Conformal Field Theories in more than Two Dimensions

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February 2016

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It is difficult to overstate the importance of conformal field theories (CFTs)

Fitzpatrick, Kaplan, Khanker
Poland, Simmons-Duffin
2014

- Pre Modern, where did it start
- Contemporary Modern
    1. When does conformal symmetry arise?
    2. Conformal Kinematics, Null Cone
    3. Quantum Fields
    4. Three point functions, Operator Product Expansion
    5. Four point functions
    6. Bootstrap
    7. Superconformal
    8. Minkowski Space Methods
- Post Modern, where might it go

THE PRINCIPLE OF RELATIVITY IN ELECTRODYNAMICS AND AN EXTENSION THEREOF

By E. Cunningham.

[Received May 1st, 1909.]*

THE TRANSFORMATION OF THE ELECTRODYNAMICAL EQUATIONS

By H. Bateman.

[Received March 7th, 1909.—Read March 11th, 1909.—Received, in revised form, July 22nd, 1909.]

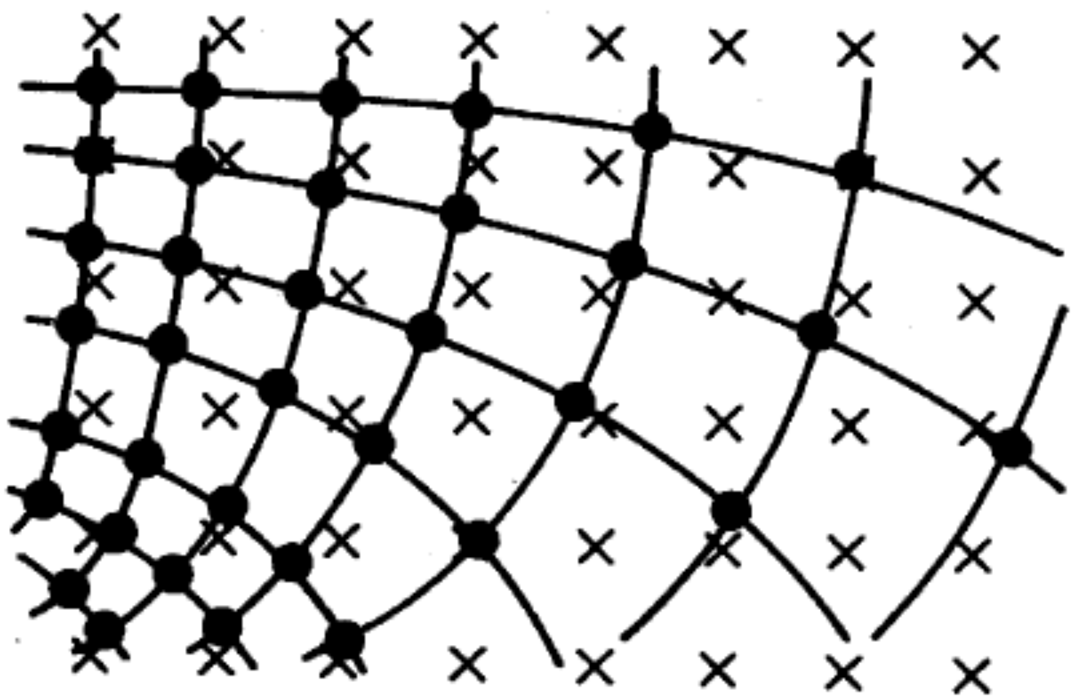
Annals of Mathematics
Vol. 37, No. 2, April, 1936

WAVE EQUATIONS IN CONFORMAL SPACE

By P. A. M. Dirac

(Received May 18, 1935)

What is Conformal Symmetry,
Conformal transformations preserve angles



$x \to x'$ $\qquad dx'^2 = \Omega(x)^2 dx^2$

Two dimensions are different

$$\delta x^\mu = v^\mu(x) \qquad \partial_\mu v_\nu + \partial_\nu v_\mu = 2\,\sigma\,\eta_{\mu\nu}$$

$$v_\mu(x) = a_\mu - \omega_{\mu\nu}x^\nu + \kappa\, x_\mu + b_\mu\, x^2 - 2\, x_\mu\, b_\nu x^\nu\,, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

translation Lorentz scale special conformal

v^μ is a conformal Killing vector $\frac{1}{2}(d+1)(d+2)$ parameters

Conformal field theories are obtained by RG flow to non trivial IR limit where the beta functions vanish

For gauge theories this is expected for a restricted range of flavours defining the conformal window

QCD

$$SU(2) \qquad 3-6 \lesssim N_f \leq 10$$
$$SU(3) \qquad 7-9 \lesssim N_f \leq 16$$

The lower bound is a strong coupling problem depending on lattice calculations

For large $N_f, N_c,\ N_f = \frac{11}{2}N_c - \epsilon$ there is the weakly coupled Banks Zaks fixed point

In $\mathcal{N}=1$SQCD by Seiberg duality conformal window is

$$\frac{3}{2}N_c < N_f < 3N_c$$

weakly coupled magnetic theory   weakly coupled electric theory

$\mathcal{N}=4$ conformal for any g_{YM}

Conserved current for any conformal Killing vector

$$J_v^\mu = T^{\mu\nu} v_\nu \quad \text{if} \qquad \partial_\mu T^{\mu\nu} = 0, \quad \eta_{\mu\nu} T^{\mu\nu} = 0$$

In general in a QFT expect

$$\eta_{\mu\nu} T^{\mu\nu} = \beta^I \mathcal{O}_I + \partial_\mu J^\mu$$

If $\qquad \beta^I = 0$

there is a conserved current for scale transformations

$$S^\mu = T^{\mu\nu} x_\nu - J^\mu, \quad \partial_\mu S^\mu = 0$$

but conformal invariance is broken

Ways out: $\quad J^\mu = \partial_\nu L^{\mu\nu} \quad$ can redefine $T^{\mu\nu}$ to make it traceless

or if $\quad \partial_\mu J^\mu = r^I \mathcal{O}_I \quad$ then get a CFT if $\quad B^I = \beta^I + r^I = 0$

This reflects potential arbitrariness in the definition of beta functions beyond a choice of scheme

Many authors have discussed whether there are scale but not conformal invariant theories

Polchinksi
Riva
Cardy
Dorigoni
Rychkov
Fortin
Grinstein
Stergiou
Nakayama
Jackiw
Pi
El-Showk
Dymarsky
Komargodski
Schwimmer
Theisen

It is very likely that for unitary theories scale invariance does imply conformal symmetry

For non unitary theories there are counterexamples

In two dimensions it is a theorem

Conformal transformations are nonlinear

$$(x-y)^2 \to \Omega(x)\Omega(y)\,(x-y)^2$$

Need 4 points to construct an invariant

It is often simpler to use homogeneous coords

$$X^\mu, X^d, X^{d+1} \qquad X \sim \lambda X$$

null projective cone (Dirac)

$$-\tfrac{1}{2} X \cdot X = \eta_{\mu\nu} X^\mu X^\nu + (X^d)^2 - (X^{d+1})^2 = 0$$

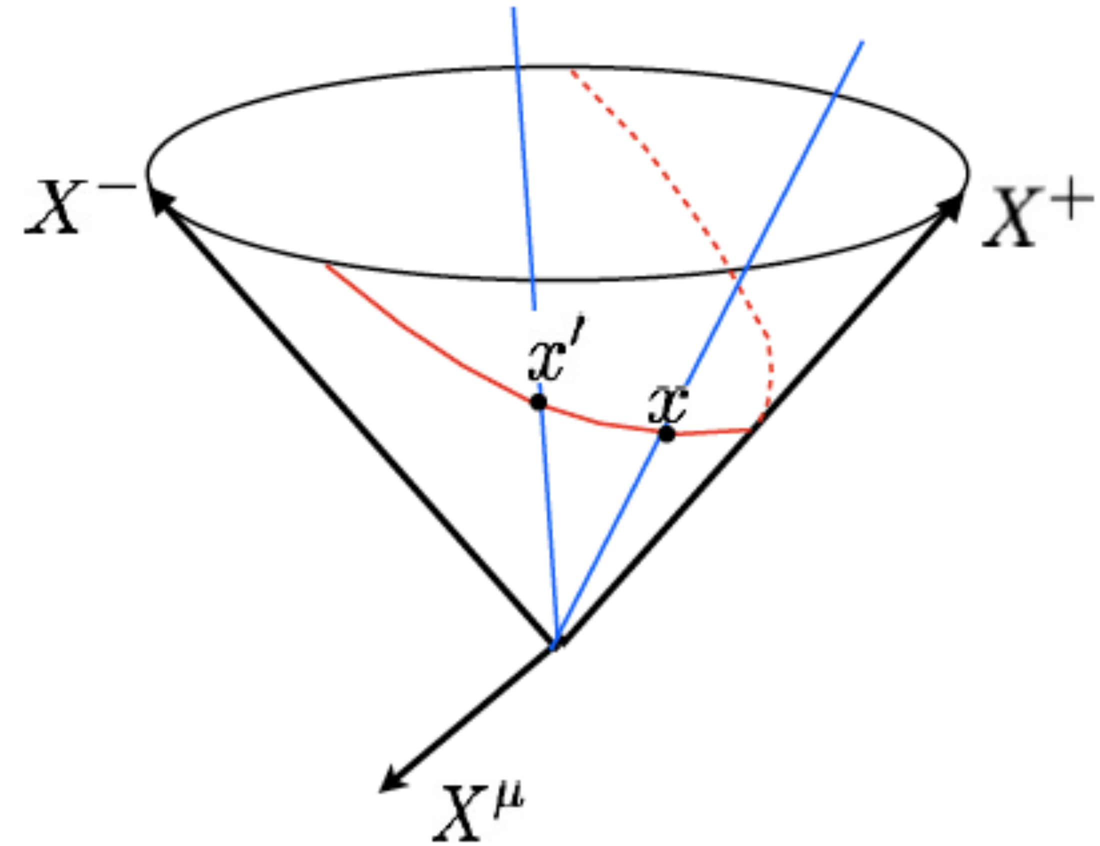
$$x^\mu = \frac{X^\mu}{X^+}, \quad X^+ = X^d + X^{d+1}, \quad (x-y)^2 = \frac{X \cdot Y}{X^+ Y^+}$$

Conformal group defined by linear transformations preserving the null cone $\quad SO(d,2) \quad SO(d+1,1)$

AdS_{d+1} may be defined by

$X \cdot X = -1$

and so has the conformal group as its isometry group



The boundary of AdS_{d+1} is the projective null cone

This is just the tip of the AdS/CFT correspondence

The inversion operation plays a crucial role in CFTs

$$x^\mu \to x^\mu / x^2, \quad X^{d+1} \to -X^{d+1}$$

like parity, parity with inversion part of connected conformal group

Algebra and Fields

Conformal Generators $M_{AB} = -M_{BA}$ $\qquad \mathfrak{so}(d+1,1)$ or $\mathfrak{so}(d,2)$

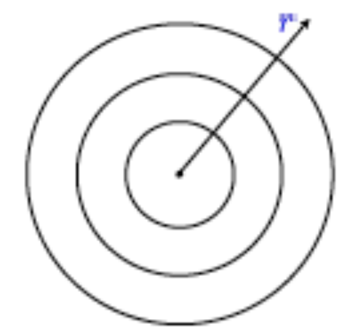
contain $\quad P_\mu \quad M_{\mu\nu} \quad D \quad K_\mu$

translation Lorentz scale special conformal

$$[D, P_\mu] = P_\mu \qquad [D, K_\mu] = -K_\mu$$

$$[M_{AB}, M_{CD}] = \eta_{AC}\, M_{BD} - \eta_{BC}\, M_{AD} - \eta_{AD}\, M_{BC} + \eta_{BD}\, M_{AC}$$

Radial Quantisation, treat D
as the Hamiltonian evolving in $\tau = \log|x|$



Fields are labelled by D eigenvalue Δ and spin

adjoint
$$\phi(x) = \Phi(X), \qquad \Phi(\lambda X) = \lambda^{-\Delta}\Phi(X)$$
$$\bar{\phi}(x) = (x^2)^{-\Delta}\phi(x/x^2)$$

states $|\phi\rangle = \phi(0)|0\rangle$ $\langle\phi| = \langle 0|\bar{\phi}(0)$ correspond to fields

Conformal primary $K_\mu|\phi\rangle = 0$ $D|\phi\rangle = \Delta|\phi\rangle$

$\langle\phi|P_\mu = 0$ $P_\mu{}^\dagger = K_\mu$

Require the fields generate unitary positive energy representations for a unitary CFT

D has positive eigenvalues, zero on the vacuum

Descendants generated by action of momentum operators $\prod(P_\mu)^{n_\mu}|\phi\rangle$ $n_\mu = 0, 1, 2, \ldots$

$K_\mu|\phi\rangle = 0$ ensures Δ is bounded below, positive energy

For fields with spin

$$\phi_{\mu_1\ldots\mu_\ell}(x)$$

$$\Phi_{A_1\ldots A_\ell}(X), \quad X^2=0$$

$$X^{A_r}\Phi_{A_1\ldots A_\ell}(X)=0, \quad r=1\ldots\ell$$

$$\Phi_{A_1\ldots A_\ell}(X)\sim\Phi_{A_1\ldots A_\ell}(X)+X_{A_r}\,\Psi_{A_1\ldots\hat{A}_r\ldots A_\ell}(X), \quad r=1\ldots\ell$$

$$\bar\phi^{\mu_1\cdots\mu_\ell}(x)=(x^2)^{-\Delta}\prod\nolimits_{r=1\ldots\ell}I^{\mu_r\nu_r}(x)\,\phi_{\nu_1\ldots\nu_\ell}(x/x^2)$$

$$I^{\mu\nu}(x)=\eta^{\mu\nu}-2\,\frac{x^\mu x^\nu}{x^2}$$

Inversion tensor

$$|\phi_{\mu_1\ldots\mu_\ell}\rangle=\phi_{\mu_1\ldots\mu_\ell}(0)|0\rangle$$

$$\langle\bar\phi^{\mu_1\cdots\mu_\ell}|=\langle 0|\bar\phi^{\mu_1\cdots\mu_\ell}(0)$$

For unitarity there are constraints on Δ and the spin

$$\big[K_\mu , P_\nu\big] = 2\,\delta_{\mu\nu}\,D + 2\,M_{\mu\nu}$$

$$\langle\phi|K_\mu\,P_\nu|\phi\rangle = 2\,\langle\phi|(M_{\mu\nu} + \delta_{\mu\nu}\,D)|\phi\rangle = 2\,\delta_{\mu\nu}\,\Delta\,.$$

$$K_\mu = {P_\mu}^\dagger \quad \Rightarrow \quad \Delta \ge 0$$

$$\Delta = 0 \quad \Rightarrow \quad P_\mu|\phi\rangle = 0$$

At the next level

$$\langle\phi|K_\sigma K_\rho\,P_\mu P_\nu|\phi\rangle = 4\Delta\big((\Delta+1)(\delta_{\sigma\mu}\,\delta_{\rho\nu} + \delta_{\sigma\nu}\,\delta_{\rho\mu}) - \delta_{\sigma\rho}\,\delta_{\mu\nu}\big)$$

$$\Delta \ge \tfrac{1}{2}(d-2) \qquad \Delta = \tfrac{1}{2}(d-2) \quad \Rightarrow \quad P^2|\phi\rangle = 0$$

For symmetric tensor fields of rank ℓ $\quad \phi_{\mu_1\dots\mu_\ell}$

$$\Delta \ge \ell + d - 2 \quad \ell = 1, 2, \dots$$

Mack
Dobrev

Equality requires $P_\mu|\phi_{\mu\mu_1\dots\mu_{\ell-1}}\rangle = 0$ conserved current

Two point functions define the normalisation of the fields

$$\langle \phi(x)\, \phi(y) \rangle = \frac{1}{(x-y)^{2\Delta}} \quad \Rightarrow \quad \langle \phi | \phi \rangle = 1$$

$$\langle \Phi(X)\, \Phi(Y) \rangle = \frac{1}{(X \cdot Y)^{\Delta}}$$

Generalisations to spin are straightforward and involve the inversion tensor

Three point functions for primary operators in CFTs are determined up to a finite number of coefficients

Three points can be mapped to any three points by a conformal transformation

Scalars

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = \frac{c_{123}}{{x_{12}}^{2\delta_3}\,{x_{23}}^{2\delta_1}\,{x_{31}}^{2\delta_2}}$$

$$x_{12} = x_1 - x_2 \qquad \delta_3 = \tfrac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3)$$

$$\langle \phi_1 | \phi_2(x) | \phi_3 \rangle \Big|_{|x|=1} = c_{123}$$

Generalisations to spin are a linear algebraic problem

Number of independent terms for spins ℓ_1, ℓ_2, ℓ_3 equal to the number of on shell amplitudes in $d+1$ dim Minkowski space

Conserved currents correspond to amplitudes for massless particles

For parity conserving amplitudes number is $\min \ell_i + 1$

Zhiboedov 2012

Hence for the em tensor three point function

$$\langle T_{\mu\nu}(x_1)\, T_{\sigma\rho}(x_2)\, T_{\alpha\beta}(x_3)\rangle$$

there are three independent terms

These correspond to the number of free CFTs in four dimensions, scalars, fermions, vectors

Three point functions for higher spin currents correspond to those of free theories in even dimensions

The spectrum of operators, scale dimensions and spins, and the three point functions determine a CFT through the operator product expansion (OPE)

The product of any two conformal primary fields is given by an expansion in terms of conformal primaries and their descendants

The expansion is *convergent* **and is determined by reproducing the three point functions**

$$\phi(x)\phi(0) = \sum_{\mathcal{O}} c_{\phi\phi\mathcal{O}} \frac{1}{(x^2)^{\frac{1}{2}(2\Delta_\phi - \Delta + \ell)}} C_{\Delta,\ell}(x,\partial)^{\mu_1\dots\mu_\ell} \mathcal{O}_{\mu_1\dots\mu_\ell}(0)$$

$$C_{\Delta,\ell}(x,0)^{\mu_1\dots\mu_\ell} = x^{\mu_1}\dots x^{\mu_\ell}$$

for two scalars $\mathcal{O}$ symmetric traceless tensor scale dimension Δ $\phi\times\phi = \sum_{\mathcal{O}} \mathcal{O}_{\Delta,\ell}$

The OPE applied to the four point functions gives non trivial constraints on the spectrum of operators

$$\langle \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) \rangle = \frac{1}{({x_{12}}^2 {x_{34}}^2)^{\Delta_\phi}}\, F(u,v)$$

$$u = \frac{{x_{12}}^2 {x_{34}}^2}{{x_{13}}^2 {x_{24}}^2} \qquad v = \frac{{x_{14}}^2 {x_{23}}^2}{{x_{13}}^2 {x_{24}}^2}$$

two conformal invariants

Crossing symmetry

$$F(u,v) = F(u/v, 1/v) \qquad 1 \leftrightarrow 2$$

$$= \left(\frac{u}{v}\right)^{\Delta_\phi} F(v,u) \qquad 2 \leftrightarrow 4$$

The OPE gives

$$\langle\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle$$

$$= \sum_{\mathcal{O}} \frac{{c_{\phi\phi\mathcal{O}}}^2}{({x_{12}}^2{x_{34}}^2)^{\frac{1}{2}(2\Delta_\phi-\Delta+\ell)}}$$

$$\times\, C_{\Delta,\ell}(x_{12},\partial_2)^{\mu_1\ldots\mu_\ell}C_{\Delta,\ell}(x_{34},\partial_4)^{\nu_1\ldots\nu_\ell}\big\langle\mathcal{O}_{\mu_1\ldots\mu_\ell}(x_2)\,\mathcal{O}_{\nu_1\ldots\nu_\ell}(x_4)\big\rangle$$

or

$$F(u,v) = 1 + \sum_{\mathcal{O}} a_{\Delta,\ell}\,G_{\Delta,\ell}(u,v) \qquad a_{\Delta,\ell} = {c_{\phi\phi\mathcal{O}}}^2 > 0$$

identity

conformal partial waves or conformal blocks

converges as

$u \to 0 \quad v \to 1$

This is analogous to the partial wave expansion for S-matrix amplitudes, conformal blocks replaced by single variable Legendre polynomials or their generalisations

The large order behaviour of $a_{\Delta,\ell}$ is constrained by reproducing the behaviour as $v \to 0 \quad u \to 1$

Pappadopulo,Rychkov
Espin, Rattazzi, 2012

In two dimensions for minimal models this can be restricted to a finite sum if the conformal blocks are extended to Virasoro conformal blocks, related to the infinite dimensional Virasoro algebra

New variables $u = z\bar{z}$ $v = (1-z)(1-\bar{z})$

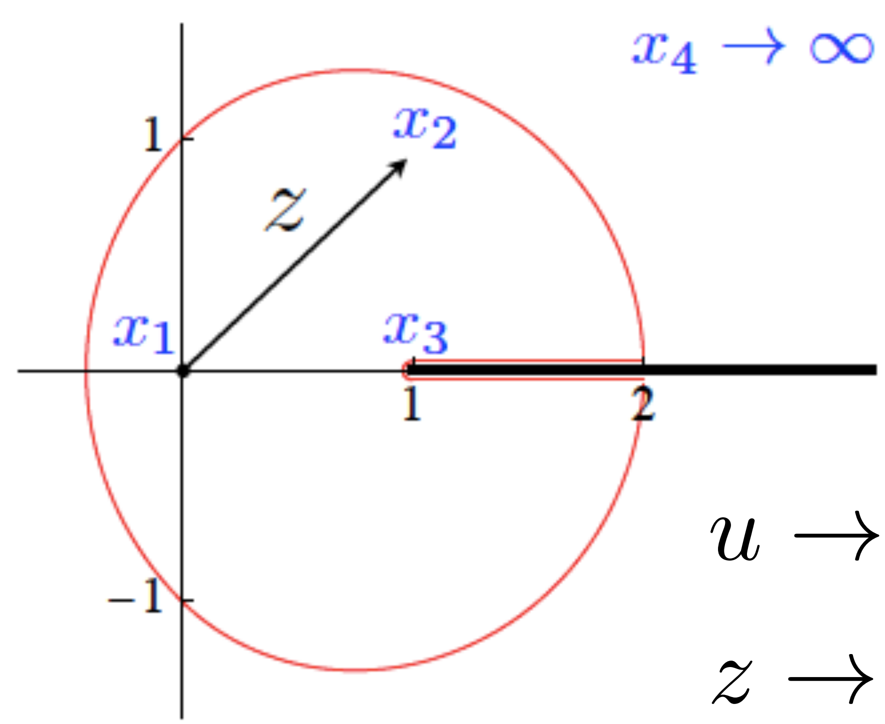
$$F(u,v) = \mathcal{F}(z,\bar{z}) = \mathcal{F}(\bar{z},z)$$

Restrict coords to a plane

$$\langle\phi|\phi(x_3)\,\phi(x_2)|\phi\rangle = \frac{1}{(z\,\bar{z})^{\Delta_\phi}}\,\mathcal{F}(z,\bar{z})$$

$$x_3 = (1,1) \qquad x_2 = (z,\bar{z}) \quad {x_2}^2 = z\bar{z}$$

2 dimensional complex plane



Crossing

$u \leftrightarrow v$

$z \to 1-z$

$u \to u/v,\ \ v \to 1/v$

$z \to z/(z-1)$

Equations for conformal blocks

Conformal blocks are non polynomial and rather non trivial functions

Conformal generators $M_{iAB} \quad i = 1, 2, 3, 4$

$$\big(M_{1AB} + M_{2AB} + M_{3AB} + M_{4AB}\big)\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle = 0$$

The Casimir operator plays a crucial role

$$\tfrac{1}{2} M_{AB} M^{BA}\,\mathcal{O}_{\Delta,\ell} = C_{\Delta,\ell}\,\mathcal{O}_{\Delta,\ell}$$

$$C_{\Delta,\ell} = \Delta(\Delta - d) + \ell(\ell + d - 2)$$

This implies

$$\tfrac{1}{2}\big(M_{1AB} + M_{2AB}\big)\big(M_1{}^{BA} + M_2{}^{BA}\big)\frac{1}{(x_{12}{}^2 x_{34}{}^2)^{\Delta_\phi}}\,G_{\Delta,\ell}$$

$$= C_{\Delta,\ell}\,\frac{1}{(x_{12}{}^2 x_{34}{}^2)^{\Delta_\phi}}\,G_{\Delta,\ell}$$

Conformal blocks are non polynomial and rather non trivial functions

In principle they are solutions of second order and fourth order PDEs

$$\tfrac{1}{2}\, M^{AB} M_{BA}\, \mathcal{O}_{\Delta,\ell} = C_{\Delta,\ell}\, \mathcal{O}_{\Delta,\ell}$$
$$\tfrac{1}{2}\, M^{AB} M_{BC} M^{CD} M_{DA}\, \mathcal{O}_{\Delta,\ell} = D_{\Delta,\ell}\, \mathcal{O}_{\Delta,\ell}$$

imply

$$\mathcal{D}_2\, G_{\Delta,\ell} = C_{\Delta,\ell}\, G_{\Delta,\ell}$$
$$\mathcal{D}_4\, G_{\Delta,\ell} = D_{\Delta,\ell}\, G_{\Delta,\ell}$$

$\mathcal{D}_2,\ \mathcal{D}_4$ 2nd**,** 4th order differential operators

Boundary condition as $u \to 0,\ v \to 1$

$$G_{\Delta,\ell}(u,v) = u^{\frac{1}{2}(\Delta-\ell)}(1-v)^{\ell}\big(1 + \mathrm{O}(u, 1-v)\big)$$

Conformal blocks were discussed by Ferrara, Gatto, Grillo, Parisi in the 1970's who obtained results in particular limits. More general expressions were obtained quite recently but we still lack compact formulae for arbitrary d

In terms of u, v variables

$\mathcal{D}_2, \mathcal{D}_4$ are complicated, simpler in terms of $z, \bar{z}$

$$\mathcal{D}_2 = D_z + D_{\bar{z}} + (d-2)\,\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\frac{\partial}{\partial z} - (1-\bar{z})\frac{\partial}{\partial \bar{z}}\right)$$

$$D_z = (1-z)z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2} - z^2\frac{\mathrm{d}}{\mathrm{d}z}$$

For $d=2$ the equation separates

$$D_z\, g_\lambda(z) = \lambda(\lambda-1)\, g_\lambda(z) \qquad g_\lambda(z) = z^\lambda F(\lambda,\lambda;2\lambda;z)$$

just a hypergeometric function

$$G_{\Delta,\ell}(u,v) = \mathcal{G}_{\Delta,\ell}(z,\bar{z})$$
$$= g_{\frac{1}{2}(\Delta+\ell)}(z)\, g_{\frac{1}{2}(\Delta-\ell)}(\bar{z}) + g_{\frac{1}{2}(\Delta+\ell)}(\bar{z})\, g_{\frac{1}{2}(\Delta-\ell)}(z)$$

symmetry under $z \leftrightarrow \bar{z}$ essential as

$u = z\bar{z},\ v = (1-z)(1-\bar{z})$ are symmetric

In four dimensions there is also a nice solution in terms of the same single variable hypergeometric functions since

$$\mathcal{D}_2 = \frac{z\bar{z}}{z-\bar{z}}\big(D_z + D_{\bar{z}} - 2\big)\frac{z-\bar{z}}{z\bar{z}}$$

This ensures the four dimensional conformal block is just

$$G_{\Delta,\ell}(u,v) = \mathcal{G}_{\Delta,\ell}(z,\bar{z})$$
$$= \frac{z\bar{z}}{z-\bar{z}}\Big(g_{\frac{1}{2}(\Delta+\ell)}(z)\, g_{\frac{1}{2}(\Delta-\ell)-1}(\bar{z}) - g_{\frac{1}{2}(\Delta+\ell)}(\bar{z})\, g_{\frac{1}{2}(\Delta-\ell)-1}(z)\Big)$$

$\frac{1}{2}(\Delta+\ell)(\frac{1}{2}(\Delta+\ell)-1) + (\frac{1}{2}(\Delta-\ell)-1)(\frac{1}{2}(\Delta-\ell)-2) - 2 = \frac{1}{2}\,C_{\Delta,\ell} \quad d=4$

Generalisations are possible in any even dimension

For general dimensions there are no simple results except in various limits

$G_{\Delta,\ell}$ has poles in Δ arising from singular vectors descendants of $|\mathcal{O}_{\mu_1\dots\mu_\ell}\rangle$

A vector is singular if it is a conformal primary and a descendant

$$P^{\mu_{k+1}}\dots P^{\mu_\ell}|\mathcal{O}_{\mu_1\dots\mu_\ell}\rangle \quad k=0,\dots\ell-1 \quad \Delta_O = d+k-1$$

which is a conformal primary tensor rank

$$\ell_s = k \qquad \Delta_s = d+\ell-1$$

$$\frac{1}{2}(\Delta_s-\ell_s)-\frac{1}{2}(\Delta-\ell)=m=1,\dots,\ell \qquad C_{\Delta_O,\ell}=C_{\Delta_s,\ell_s}$$

e.g. angular momentum

$J_+|j,j\rangle=0,\ J_-{}^{2j+1}|j,j\rangle$ is a singular vector if $2j=0,1,2,\dots$

Two more sets of singular vectors

$$(\Delta_O, \ell) = (\tfrac{1}{2}d - m, \ell) \to (\Delta_s, \ell_s) = (\tfrac{1}{2}d + m, \ell)$$

$$(\Delta_O, \ell) = (1 - \ell - m, \ell) \to (\Delta_s, \ell_s) = (1 - \ell, \ell + m)$$

$$m = 1, 2, \ldots$$

$$\tfrac{1}{2}(\Delta_s - \ell_s) - \tfrac{1}{2}(\Delta - \ell) = m$$

poles of the form $G_{\Delta,\ell} \sim \dfrac{c_s}{\Delta - \Delta_O} G_{\Delta_s,\ell_s}$ $\quad c_s$ calculable

$$H_{\Delta,\ell}(\rho,\bar\rho) = |\rho|^{-\Delta+\ell} G_{\Delta,\ell}(u,v)$$

analytic in Δ well defined at infinity
no branch cuts

$$\rho = \frac{z}{(1+\sqrt{1-z})^2} \quad u = z\bar z \quad v = (1-z)(1-\bar z) \qquad z = \frac{4\rho}{(1+\rho)^2}$$

$$H_{\Delta,\ell}(\rho,\bar\rho) = H_{\infty,\ell}(\rho,\bar\rho) + \sum_{\text{singular vectors}} \frac{c_s}{\Delta - \Delta_O} |\rho|^{2m} H_{\Delta_s,\ell_s}(\rho,\bar\rho)$$

known by solving the Casimir equation for large Δ

iterating gives a rational approximation

Kos
Poland
Simmons-Duffin
following
Zamolodchikov in 2d

Bootstrap !

Baron von Munchausen
1720-1797
fantasist
and hero of
the romantic
age

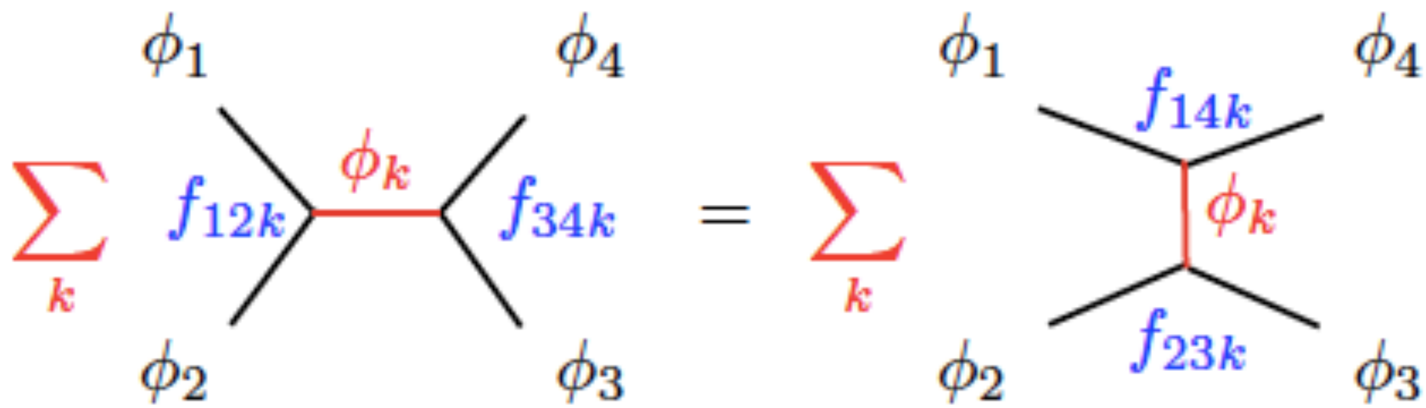
the Baron
escapes
a swamp
by
pulling on
his
hair,
in later
myth
his
bootstraps

as in The Surprising Adventures of Baron von Munchhausen, published in English in 1785

Bootstrap equation

Combining crossing with the conformal block expansion

proposed by Polyakov in the context of CFTs in 1971
revived by Rychkov, Rattazzi, Tonni, Vichi in 2008
mini Milner prize to Rychkov in 2013



$$v^{\Delta_\phi} F(u,v) = u^{\Delta_\phi} F(v,u) \qquad 1 = \sum_{\mathcal{O}} a_{\Delta,\ell}\, F^{\Delta_\phi}_{\Delta,\ell}(z,\bar{z})$$

> 0 known function

$$F^{\Delta_\phi}_{\Delta,\ell}(z,\bar{z}) = \frac{v^{\Delta_\phi} G_{\Delta,\ell}(u,v) - u^{\Delta_\phi} G_{\Delta,\ell}(v,u)}{v^{\Delta_\phi} - u^{\Delta_\phi}}$$

There is a region in the neighbourhood of the crossing symmetric point

$$u = v = \frac{1}{4} \qquad z = \bar{z} = \frac{1}{2} \qquad \rho = \bar{\rho} = 3 - 2\sqrt{2} \approx 0.18$$

where this expansion converges.

Can truncate the expansion by considering a finite Taylor expansion around $z = \bar{z} = \frac{1}{2}$
in powers of $z + \bar{z} - 1, \quad (z - \bar{z})^2$

$$F^{\Delta_\phi}_{\Delta,\ell}(z,\bar{z}) \to F^{\Delta_\phi}_{\Delta,\ell;n,m} = \partial_z^n \partial_{\bar{z}}^m F^{\Delta_\phi}_{\Delta,\ell}(z,\bar{z})\Big|_{z=\bar{z}=\frac{1}{2}}$$

$$\delta_{n0}\delta_{m0} = \sum_{\Delta,\ell} a_{\Delta,\ell}\, F^{\Delta_\phi}_{\Delta,\ell;n,m} \quad a_{\Delta,\ell} \geq 0$$

$$\text{restrict } m \geq n, m+n \text{ even}, \text{ truncate to } m+n \leq N$$

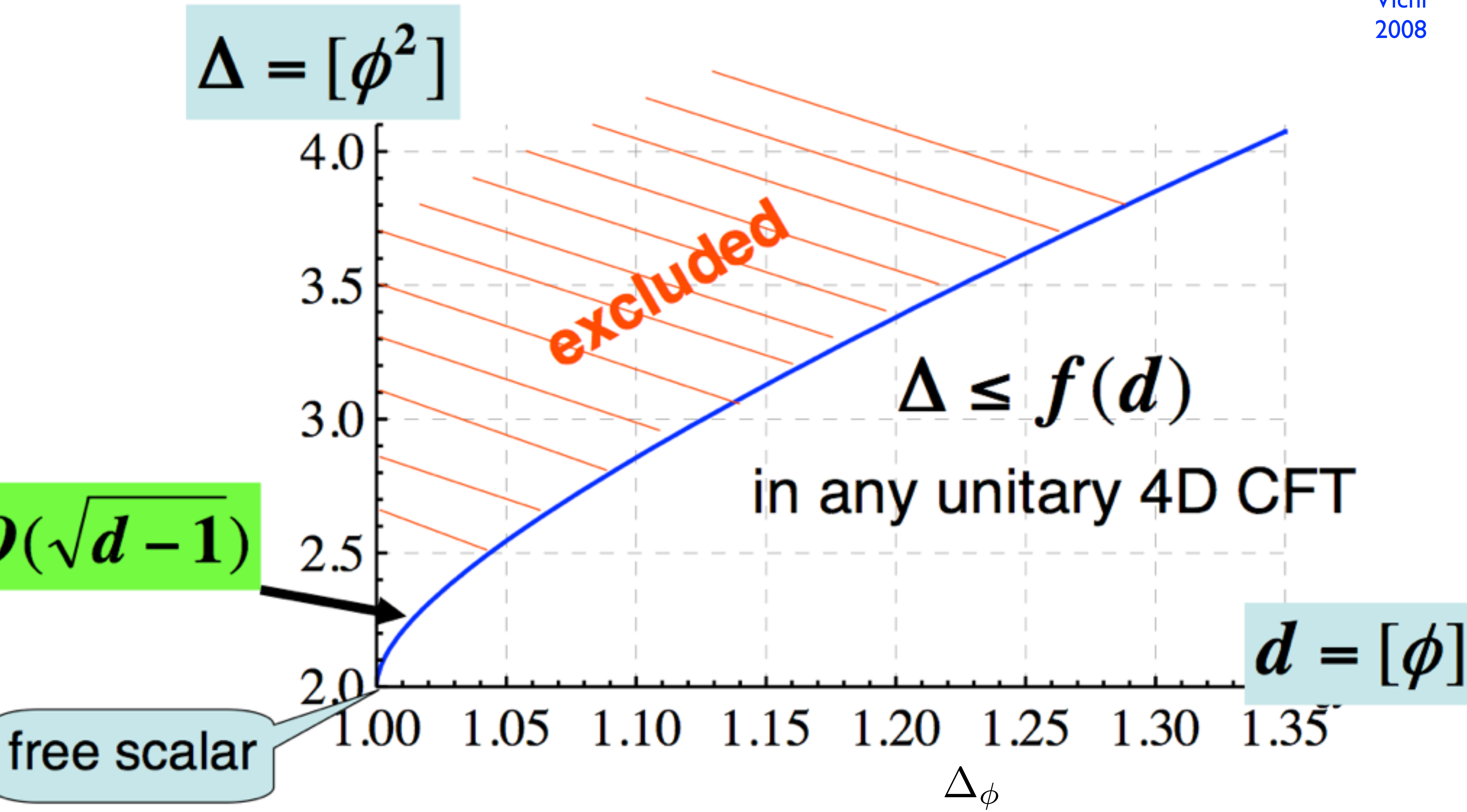
Truncate the spectrum of operators,

These equations do not have solutions unless there are restrictions or bounds on Δ, ℓ

This is a problem in linear programming

Scalar field theory in 4 dimensions

Rattazzi
Rychkov
Tonni
Vichi
2008



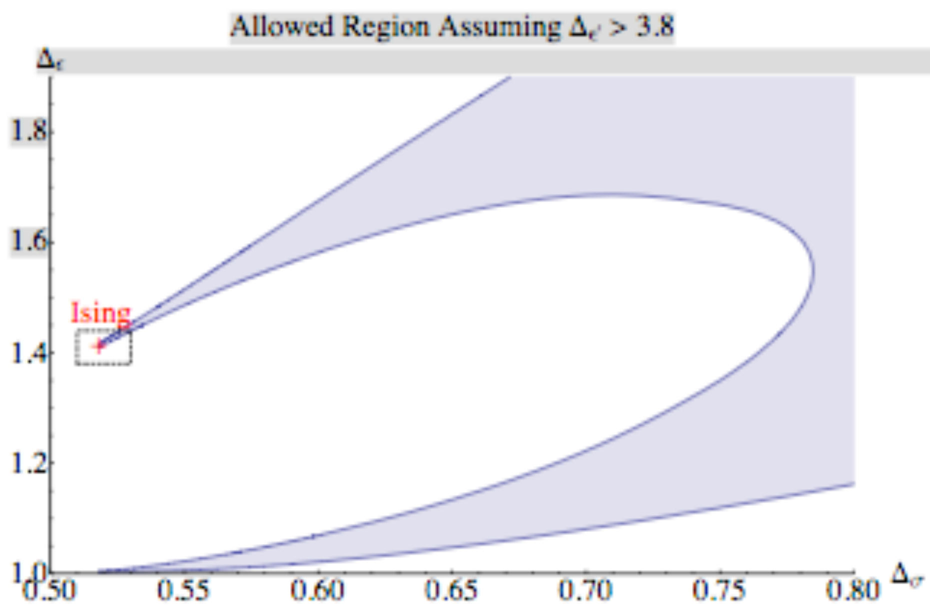
3D Ising Model

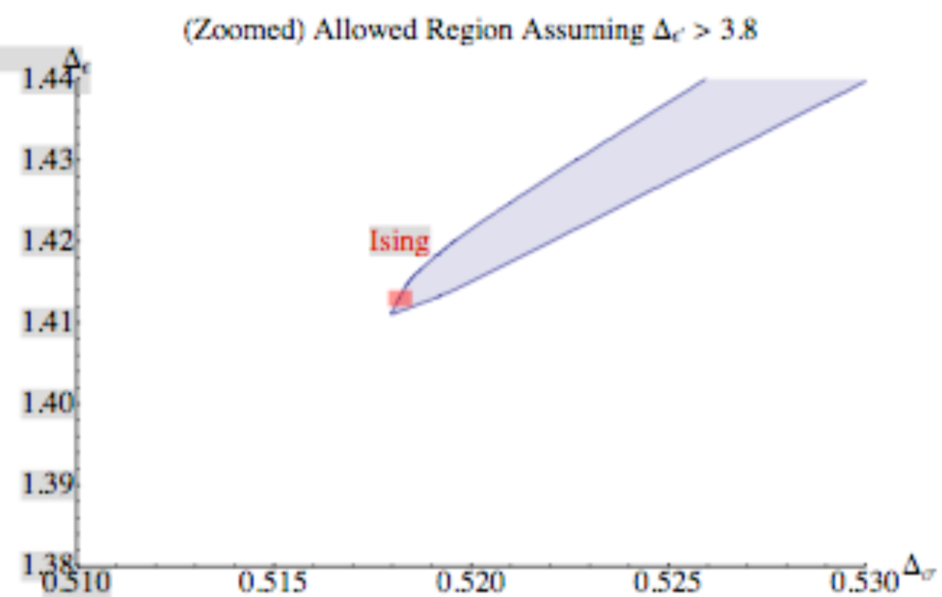
El-Showk
Paulos
Poland
Rychkov
Simmons-Duffin
Vichi
2012

Some notation: OPE

$$\sigma \times \sigma \sim 1 + \epsilon + \epsilon' + \epsilon'' + \ldots \qquad L = 0$$
$$+ \quad T_{\mu\nu} + T' + \ldots \qquad L = 2$$
$$+ \quad C + C' + \ldots \qquad L = 4$$

Allowed regions in Δ_σ, Δ_ϵ plane if $\Delta_{\epsilon'} \geq 3.8$?





Operator	Spin l	$\mathbb{Z}_2$	Δ
σ	0	$-$	0.5182(3)
σ'	0	$-$	$\gtrsim 4.5$
ε	0	$+$	1.413(1)
ε'	0	$+$	3.84(4)
ε''	0	$+$	4.67(11)
$T_{\mu\nu}$	2	$+$	3
$C_{\mu\nu\kappa\lambda}$	4	$+$	5.0208(12)

The x axis is Δ_σ the y axis is the Δ lowest in the OPE

ϕ^4 scalar theory

$\sigma \sim \phi \quad \epsilon \sim \phi^2 \qquad d = 3$ free theory $\Delta_\phi = \frac{1}{2} \quad \Delta_{\phi^2} = 1$

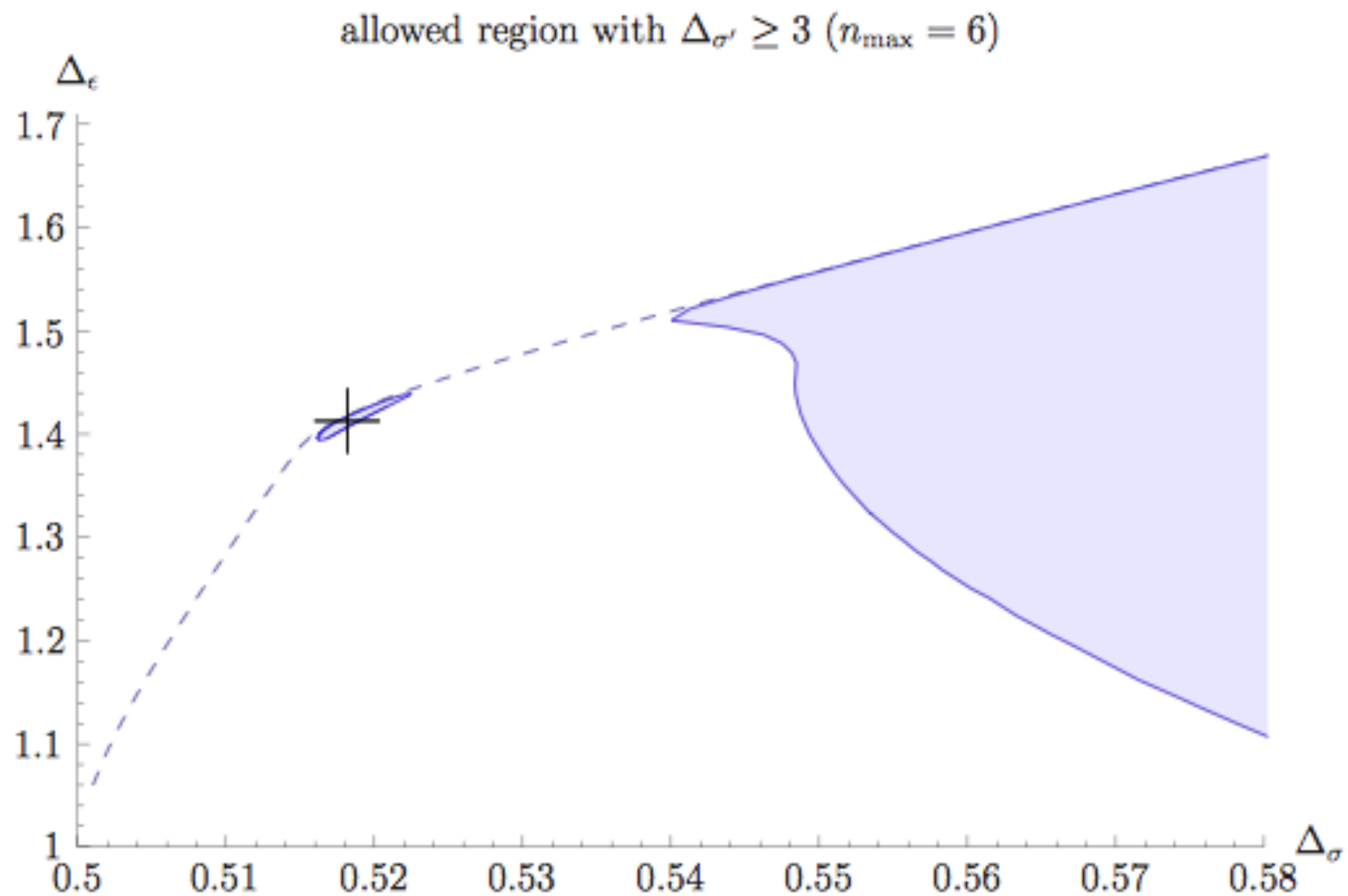


Figure 2: Allowed region of $(\Delta_\sigma, \Delta_\epsilon)$ in a $\mathbb{Z}_2$-symmetric CFT_3 where $\Delta_{\sigma'} \geq 3$ (only one $\mathbb{Z}_2$-odd scalar is relevant). This bound uses crossing symmetry and unitarity for $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$, and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, with $n_{\max} = 6$ (105-dimensional functional), $\nu_{\max} = 8$. The 3D Ising point is indicated with black crosshairs. The gap in the $\mathbb{Z}_2$-odd sector is responsible for creating a small closed region around the Ising point.

Kos
Poland
Simmons-Duffin
2014

$\mathbb{Z}_2$ symmetry

$$\sigma \times \sigma = \sum_{O^+} \lambda_{\sigma\sigma O^+} O^+$$

$$\sigma \times \epsilon = \sum_{O^-} \lambda_{\sigma\epsilon O^-} O^-$$

$$\epsilon \times \epsilon = \sum_{O^+} \lambda_{\epsilon\epsilon O^+} O^+$$

Analyse

$\langle\sigma\sigma\sigma\sigma\rangle$

$\langle\sigma\sigma\epsilon\epsilon\rangle$

$\langle\epsilon\epsilon\epsilon\epsilon\rangle$

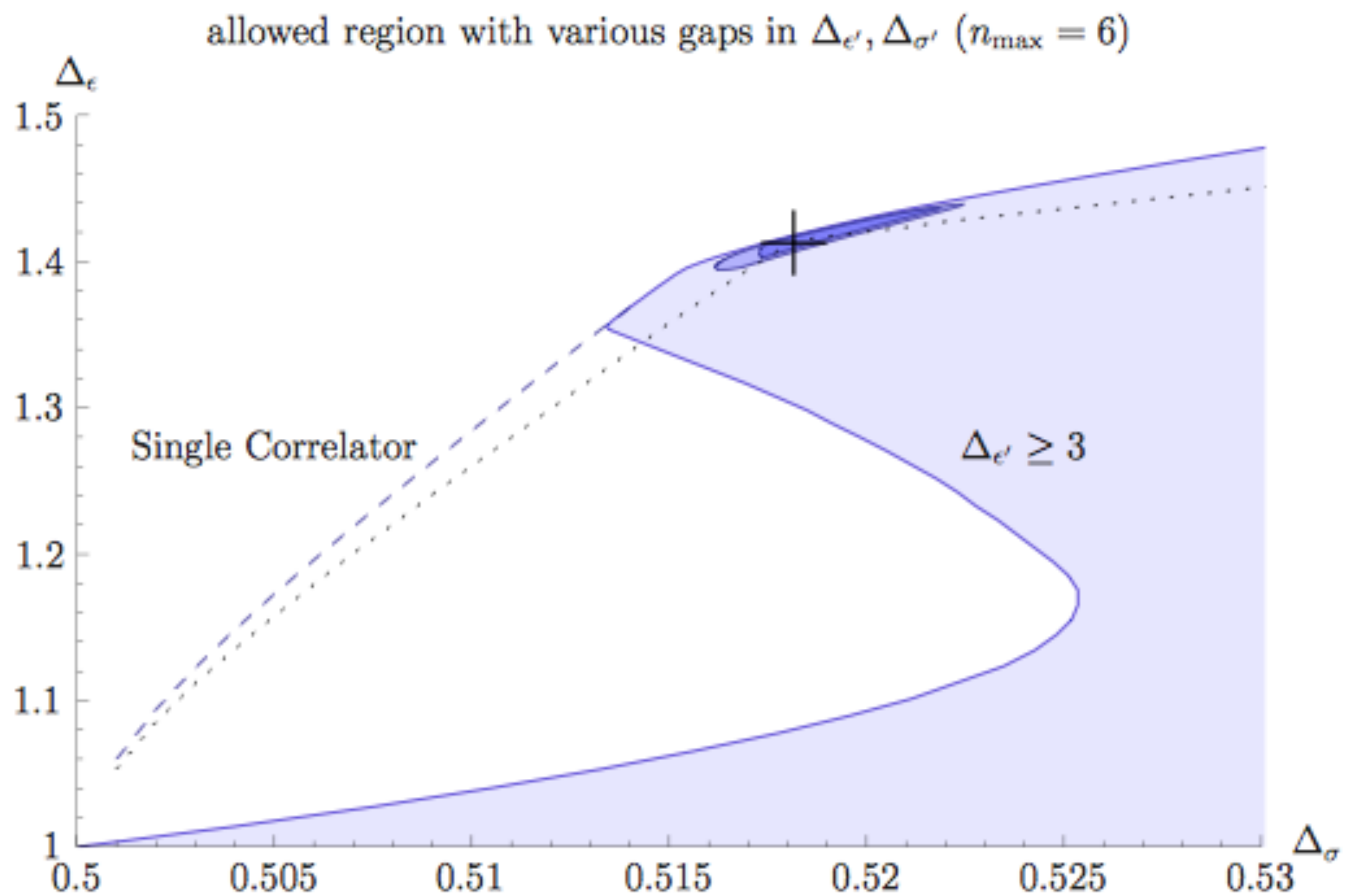


Figure 3: Allowed regions in a $\mathbb{Z}_2$-symmetric CFT_3, assuming various gaps in the scalar spectrum. The dashed line is an upper bound on Δ_ϵ using crossing symmetry and unitarity of $\langle\sigma\sigma\sigma\sigma\rangle$, with no assumptions about gaps, at $n_{\max} = 6$. The black dotted line is the same bound with $n_{\max} = 10$. The light blue shaded region assumes a gap $\Delta_{\epsilon'} \geq 3$ in the $\mathbb{Z}_2$-even sector. The medium blue shaded region assumes a gap $\Delta_{\sigma'} \geq 3$ in the $\mathbb{Z}_2$-odd sector, and uses crossing symmetry for the system of correlators $\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle$ (same as figure 2). The dark blue region assumes both $\Delta_{\sigma'}, \Delta_{\epsilon'} \geq 3$, and uses the system of multiple correlators. All bounds other than the black dotted line are computed with $n_{\max} = 6$, $\nu_{\max} = 8$ (21 components for single correlator bounds, 105 components for multiple correlator bounds). The 3D Ising point is indicated with black crosshairs.

Crucial assumption, no relevant $\Delta < 3$ operators other than σ, ϵ

Simmons Duffin
2015

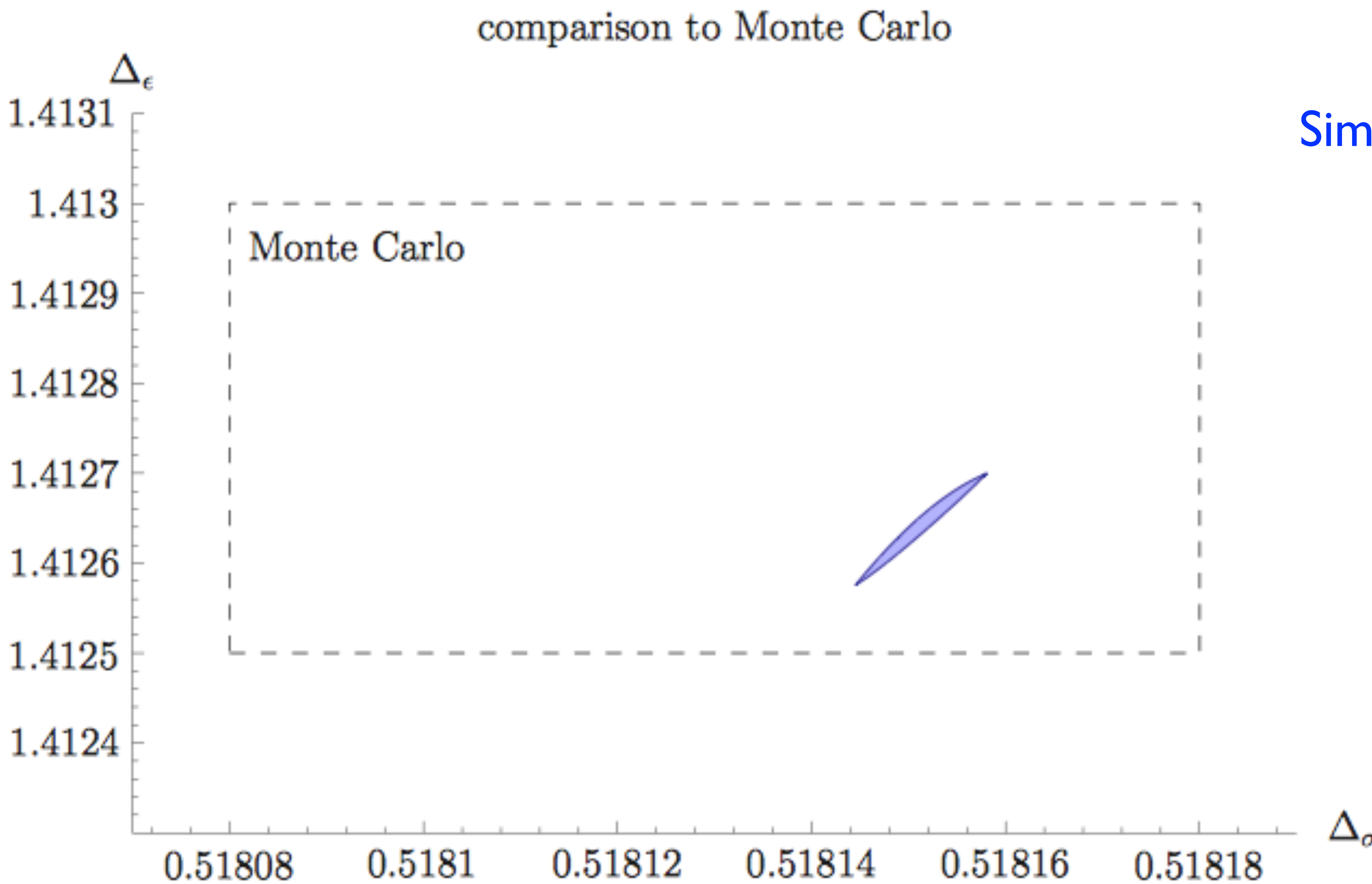


Figure 3: Comparison between the allowed region for the 3d Ising CFT using SDPB with $\Lambda = 43$ (blue) and Monte Carlo determinations of critical exponents (dashed rectangle) [67]. The size of the Monte Carlo rectangle is set by statistical and systematic errors associated with the simulation. By contrast, the blue region is a rigorous bound with sharp edges.

Energy Momentum Tensor, this plays a critical role in CFTs

The normalisation is fixed by Ward identities

$$S_d\, T^{\mu\nu}(x)\, \mathcal{O}(0) \sim -\frac{d\,\Delta_{\mathcal{O}}}{d-1}\, \frac{1}{(x^2)^{\frac{1}{2}d}} \left(\frac{x^\mu x^\nu}{x^2} - \frac{1}{d}\,\eta^{\mu\nu} \right) \mathcal{O}(0)$$

generalises to non zero spin

$$S_d^{\,2} \big\langle T^{\mu\nu}(x)\, T^{\sigma\rho}(0) \big\rangle = C_T\, \frac{1}{(x^2)^d}\, \mathcal{I}^{\mu\nu,\sigma\rho}(x)$$

inversion tensor

C_T is a measure of the numbers of degrees of freedom, $C_T > 0$ for unitary theories

$$d = 2 \quad C_T = 2c$$

$$d = 4 \quad C_T = 160c = \tfrac{4}{3}\, n_S + 4\, n_W + 16\, n_A$$

$d = 3$

$C_T = \frac{3}{2}\, n_S + 3\, n_F$

C_T can be determined in the conformal bootstrap from the contribution of the $\ell = 2,\ \Delta = d$ conformal block

$$c_{\text{Ising}}/c_{\text{free}} = 0.9465$$

Higher dimensional 'perturbative' CFTs
Non linear sigma model

$$\mathcal{L} = \tfrac{1}{2}\,\partial\phi_i\partial\phi_i \quad \phi_i\phi_i = 1 \quad i = 1,\ldots,N$$

This defines a CFT for any d in a 1/N expansion

$$\phi_i \;\; N\text{-vector} \;\; \Delta_\phi = \tfrac{1}{2}(d-2) + \mathrm{O}(1/N)$$

$$\sigma \;\; \text{singlet} \;\; \Delta_\sigma = 2 + \mathrm{O}(1/N)$$

violates unitarity bound for d>6

For d even there is a renormalisable Lagrangian which gives equivalent results in an epsilon expansion with small couplings

$$\mathcal{L}_4 = \tfrac{1}{2}\big(\partial\phi_i\partial\phi_i + \sigma^2 + g\,\sigma\phi_i\phi_i\big)$$

$$\mathcal{L}_6 = \tfrac{1}{2}\big(\partial\phi_i\partial\phi_i + \partial\sigma\partial\sigma + g\,\sigma\phi_i\phi_i + \tfrac{1}{3}\lambda\,\sigma^3\big)$$

$$\mathcal{L}_8 = \tfrac{1}{2}\big(\partial\phi_i\partial\phi_i + \partial^2\sigma\partial^2\sigma + g\,\sigma\phi_i\phi_i + \lambda'\,\sigma^2\partial^2\sigma + \tfrac{1}{12}\lambda\,\sigma^4\big)$$

Diab
Fei
Giombi
Klebanov
Tarnopolsky
and
Gracey

Results for C_T

$$d=4 \quad C_T = C_{T\,\text{free scalar}}\, N$$

$$d=6 \quad C_T = C_{T\,\text{free scalar}}\, (N+1)$$

$$d=8 \quad C_T = C_{T\,\text{free scalar}}\, (N-4)$$

+1 from dynamical sigma

-4 from higher derivative sigma

There may exist non trivial unitary CFTs for d=5 (at least for sufficiently large N) but also perhaps non unitary non trivial CFTs when d=7

Similar considerations apply to the Gross-Neveu model in the large N limit but here the sigma field has dimension 1 and the theory is non unitary for d>4

Superconformal Symmetry

Extension of conformal group to include supersymmetry

In addition to the usual fermionic charges $Q, \bar{Q}$
there are additional charges $S, \bar{S}$

$$\{Q, \bar{Q}\} = P \qquad \{S, \bar{S}\} = K \qquad \{Q, S\} = \underset{\text{Lorentz}}{M} + \underset{\text{Scale}}{D} + \underset{\text{R-charge}}{R}$$

R-symmetry is an essential part of the superconformal group

$$\mathcal{N} = 1 \quad U(1)_R \qquad \mathcal{N} = 2 \quad U(2)_R \qquad \mathcal{N} = 4 \quad SU(4)_R$$

For $|\phi\rangle$ a superconformal primary $K|\phi\rangle = S|\phi\rangle = \bar{S}|\phi\rangle = 0$

Supermultiplet generated by $\prod_{n,j,k} P^n \, Q^j \, \bar{Q}^k |\phi\rangle$

Shortening conditions

One or more of the $Q, \bar{Q}$ acting on $|\phi\rangle$ may give zero

This gives rise to short or semi-short $\frac{1}{n}$-BPS multiplets

$n = 2, 4, 8, 16$

Such multiplets are protected Δ is determined in terms of ℓ and R-symmetry representation

For short multiplets $\ell = 0$ and the R-symmetry representations are restricted

For 4 point functions the conformal partial wave expansion extends to one in terms of superconformal blocks

For $\mathcal{N} = 1$ short multiplets in d=4 correspond to chiral superfields with $\Delta = \frac{3}{2}\, r$

$\mathcal{N} = 4$ $\quad \mathcal{O}_{20}^{I}$ half BPS 20 dim short supermultiplet

$\Delta = 2$

Beem
Rastelli
van Rees
2013

$$\left\langle \mathcal{O}_{20}^{I_1}(x_1)\,\mathcal{O}_{20}^{I_2}(x_2)\,\mathcal{O}_{20}^{I_3}(x_3)\,\mathcal{O}_{20}^{I_4}(x_4)\right\rangle = \frac{1}{({x_{12}}^2\,{x_{34}}^2)^2}\,A^{I_1 I_2 I_3 I_4}(u,v)$$

$A^{I_1 I_2 I_3 I_4}(u,v) \to F(u,v)$ $\quad$ superconformal Ward identity

$$\frac{v^2 F(u,v) - u^2 F(v,u)}{v^2 - u^2} = 1 + \frac{1}{a(u+v)}$$

bootstrap equation

a central charge

protected short multiplet contribution

$$c = a = \tfrac{1}{4}\dim G \qquad G$$ gauge group

Expand $F(u,v)$ over the sum of protected and unprotected superconformal blocks with positive coefficients

Get constraints on potential Δ_ℓ

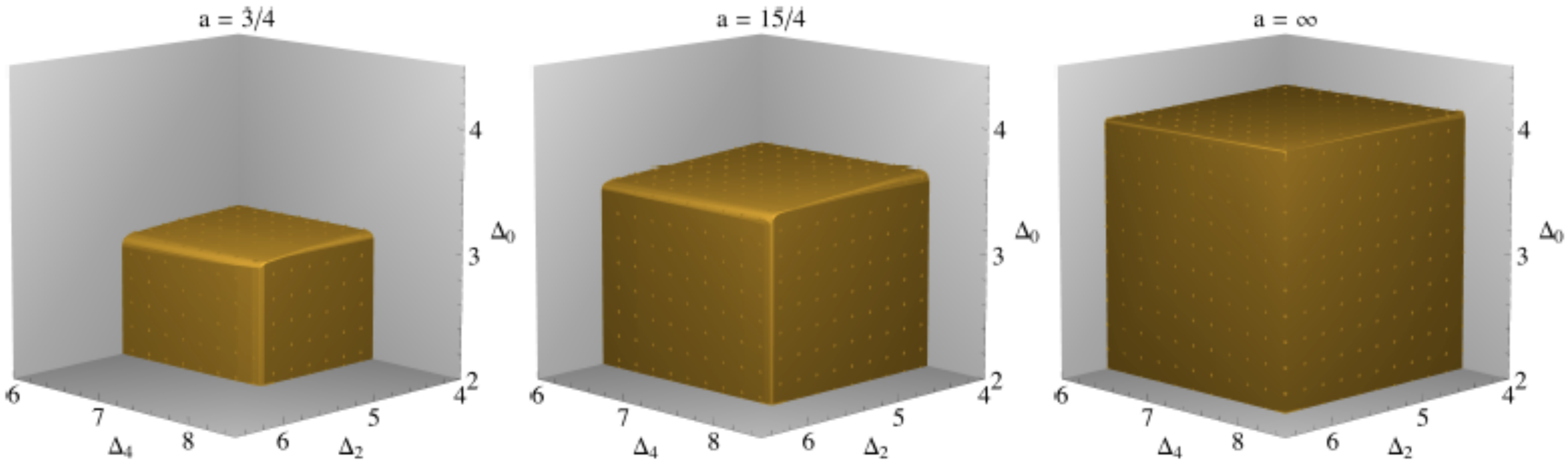


FIG. 1. Exclusion plots in the space of leading twist gaps Δ_0, Δ_2, and Δ_4. Central charges $a = 3/4$, $a = 15/4$, and $a = \infty$ are shown, corresponding to $\mathcal{N} = 4$ SYM with gauge group $SU(2)$, $SU(4)$, and $SU(\infty)$, respectively. The area outside of a cube-shaped region is excluded.

Conjecture: the corner of the cube corresponds to a self dual point under $Sl(2,\mathbb{Z})$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} = e^{\frac{1}{2}\pi i}, e^{\frac{1}{3}\pi i}$$

Such self dual points are difficult to access by other methods

Minkowski approach

$$\phi(x)\phi(0) = \sum_{\mathcal{O}} c_{\phi\phi\mathcal{O}} \frac{1}{(x^2)^{\frac{1}{2}(2\Delta_\phi - \Delta + \ell)}} C_{\Delta,\ell}(x,\partial)^{\mu_1\ldots\mu_\ell} \mathcal{O}_{\mu_1\ldots\mu_\ell}(0)$$

$$C_{\Delta,\ell}(x,0)^{\mu_1\ldots\mu_\ell} = x^{\mu_1}\ldots x^{\mu_\ell}$$

In the light cone limit $x^2 \to 0$ the OPE is essentially an expansion in which operators of equal twist τ contribute equally

$$\tau_\ell = \Delta - \ell$$

This limit is relevant for deep inelastic scattering where the twist controls the approach to scaling in the deep inelastic limit

Positivity requires that the twist is a convex function

Nachtmann
1973

$$\frac{\tau_{\ell_3} - \tau_{\ell_1}}{\ell_3 - \ell_1} \leq \frac{\tau_{\ell_2} - \tau_{\ell_1}}{\ell_2 - \ell_1} \quad \ell_1 < \ell_2 < \ell_3$$

Such results have been refined using a bootstrap type approach relating s and t channel expansions in the light cone limit

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \bar{\mathcal{O}}_1(x_3)\, \bar{\mathcal{O}}_2(x_4) \rangle$$

Fitzpatrick
Kaplan
Poland
Simmons-Duffin
Komargodski
Zhiboedov
2012(Dec)

has an s channel expansion in terms of operators with leading twist

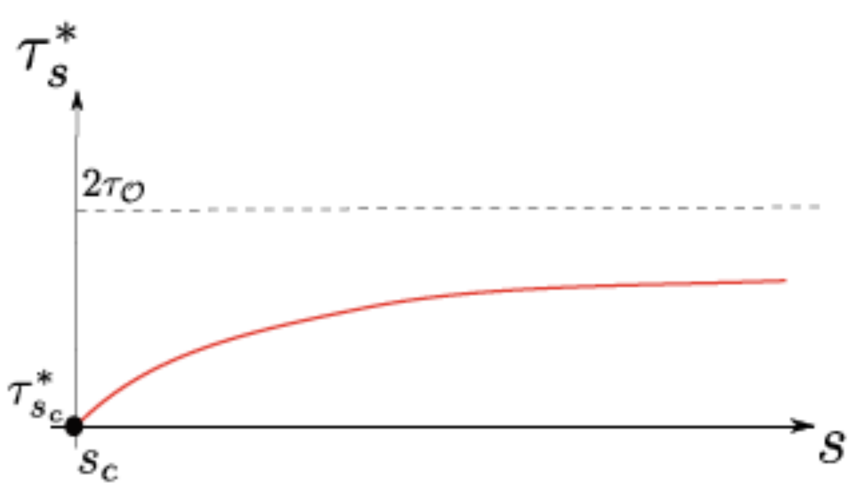
multi trace operators

$$\tau_\ell = \tau_{\mathcal{O}_1} + \tau_{\mathcal{O}_2} - \frac{C}{\ell^{\,\tau_{\min}}}$$

$d > 2$

large ℓ

$\tau_{\min}$ is the minimum twist of operators appearing in the t channel expansion



C is also calculable

Emergent symmetries

In d=3 the Ising model appears to be the unique CFT with two relevant fields σ, ϵ and $\mathbb{Z}_2$ symmetry

Does $\mathbb{Z}_2$ emerge in the RG at the fixed point?

no ϕ^3 interaction

In d=4 Lagrangian theories with IR fixed points defining a CFT require gauge fields and large numbers of fermions and so should have a large flavour symmetry group

In d=6 a conjecture would be that all non trivial CFTs have superconformal symmetries. d=6 superconformal theories have no relevant scalar

1. Understand the origin of kinks in bootstrap bounds in terms of decoupling of particular states. Why are there islands? Maybe there is a more analytic approach.

2. Show there are no non trivial unitary CFTs for d>6. This is true for SCFTs for algebraic, representation theory reasons (Nahm).

3. Are there CFTs with large anomalous dimensions or which are not a deformation of a free Lagrangian theory? This might be relevant to extending ideas of naturalness.

4. Construct conformal blocks with external spins. No succinct formulae exist as yet, but there are now bootstrap calculations for fermions in d=3 and significant results for d=4.

Echeverri
Elkhidir
Karateev
Serone
2016

5. Bootstrap < T T T T>, very hard.

$\mathcal{N}=1$ bootstrap $\phi, \bar\phi$ chiral superfields

Apply bootstrap to $\langle \phi\, \bar\phi\, \phi\, \bar\phi \rangle$

Poland Stergiou 2015

free theory point

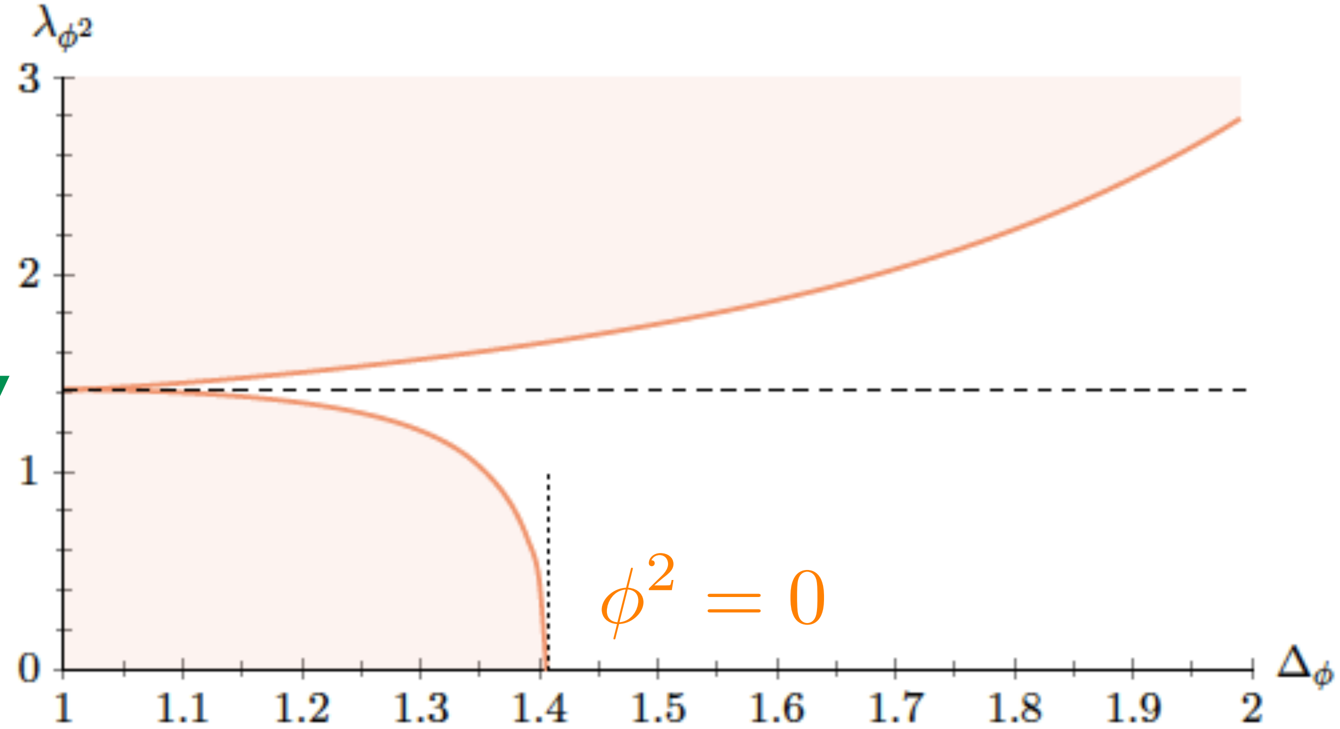


Fig. 2: Lower and upper bounds on the OPE coefficient of the operator ϕ^2 in the $\phi \times \phi$ OPE.

SCFT

$\mathcal{N}=1 \quad d=4$

$24c = n_C + 3n_V$

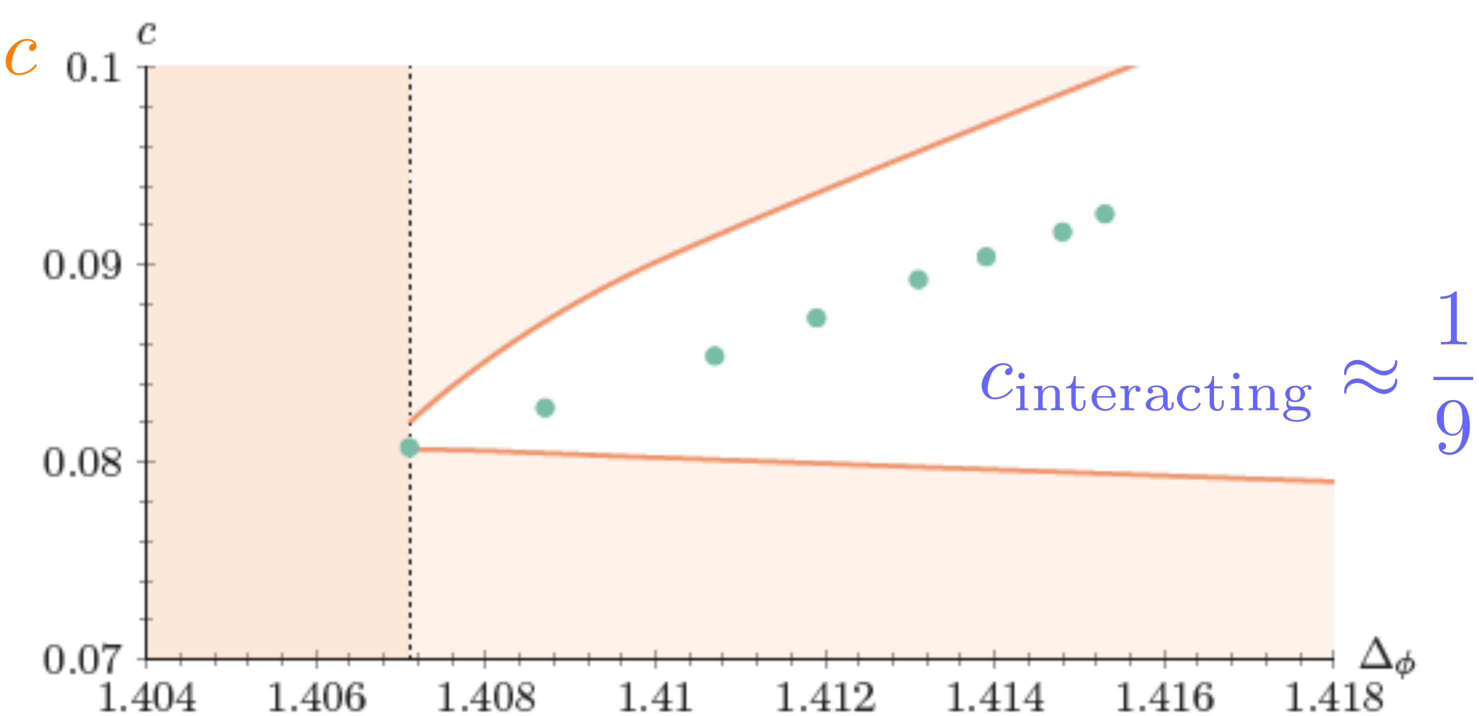
$SU(N_c)$ N_f flavours SQCD

$24c = 2N_cN_f + 3(N_c^2-1)$

$\frac{3}{2}N_c \le N_f \le 3N_c$

$C_T = 160c$

$c_{\mathrm{free}} = \dfrac{1}{24}$

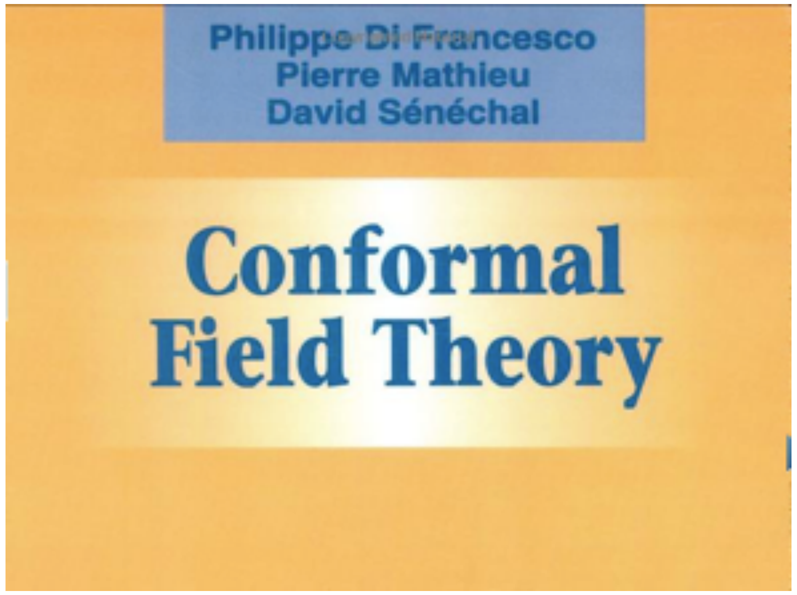


Difficult to accommodate with gauge fields

Fig. 8: Lower and upper bounds on the central charge as a function of the dimension of ϕ, with the assumptions that there is no ϕ^2 operator and that all vector operators but the first one obey

Two dimensional CFTs are not always a good guide to higher dimensions. We have no idea as to any classification in d=3,4

Perhaps we will have a new yellow book, the bible for 2dim CFTs, but not yet



I would like to remember Francis Dolan with nearly all my work on CFTs was done