u_\alpha C^{\alpha\mu\nu\beta} u_\beta\end{aligned}$$

$C^{\alpha\beta\gamma\delta}$... Weyl tensor, $\Omega^{\alpha\beta}$... antisymmetric vorticity tensor

[Baier, Romatschke, Son, Starinets, Stephanov 07]

[Bhattacharyya, Hubeny, Minwalla, Rangamani 07, Loganayagam 08]

# Bekenstein- Hawking entropy

thermodynamics: $dS = \frac{dE}{T}$

- Schwarzschild BH ($G = G_{(4)}, T_H = \frac{1}{4\pi r_0}$)

$$E = M = \frac{r_0}{2G}\ ,\ dE = \frac{dr_0}{2G}\ ,\ dS = \frac{4\pi}{2G} r_0 dr_0\ ,$$

entropy

$$S = \frac{\pi r_0^2}{G} = \frac{A}{4G}$$

a universal relation with $A = 4\pi r_0^2 ...$ area of the horizon

# Bekenstein-Hawking entropy, cont.

- $AdS_5 \otimes S^5$: $T_H = \frac{r_0}{\pi R^2}$ , $S = \frac{A_{(8)}}{4G_{(10)}}$

$$A_{(8)} = (\frac{r_0}{R})^3 V_3 R^5 \Omega_{(5)} = \pi^6 T_H^3 R^8 V_3 \ , \ G_{(10)} = \frac{\pi^4 R^8}{2N_c^2}$$

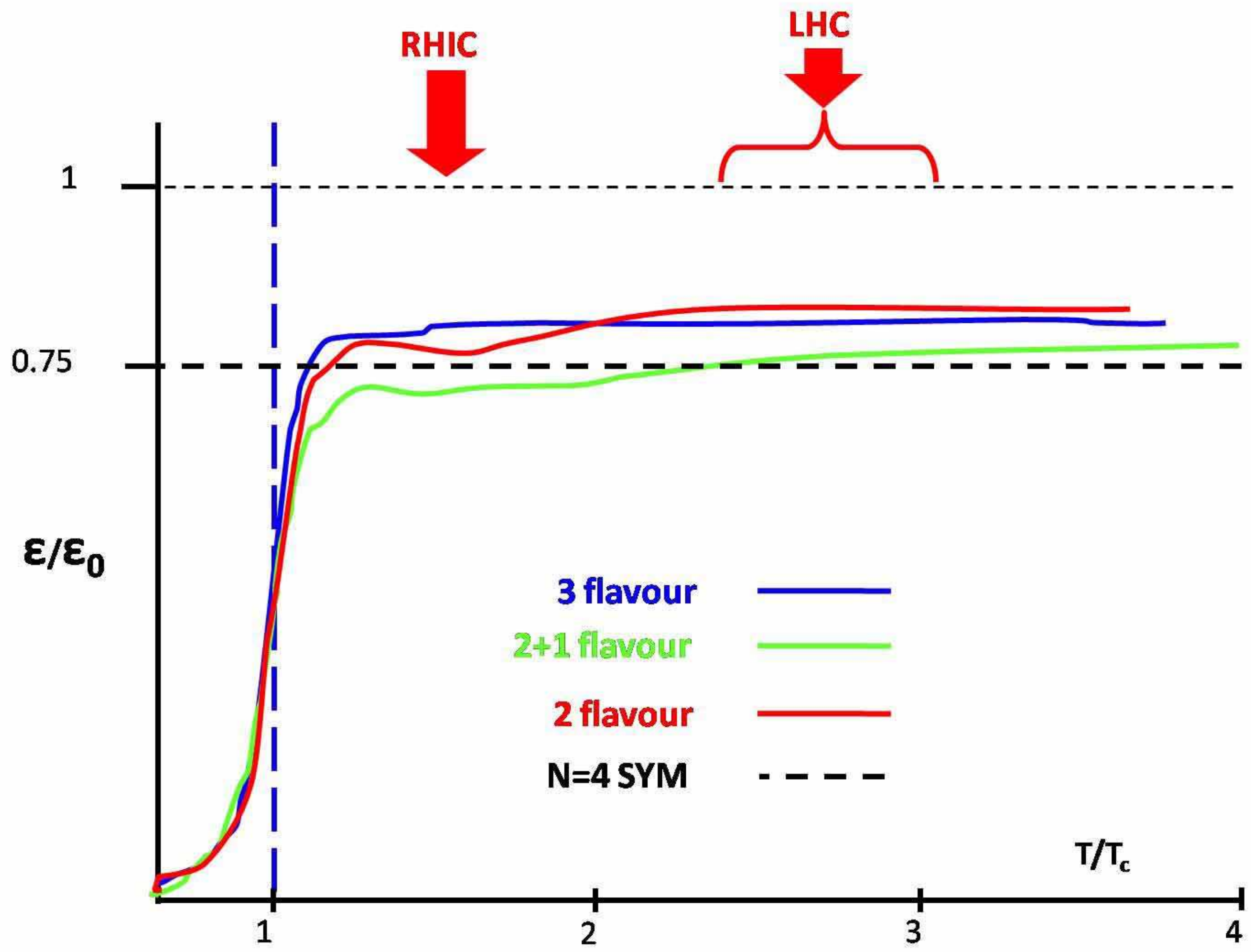
gives for the entropy density

$$s_{BH} = \frac{S}{V_3} = \frac{\pi^2 N_c^2}{2} T_H^3$$

NOTE:

$$s_{Boltzmann} = \frac{4p}{T} = \frac{4\pi^2 T^3}{90}[\frac{7}{4} \cdot 4 + 2 + 6] N_c^2 = \frac{2\pi^2 N_c^2}{3} T^3$$

$$s_{BH} = \frac{3}{4} s_{Boltzmann}$$




energy density of QCD and SYM - via BH entropy

[from Myers and Vazquez 08]

# master formula for AdS/CFT

schematically in terms of coinciding partition functions:

$$\int e^{iS_{4d}^{gauge}+i\Phi_0 O} = \int e^{iS_{5d}[\Phi]} \simeq e^{iS^{classical}[\Phi_0]}$$

$S_{5d}$ is computed with non-trivial boundary condition (holography)

$$\Phi(t,\vec{x},r) \overset{r->\infty}{=} \Phi_0(t,\vec{x})$$

$$\Longrightarrow$$

quantum correlation $=$ classical two-point function

$$G_R(x,y) = -i < O(x)O(y) >= -\frac{\delta^2 S^{classical}}{\delta\Phi_0(x)\delta\Phi_0(y)}|_{r=\infty}$$

gauge: $O = T_{\mu\nu}$ .. energy-momentum tensor
gravity: $\phi = g_{\mu\nu}$ .. graviton

# correlators from gravity, $d = 4$

response of the fluid to small metric perturbations
example: $g_{xy} = h_{xy}(t, z, r) \neq 0,\ u^0 = 1,\ T =$ const
e.g. it leads from Christoffel symbols in the covariant derivatives to linear approximation:

$$\sigma^{xy} = \sigma_{xy} \approx \frac{1}{2}(\Gamma^y_{x0} + \Gamma^x_{y0}) \approx \frac{1}{2}\partial_t h_{xy}$$

and for the stress tensor

$$\delta\Pi^{xy} \approx -\eta\partial_t h_{xy} + \eta\tau_\pi\partial_t^2 h_{xy} - \frac{\kappa}{2}[\partial_t^2 h_{xy} + \partial_z^2 h_{xy}]$$

Fourier transform $h(t, z) = \exp\left(-i\omega t + ikz\right)\, h(\omega, k)$, etc.

$$\delta\Pi^{xy}(\omega, k) = -G_R^{xy,xy}(\omega, k) h_{xy}(\omega, k)$$

via linear response

# correlators, cont.

retarded Green function:

$$G_R^{xy,xy}(\omega, k) = -i\eta\omega + \eta\tau_\pi\omega^2 - \frac{\kappa}{2}(\omega^2 + k^2) + \ldots.$$

i.e. Kubo formula

$$\eta = \lim \frac{1}{\omega}\, iG_R^{xy,xy}(\omega, \vec{0})|_{\omega=0}$$

$$= \lim \frac{1}{\omega} \int dt d^3x \;\exp{(i\omega t)} < [T_{xy}(t, \vec{x}), T_{xy}(0, \vec{0})] > |_{\omega=0}$$

# quasinormal mode

$G_R$ from gravity action with $AdS_5$ metric:

$$S_{5d} = \frac{N_c^2}{8\pi^2 R^3} \int d^5x \, [\sqrt{-g} R_{5d} + ..] \, ,$$

factor from $\frac{1}{16\pi G_{(10)}} R^5 \Omega_{(5)}, \;\; \Omega_{(5)} = \pi^3$

and "dictionary" $G_{(10)} = 8\pi^6 g_s^2 l_s^8 \; , \;\; g_{YM}^2 N_c = 4\pi g_s N_c = R^4/l_s^4$

scalar mode and EoM (of Heun's type)

$$\sqrt{-g} R_{(5d)} \to -\frac{1}{2}\sqrt{-g} g^{\mu\nu} \; \partial_\mu \Phi \partial_\nu \Phi \; , \;\; \Phi \equiv h_{xy} \;\; :$$

$$\Phi(t, z, u = \tfrac{r_0^2}{r^2}) = \exp\left(-i\omega t + ikz\right)\Phi_k(u) \; , \; \Phi_k' = \tfrac{d}{du}\Phi_k \; ,$$

$$\Phi_k'' - \frac{1+u^2}{u(1-u^2)}\Phi_k' + \frac{1}{4\pi^2 T^2} \, \frac{\omega^2 - k^2(1-u^2)}{u(1-u^2)^2}\Phi_k(u) = 0$$

# quasinormal mode, cont.

solution with ansatz $\Phi_k(u) = f_k(u)\Phi_0(k)\ ,\ f_k(0) = 1$ and boundary condition: only the incoming wave, the one which moves toward the horizon at $r- > r_0 \equiv u = 1$, i.e. nothing comes out the horizon

$$f_k(u) = (1-u^2)^{-i\omega/(4\pi T)} F_k(u) \approx 1 + i\frac{\omega}{4\pi T}u^2 + ...$$

note, near $u \sim 1$:

$$\exp\left(-i\omega t\right) f_k(u) \to \exp\left[-i\omega(t + r^*)\right]\ ,\ r^* = \frac{\ln(1-u)}{4\pi T}$$

moves from $u = 0$ at $t = 0$ to $u = 1$ at $t- > \infty$

# classical gravity action

inserting the solution into $S_{(5d)}$

$$S_{(5d)} \approx \frac{N_c^2}{8\pi^2 R^3} \int d^4x \int_0^1 \; du \, \sqrt{-g}(-\frac{1}{2}) g^{uu} \partial_u \Phi \partial_u \Phi$$

and keeping only the surface contribution at $u = 0$, $\Rightarrow$

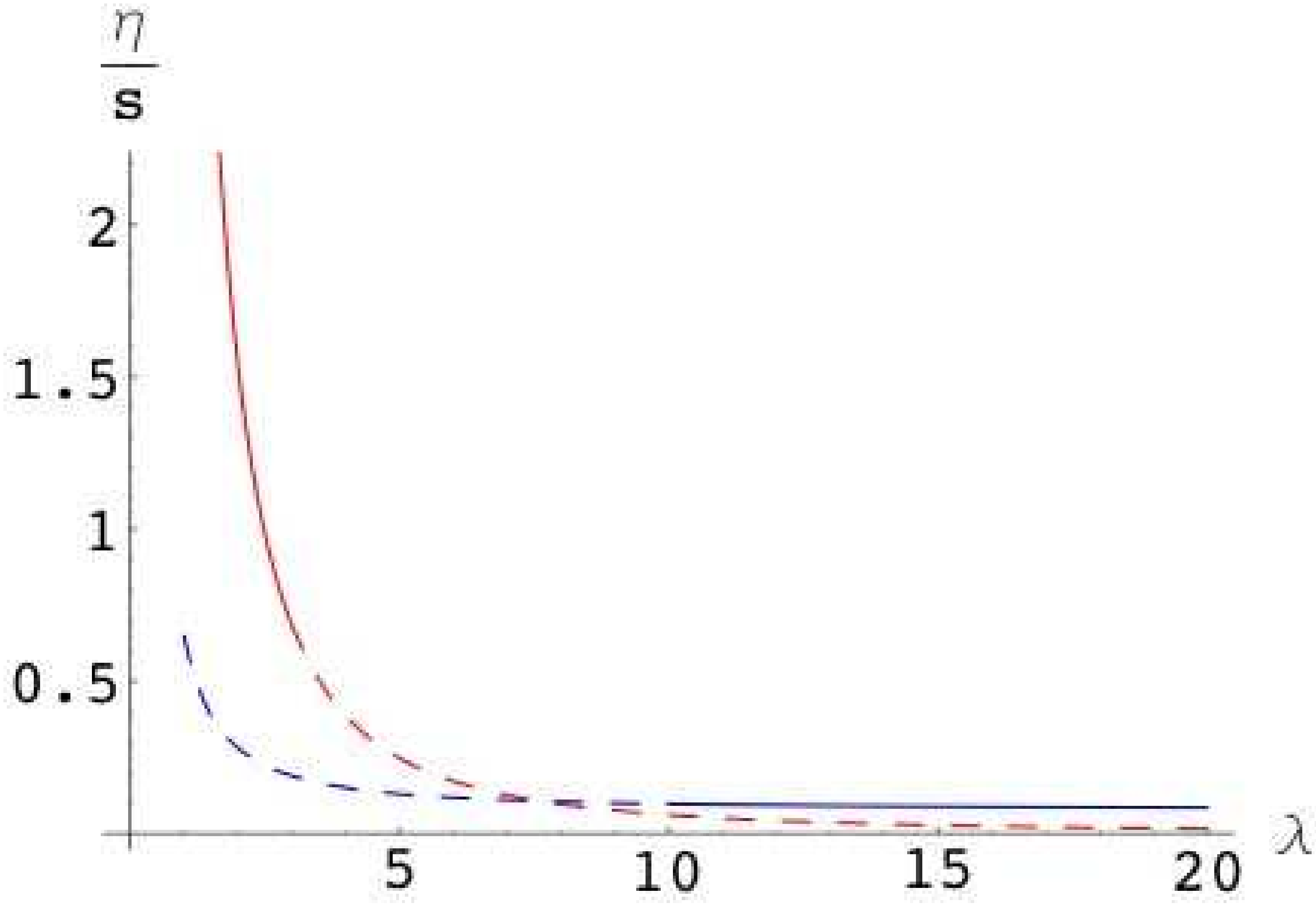
$$S^{classical} = -\frac{\pi^2 N_c^2 T^4}{8} \int \frac{d^4 k}{(2\pi)^4} \, \Phi_0(-k) [\frac{(1-u^2)}{u} f_k'(u) f_k(u)]|_{u=0} \Phi_0(k)$$

and finally

$$G_R^{xy,xy}(\omega, k) = \frac{\pi N_c^2 T^3}{8} [-i\omega + \frac{1 - 2\ln 2}{2\pi T} \, \omega^2 - \frac{1}{2\pi T} \, k^2 \pm \ldots\ldots] \, ,$$

$$\eta = \frac{\pi N_c^2 T^3}{8} \, , \; \tau_\pi = \frac{2 - \ln 2}{2\pi T} \, , \; \kappa = \frac{\eta}{\pi T}$$

# $\eta/s$




behaviour of $\eta/s$ as a function of the 't Hooft coupling

# shear viscosity $\eta$

near equilibrium: $\eta \ \simeq \ \epsilon\bar{v}\lambda_f, \quad entropy\ density \ \ s \simeq \epsilon/m \quad \rightarrow$

$$\frac{\eta}{s} \simeq m\bar{v}\lambda_f \simeq \hbar\,\frac{mean\ free\ path}{deBroglie\ wavelength}$$

- dilute system (QFT $->$ kinetic theory $->$ hydro):
  scale $\lambda_f \ \rightarrow \frac{\eta}{s} >> \hbar$, e.g. pQCD $(N_f = 0)$

$$\frac{\eta}{s} \simeq 3.8\frac{1}{g^4 \ln(2.8/g)} \simeq O(1) \ \ for\ g = 2.5$$

BUT with $\ln(2.8/g) \simeq O(1) : \frac{\eta}{s} \simeq 0.1 \quad \rightarrow$ sensitive to constant under the log !

[Arnold, Moore and Yaffe 03]

- strongly coupled system (QFT $->$ hydro):
  scale $1/T \ \rightarrow \frac{\eta}{s} = \frac{\hbar}{4\pi} \simeq 0.08$

[Policastro, Son and Starinets 01]

# sound mode

consistency requirement from sound channel:

$$\omega_{1,2} = \pm c_s k - i\Gamma\, k^2 \pm \frac{\Gamma}{c_s}(c_s^2\tau_\pi - \frac{\Gamma}{2})\, k^3 + \mathcal{O}(k^3)\ ,$$

with $\Gamma = \frac{2}{3}\frac{\eta}{\epsilon+p}$

from gravity perturbation $g_{tz} = h_{tz} << 1$

$$\omega_{1,2} \pm \frac{1}{\sqrt{3}}\, k - i\frac{1}{6\pi T}\, k^2 \pm \frac{3 - 2\ln 2}{24\pi^2 T^2\sqrt{3}}\, k^3$$

gives sound velocity $c_s = \frac{1}{\sqrt{3}}$ and

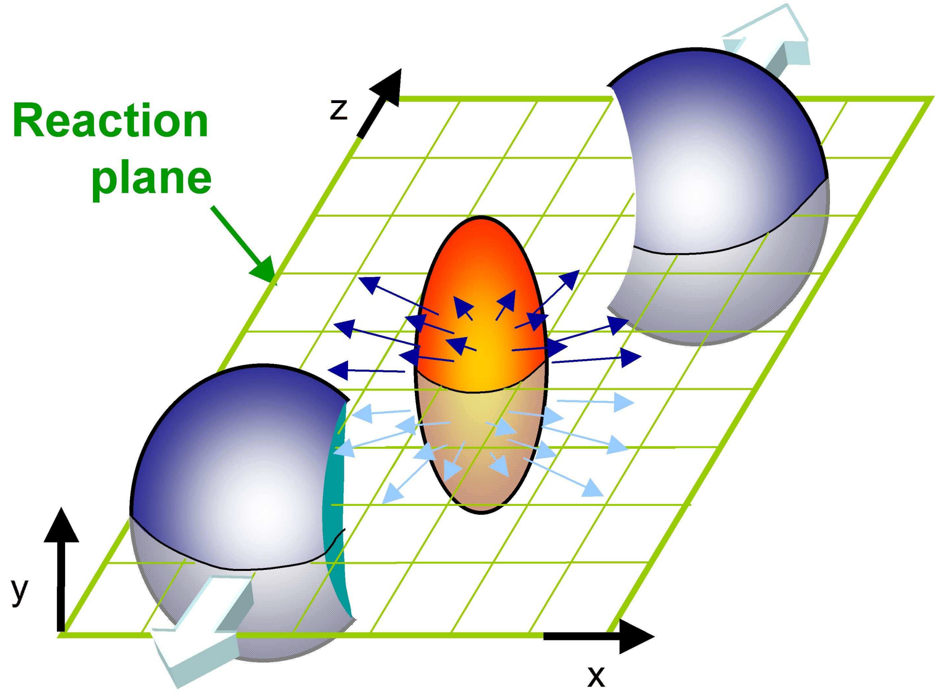
$$\frac{\eta}{s} = \frac{1}{4\pi}\ ,\ \ \tau_\pi = \frac{2 - \ln 2}{2\pi T}$$

# hydro + RHIC

heavy-ion collisions require beyond hydrodynamics:

- hydrodynamics = differential equations
  initial conditions !
- initial = equilibration time
- distribution of energy density (Glauber? CGC ?)
- QCD equation of state
- hadronisation prescription (Cooper-Frye?)

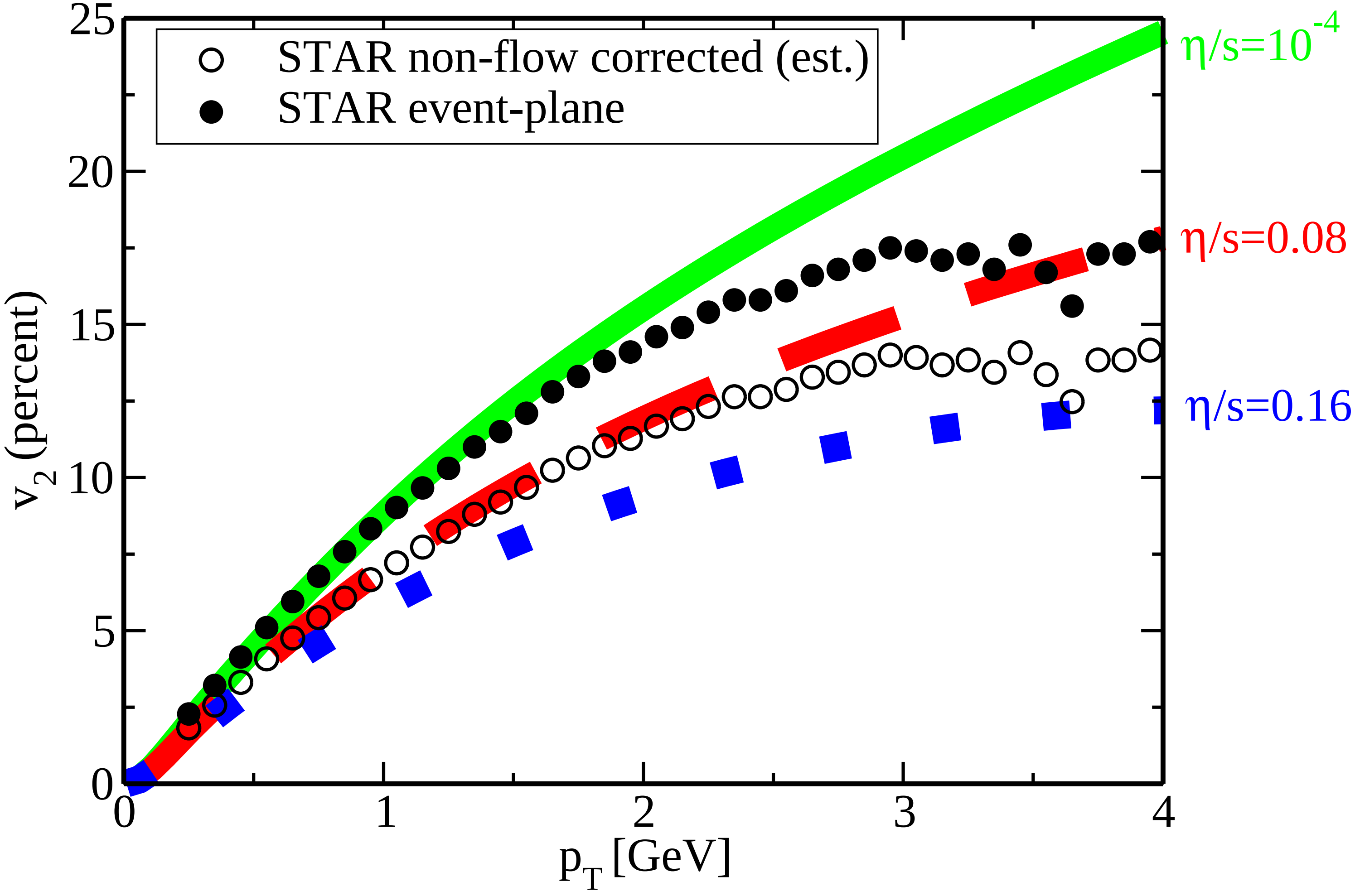
# as important example




spatial asymmetry

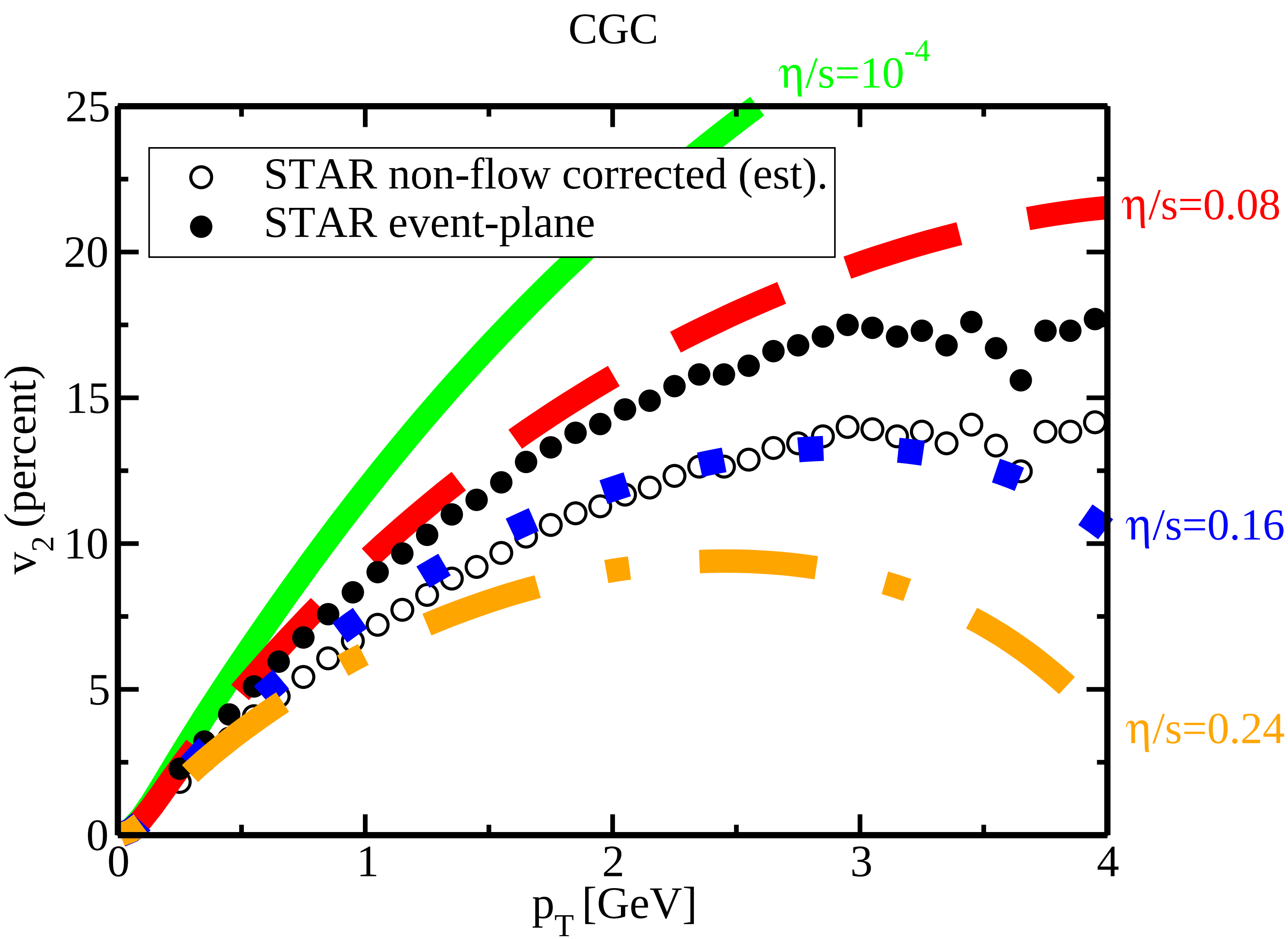
- $$\frac{dN}{dy dp_\perp d\phi} = \langle \frac{dN}{dy dp_\perp d\phi} \rangle_\phi \left(1 + 2 v_2(p_\perp) \cos(2\phi) + \ldots \right)$$
- elliptic flow: $v_2(p_\perp)$

Glauber




elliptic flow

[Luzum and Romatschke 08]




elliptic flow

[Luzum and Romatschke 08]

## main results [Luzum and Romatschke 08]

- viscous hydrodynamics simulation give a good description of RHIC data for

$$\frac{\eta}{s} = 0.1 \pm 0.1(theory) \pm 0.08(experiment)$$

- modest estimate:

$$\frac{\eta}{s} < 0.5$$

- weak dependence on the values of the second-order parameters $\tau_\pi, \lambda_1, ..$

- early thermalisation time is questioned, but

$$\tau_0 < 2\ fm$$

# transport coefficients

kinetic QCD theory (weak coupling):

$$\frac{\eta/s}{\tau_\pi T} \simeq 0.17{-}0.2,\ \lambda_1 \simeq -(2.0{-}2.2)\frac{\eta^2}{sT},\ \lambda_2 = -2\tau_\pi \eta,\ \lambda_3 = \kappa = 0$$

[York and Moore 08]

finite 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c, \quad \lambda >> 1$ corrections to coefficients by gauge/gravity duality, e.g.:

[Buchel and Paulos 08]

$$\frac{\eta/s}{\tau_\pi T} = 0.383\,(1 - 3.52\,\lambda^{-3/2} + ...),$$

$$\kappa = \frac{\eta}{\pi T}\,(1 - \frac{145\zeta(3)}{8}\lambda^{-3/2} + ...)$$

excitement about gauge/gravity correspondence:

mainly to gain intuition into STRONG COUPLING

ASKING FOR MORE:

Is there an experiment whose outcome could cast strong doubts on the relevance of AdS/CFT to understand QCD ?

[P. Jacobs 08]

JET PHYSICS ? [Hatta, Iancu and Mueller 07 - 08]

MOST CHALLENGING TASK of the theory is to find the microscopic mechanism for the rather
RAPID EQUILIBRATION of matter in RHIC collisions

# EXTRAS

## COMPARISON

| | QCD | $\mathcal{N}$=4 SYM |
|---|---|---|
| T=0 | $N_c=3=N_f$, confinement, discrete spectrum, scattering, . . . . | $N_c$ large, $N_f/N_c$ small, deconfined, conformal, supersymmetric, . . . . |
| | **very different !!** | |
| $T>T_c$ | strongly-coupled plasma of gluons & fundamental matter<br>deconfined, screening, finite corr. lengths, . . . | strongly-coupled plasma of gluons & adjoint and fundamental matter<br>deconfined, screening, finite corr. lengths, . . . |
| | **very similar !!** | |
| $T>>T_c$ | runs to weak coupling | remains strongly-coupled |
| | **very different !!** | |

QCD and $\mathcal{N} = 4\ SYM$ as a function of temperature

[from Myers and Vazquez 08]

# entropy current

Israel - Stewart 79: $s^\mu = (s - \frac{\tau_\pi}{4\eta T}\Pi_{\alpha\beta}\Pi^{\alpha\beta})u^\mu$

$\Rightarrow$ second law:

$$\nabla_\mu s^\mu = \frac{\Pi_{\alpha\beta}\Pi^{\alpha\beta}}{2\eta T} \geq 0$$

instead in causal viscous and conformal hydrodynamics
a more general current in terms of $u^\mu$ and its derivatives:

$$s^\mu = su^\mu + (\sharp\, \sigma^2 + \sharp\, \Omega^2)u^\mu + O(u\nabla^2 u)$$

with

$$\nabla_\mu s^\mu = \frac{\eta}{2T}\sigma^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{4T}(\kappa - 2\lambda_1)\, \sigma^\mu_\nu \sigma^\nu_\lambda \sigma^\lambda_\mu$$

in $\mathcal{N} = 4$ SYM: $\kappa = 2\lambda_1$

[Loganayagam 08]

# Bjorken flow

boost-invariant (irrotational) $1+1$ flow [Bjorken 83]
second-order equations (proper time $\tau$, $\Phi$... viscous flow):

$$\partial_\tau \epsilon = -\frac{4\epsilon}{3\tau} + \frac{\Phi}{\tau}$$

$$\tau_\pi \partial_\tau \Phi = \frac{4\eta}{3\tau} - \Phi - \frac{4\tau_\pi}{3\tau}\Phi - \frac{\lambda_1}{2\eta^2}\Phi^2$$

non-linear term NOT in MIS theory! [BRSSS 07]

compare with AdS/CFT calculation:

$$\frac{\lambda_1 T}{\eta} = \frac{1}{2\pi}\left[1 + \frac{215\,\zeta(3)}{8}\lambda^{-3/2} + \ldots\right]$$

[Janik, Peschanski, Heller 06; Buchel, Paulos 08]

# Müller - Israel - Stewart theory

keeping esssentially one term in the derivative expansion
up to second order

$$\Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} + 2\eta\,\tau_\pi\,{}^{\langle} D\sigma^{\mu\nu\rangle} \quad , \quad D = u\cdot\nabla$$

remark: does not match with AdS/CFT $\mathcal{N}=4$ SYM

$$sound: \quad \tau_\pi = \frac{2-\ln 2}{2\pi T}$$

$$Bjorken\ flow: \quad \tau_\pi = \frac{1-\ln 2}{2\pi T}$$

$\Rightarrow$ all second-order terms consistent with conformal symmetry have to be included !

# equilibration time

REMINDER:

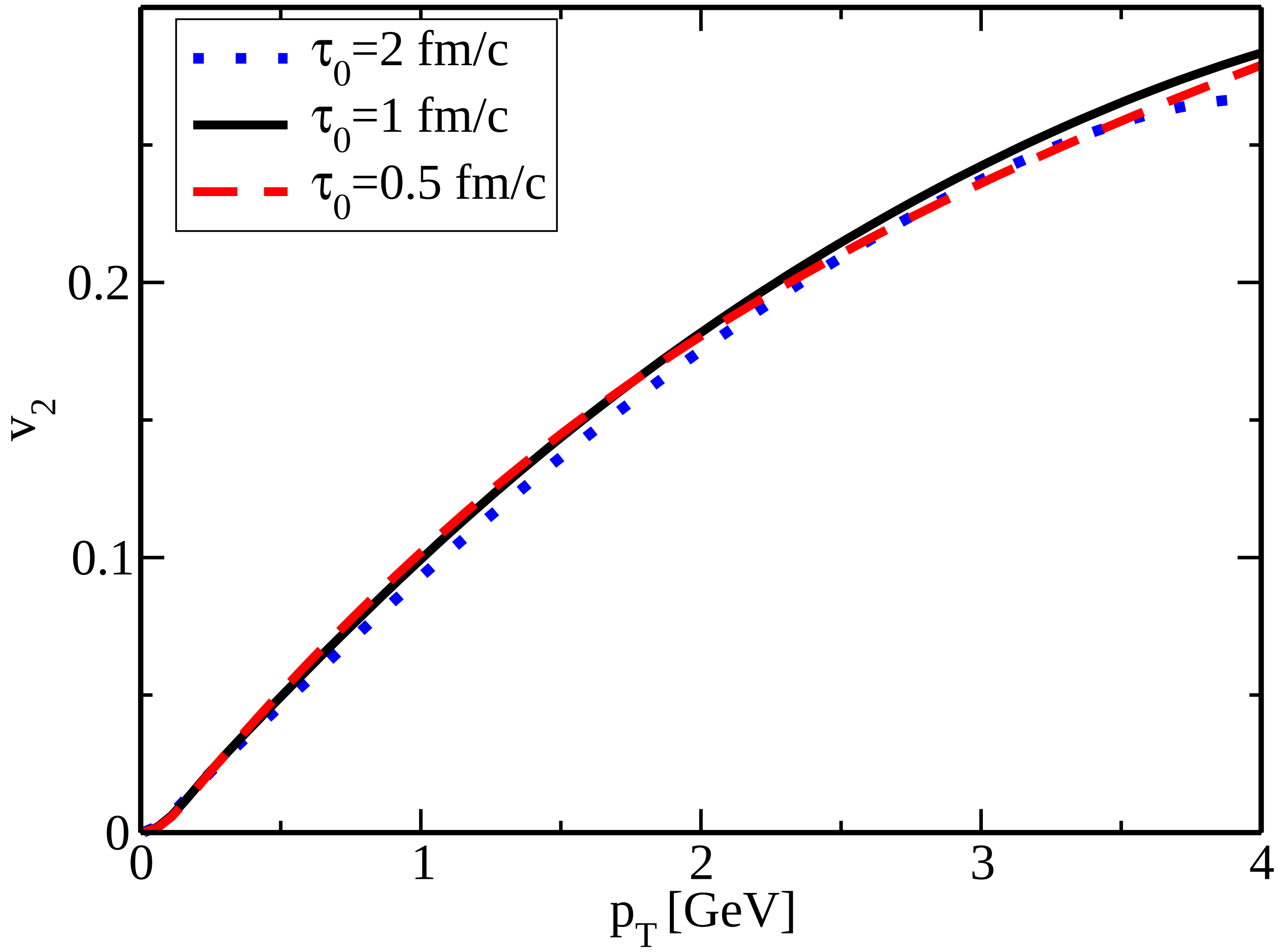
claim of short $\tau_0 \equiv \tau_{eq} \leq 0.5$ fm at RHIC

OPEN QUESTION:

$$CGC\ (\alpha_s << 1)\ \rightarrow\ sQGP\ (\alpha_s > O(1))$$

within a very short time $< 0.5$ fm ?

(b)




early thermalisation ?

[Luzum and Romatschke 08]

# parametric **pQCD estimate**

for thermalisation in an expanding gluonic medium

near equilibrium at $T(\tau)$: Knudsen number *Kn*

$$\frac{1}{Kn} = \frac{longitudinal\ expansion\ time}{mean\ free\ path} = \frac{\tau}{\lambda_f} >> 1$$

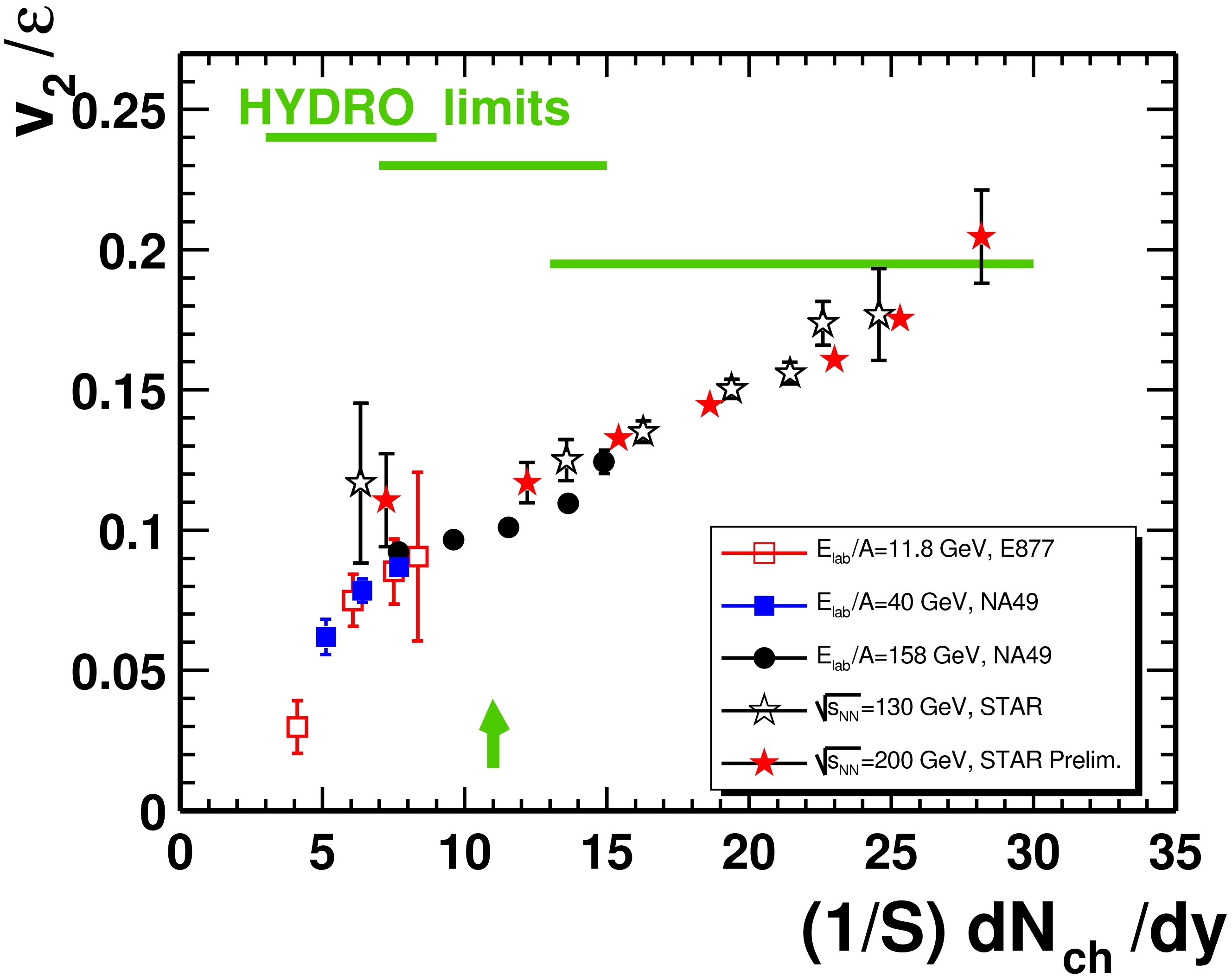
from many gluon interactions (including saturation):

Arnold et al.: $\tau_{eq} Q_s \geq {\alpha_s}^{-7/3}$

'bottom-up' [Baier, Son, Mueller and Schiff 01]

$$\tau_{eq} Q_s \geq \alpha_s^{-13/5}$$

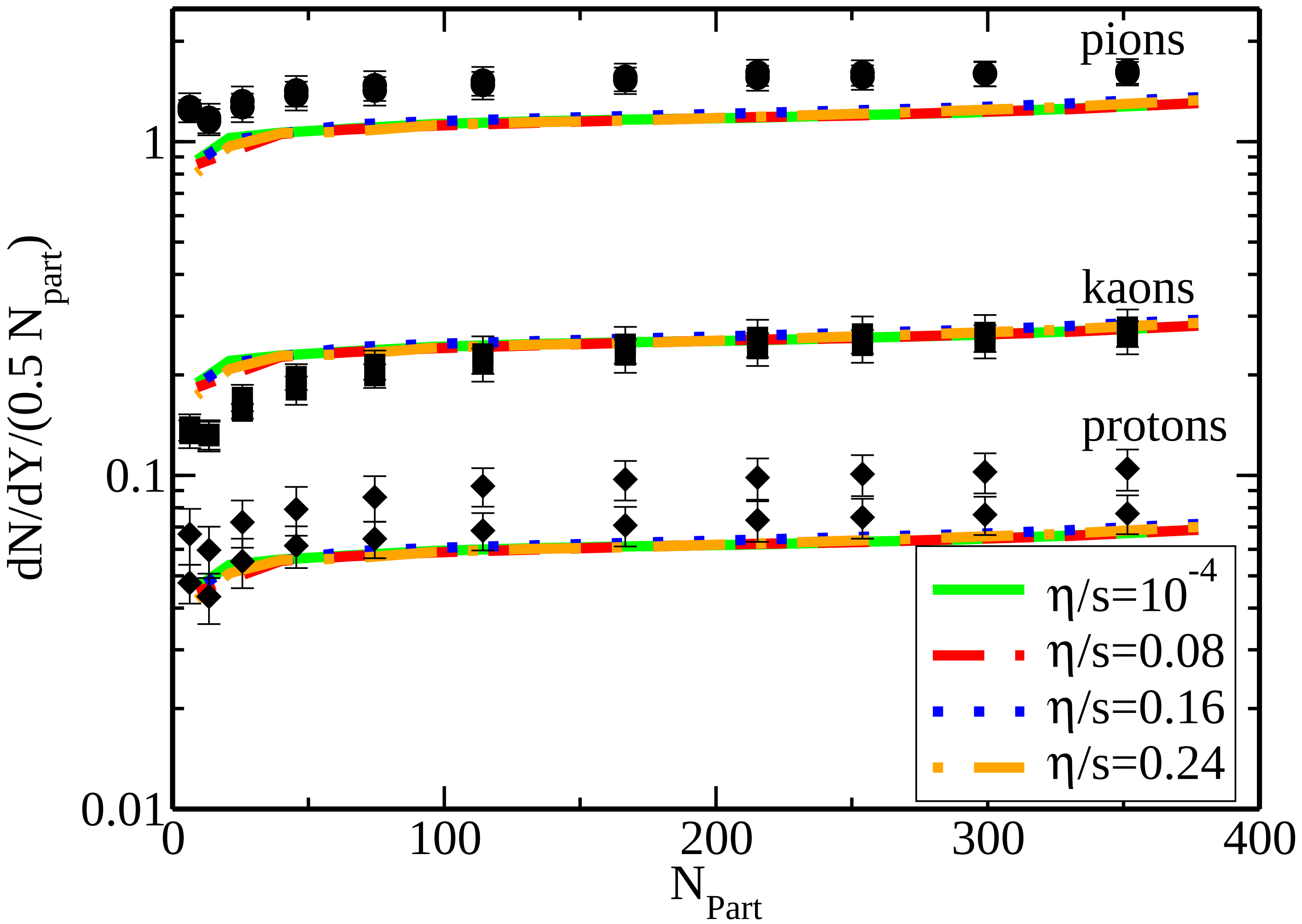
RHIC: $\tau_{eq} \geq 2 - 3$ fm




$v_2$ : experiment vs. perfect hydro

[Heinz 04]

CGC




multiplicity

[Luzum and Romatschke 08]