RG Flow and the a/c Theorem

Hugh Osborn

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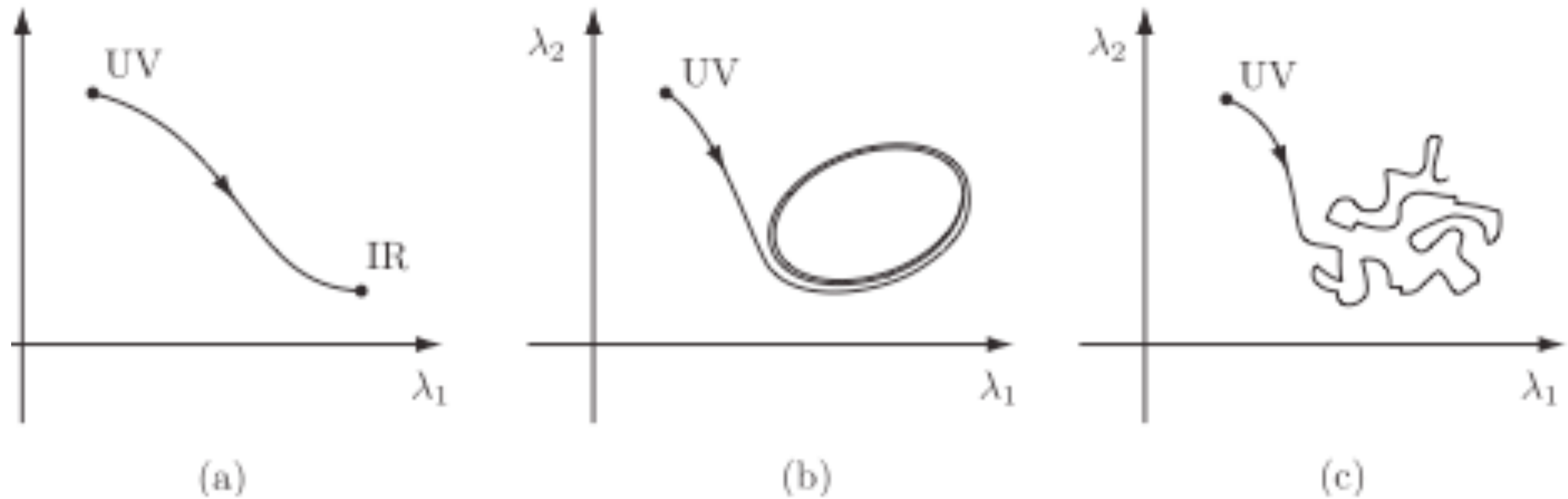
Since Wilson we know that QFTs should be considered as belonging to a space of QFTs with differing couplings

- RG flow provides a topology on the space of quantum field theories
- Is RG flow always between isolated fixed points?
- Are fixed points CFTs?
- Is more exotic behaviour such as limit cycles possible?
- Is RG flow gradient flow?
- Is Unitarity crucial for these properties?

Possible RG flows for two couplings



UV, IR fixed points	UV fixed point, IR limit cycle	UV fixed point, IR chaos

Most of of our understanding of RG flow is based on fixed points, but limit cycles can exist in non unitary theories

Zamolodchikov c-theorem in 2 dimensions provides strong constraints on RG flow

Based just on em tensor conservation and unitarity

$$T_{\mu\nu} \to T_{zz}, T_{z\bar{z}} = T_{\bar{z}z}, T_{\bar{z}\bar{z}}$$

Rotational scalars

Let

$$C(\mu^2 x^2) = 2z^4 \langle T_{zz}(x)\, T_{zz}(0)\rangle$$

$$H(\mu^2 x^2) = z^2 x^2 \langle T_{z\bar{z}}(x)\, T_{zz}(0)\rangle$$

$$G(\mu^2 x^2) = x^2 x^2 \langle T_{z\bar{z}}(x)\, T_{z\bar{z}}(0)\rangle > 0$$

$$x^2 = z\bar{z} \qquad \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0 \qquad \Theta = \tfrac{1}{4} T_{\bar{z}z} = \beta^I \mathcal{O}_I$$

$$t = \tfrac{1}{2}\,\ln \mu^2 x^2 \qquad \tilde{C} = C - 4H - 6G$$

$$\frac{d}{dt}\tilde{C} = -\beta^I \frac{\partial}{\partial g^I}\tilde{C} = -24\,G = -\frac{3}{2}\,G_{IJ}\beta^I\beta^J < 0$$

Crucial properties

$\tilde{C} =$ Virasoro central charge at fixed points

$$\tilde{C}(0) = c_{UV} \qquad \tilde{C}(\infty) = c_{IR} \qquad c_{UV} > c_{IR}$$

Zamolodchikov metric $G_{IJ} = x^2 x^2 \langle \mathcal{O}_I(x)\, \mathcal{O}_J(0) \rangle$

positive for unitary theories

Cardy sum rule $\qquad \Theta = \beta^I \mathcal{O}_I$

$$c_{UV} - c_{IR} = \frac{3}{2} \int_0^\infty dr\, r^3 \, \langle \Theta(x)\, \Theta(0) \rangle \qquad r^2 = x^2$$

Basic eqs for a-theorem

$$-\frac{d}{dt}g^I = \beta^I(g)$$ t increases to the IR

Fixed point $\beta^I(g_*) = 0$

a-theorem (minimal) $a_{UV} - a_{IR} > 0$

a-theorem (strong) $$\frac{d}{dt}\tilde{a}(g) = -G_{IJ}(g)\beta^I(g)\beta^J(g)$$

G_{IJ} is a positive metric on couplings

$\tilde{a}(g)$ is stationary at a fixed point

$$\tilde{a}(g) = a(g) + \mathrm{O}(\beta)$$

gradient flow $$\frac{\partial}{\partial g^I}\tilde{a}(g) = T_{IJ}(g)\beta^J(g)$$

$T_{(IJ)} = G_{IJ}$ gradient flow requires integrability conditions on beta functions

Cardy proposal in 4 dimensions

For a CFT on curved space

$$\gamma^{\mu\nu}\langle T_{\mu\nu}\rangle = c\,\text{Weyl tensor}^2 - a\,\text{Euler density}$$

For conformally flat spaces, e.g. a sphere, the Weyl tensor vanishes

Free theories
$$c = 12n_V + n_S + 3n_F$$
$$a = \tfrac{1}{3}\big(62n_V + n_S + \tfrac{11}{2}n_F\big)$$

On flat space for a CFT

$$\langle T_{\mu\nu}\,T_{\sigma\rho}\rangle \propto c \qquad c > 0$$
$$\langle T_{\mu\nu}\,T_{\sigma\rho}\,T_{\alpha\beta}\rangle \propto A\,c + B\,a + C$$

In simple versions of AdS/CFT $\quad a = c$

Is $a > 0$?

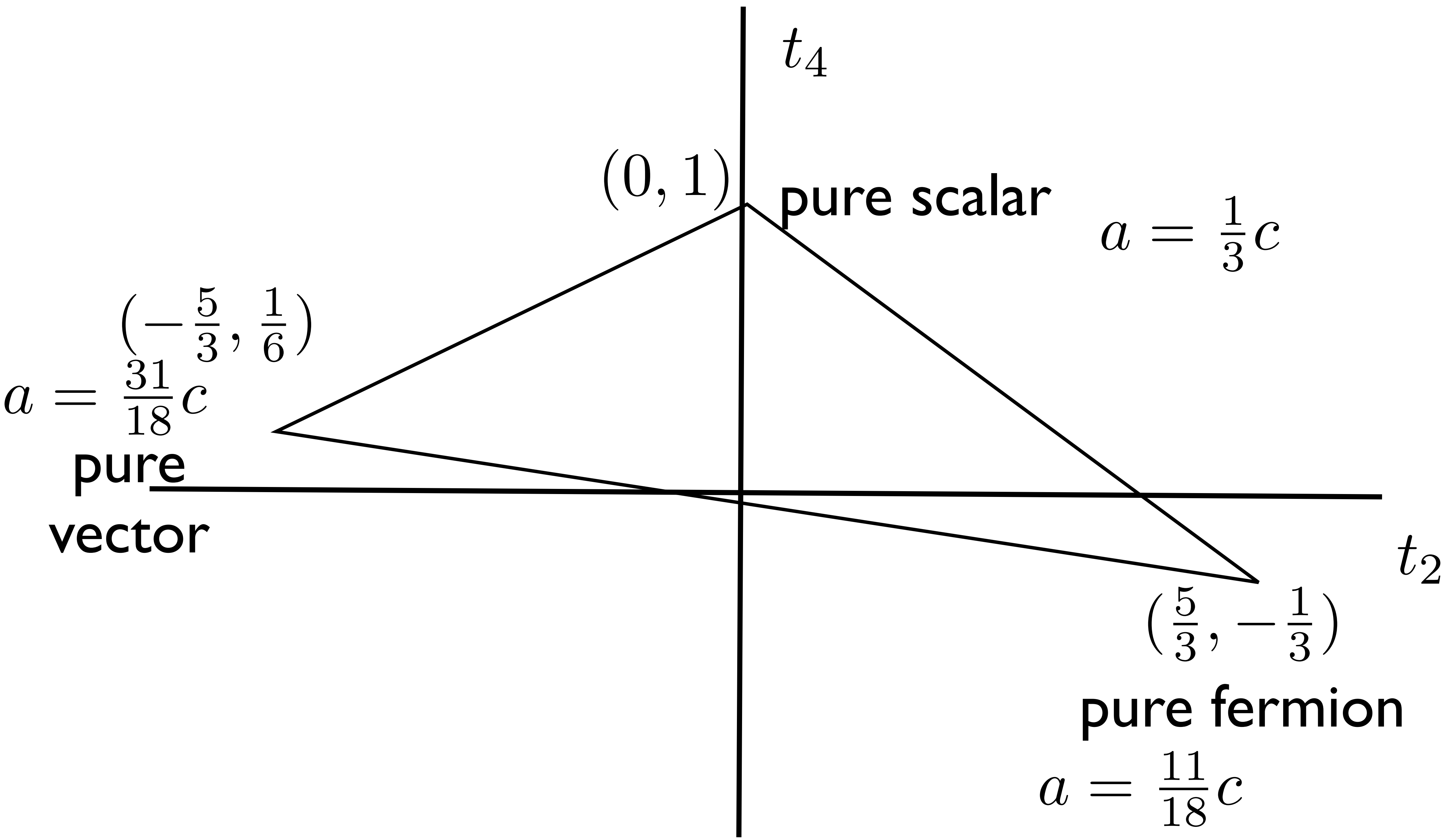
First proper argument given by Hofman and Maldacena in 2008

Look at energy flux at infinity

$$\mathcal{E}(\mathbf{n}) = \lim_{r\to\infty} \int_{-\infty}^{\infty} dt\, n^i T^0{}_i(t, r\mathbf{n})$$

$$\langle \epsilon_{ij} T_{ij}\, \mathcal{E}(\mathbf{n})\, \epsilon_{kl} T_{kl} \rangle \sim c\big(1 + t_2 Y_2(\mathbf{n}) + t_4 Y_4(\mathbf{n})\big) \geq 0$$

Requiring positivity in all directions restricts t_2, t_4 to a triangular region, which implies $a > 0$



scalars, vectors give the extreme values of a/c

for Susy $t_4 = 0$

Derivation of 4 dim RG eqs

Consider Weyl rescalings $\gamma_{\mu\nu} \to e^{2\sigma}\gamma_{\mu\nu}$

Couplings $g^I(x) \quad \gamma_{\mu\nu}(x)$

W vacuum functional

$E^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}R$

Einstein tensor

$$2\int \sigma\,\gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}}W = \int \sigma\,\beta^I\frac{\delta}{\delta g^I}\,W$$
$$-\int \sigma\Big(c\,\text{Weyl tensor}^2 - a\,\text{Euler density}$$
$$-\tfrac{1}{2}G_{IJ}E^{\mu\nu}\partial_\mu g^I\partial_\nu g^J + \ldots\Big)$$
$$+\int \partial_\mu\sigma\Big(E^{\mu\nu}w_I\partial_\nu g^I + \ldots\Big)$$

Defines G_{IJ}, w_I as well as c, a

There are crucial integrability conditions by commuting Weyl rescalings for different σ

$$\partial_I a = G_{IJ}\beta^J - \mathcal{L}_\beta w_I$$
$$\mathcal{L}_\beta w_I = \beta^J \partial_J w_I + \partial_I \beta^J \, w_J$$

If $\partial g = 0$ then the equation is equivalent to

$$\gamma^{\mu\nu}\langle T_{\mu\nu}\rangle = \beta^I \langle \mathcal{O}_I \rangle + c \text{ Weyl tensor}^2 - a \text{ Euler density}$$

In two dimensions there are very similar eqs

$$2\int \sigma\,\gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}}W = \int \sigma\,\beta^I\frac{\delta}{\delta g^I}\,W$$

$$-\int \sigma\Big(\tfrac{1}{2}c\,R - \tfrac{1}{2}G_{IJ}\gamma^{\mu\nu}\partial_\mu g^I\partial_\nu g^J + \ldots\Big)$$

$$+\int \partial_\mu\sigma\Big(\gamma^{\mu\nu}w_I\partial_\nu g^I + \ldots\Big)$$

Consistency $\partial_I c = G_{IJ}\beta^J - \mathcal{L}_\beta w_I$

$$\mathcal{L}_\beta w_I = \beta^J\partial_J w_I + \partial_I\beta^J\,w_J$$

This is equivalent to Zamolodchikov's eqs

Let

$$\tilde{a} = a + w_I \beta^I$$

Then

$$\partial_I \tilde{a} = T_{IJ} \beta^J \qquad T_{IJ} = G_{IJ} + \partial_I w_J - \partial_J w_I$$

$$\beta^I \partial_I \tilde{a} = G_{IJ} \beta^I \beta^J$$

$\tilde{a} = a$ at fixed points

Ambiguities

$$G_{IJ} \sim G_{IJ} + \mathcal{L}_\beta D_{IJ} \qquad w_I \sim w_I + D_{IJ} \beta^J$$

$$\tilde{a} \sim \tilde{a} + D_{IJ} \beta^I \beta^J$$

For irreversible RG flow require

G_{IJ} positive

In two dimensions

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_\beta\right)\langle \mathcal{O}_I(x)\,\mathcal{O}_J(0)\rangle = G_{IJ}\,\partial^2\delta^2(x)$$

From this it follows that

$$G_{IJ} \sim (x^2)^2\langle \mathcal{O}_I(x)\,\mathcal{O}_J(0)\rangle > 0$$

equivalent to Zamolodchikov metric

In four dimensions at a fixed point

$$\langle \mathcal{O}_I(x)\, \mathcal{O}_J(0)\rangle = g_{IJ}\, \mathcal{R} \frac{1}{(x^2)^4}$$

$$= -\frac{1}{3\times 4^4}\, g_{IJ} (\partial^2)^3 \left(\frac{1}{x^2}\, \ln \mu^2 x^2 \right)$$

$$\mu \frac{\partial}{\partial \mu} \langle \mathcal{O}_I(x)\, \mathcal{O}_J(0)\rangle \propto g_{IJ} (\partial^2)^2 \delta^4(x)$$

$G_{IJ} \propto g_{IJ} > 0$ unitarity

Away from a fixed point G_{IJ} and g_{IJ} differ

$$Im \langle \tilde{\mathcal{O}}_I(p)\, \tilde{\mathcal{O}}_I(-p)\rangle \geq 0$$

In four dimensions for general renormalisable field theories

with gauge couplings g Yukawa couplings Y

quartic scalar couplings λ

using perturbation theory

$$G_{IJ}dg^I dg^J = 2n_V \frac{dg^2}{g^2} + \frac{1}{3}dY^2 + \frac{1}{144}d\lambda^2$$

1 loop 2 loops 3 loops

Integrability gives constraints on beta functions

$$\beta_\lambda^{(1)} = -24\,Y^4 \quad \Leftrightarrow \quad \beta_Y^{(2)} = -2\,\lambda Y^3$$

$$\beta_Y^{(1)} = 6\,g^2 Y \quad \Leftrightarrow \quad \beta_g^{(2)} = \frac{2}{n_V} g^3 Y$$

Possible proof of a-theorem in 4 dimensions

Komargodski & Schwimmer 2011,
Luty, Polchinski & Rattazzi 2012

Based on constructing effective Lagrangians for the dilaton. This couples to the trace of the energy momentum tensor. In a CFT if conformal symmetry is spontaneously broken then the dilaton is a physical Goldstone boson.

The construction assumes a lot of folklore about effective Lagrangians and the role of anomalies.

$$\langle \dots \tilde{T}_{\mu\nu}(p) \dots \rangle \sim \frac{p_\mu p_\nu}{p^2} \langle \dots \tau(p) \dots \rangle \Big|_{p^2 \to 0}$$

The pole determines a non zero amplitude in a CFT for a massless dilaton τ

By assumption the CFT is invariant under Weyl rescaling of the metric and a shift of the dilaton field

$$W[e^{2\sigma}\gamma_{\mu\nu}, \tau + \sigma] = W[\gamma_{\mu\nu}, \tau]$$

A trivial solution is one which depends only on

$$\tilde{\gamma}_{\mu\nu} = e^{-2\tau}\gamma_{\mu\nu}$$

but W should be invariant under diffeomorphisms

$$\tilde{\gamma}_{\mu\nu} \to \tilde{R}_{\alpha\beta\gamma\delta}$$

$$W[\gamma_{\mu\nu},\tau] = W[\tilde{\gamma}_{\mu\nu}] + \int d^4x\sqrt{-\gamma}\,\mathcal{L}_{\text{anomaly}}$$

$$\begin{aligned}\mathcal{L}_{\text{anomaly}} = \tau\Big(c\ \text{Weyl tensor}^2 - a\ \text{Euler density}\Big) \\ - 4a\Big(E^{\mu\nu}\partial_\mu\tau\partial_\nu\tau \\ - \partial^\mu\tau\partial_\mu\tau\nabla^2\tau + \tfrac{1}{2}\big(\partial^\mu\tau\partial_\mu\tau\big)^2\Big)\end{aligned}$$

This can be used to construct a low energy effective action for τ

If there are couplings g to operators with dimension Δ then $g \to e^{(4-\Delta)\tau} g$

The trace anomaly generates contributions involving τ on flat space

$$\mathcal{L}_{\text{anomaly}} = a\Big(4\,\partial^\mu\tau\partial_\mu\tau\,\nabla^2\tau - 2\,(\partial^\mu\tau\partial_\mu\tau)^2\Big)$$

A effective kinetic term is may be constructed from $\tilde{R}$

$$e^{-\tau} = 1 - \frac{\varphi}{f}$$

$$\mathcal{L}_{\text{kinetic}} = -\tfrac{1}{2} f^2\, e^{-2\tau}\partial^\mu\tau\partial_\mu\tau = -\tfrac{1}{2}\,\partial^\mu\varphi\partial_\mu\varphi$$

On shell $\nabla^2\tau = \partial^\mu\tau\partial_\mu\tau$ or $\nabla^2\varphi = 0$

$$\mathcal{L}_{\text{anomaly}} \to 2a\,(\partial^\mu\tau\partial_\mu\tau)^2 = 2a\,\nabla^2\tau\nabla^2\tau$$
$$= \frac{a}{2f^4}\,\nabla^2\varphi^2\,\nabla^2\varphi^2 + \ldots$$

This determines the leading low energy $E \ll f$ contribution to the dilaton scattering amplitude

$$\mathcal{A}(s,t) \sim \frac{a}{2f^4}\langle -p_3, -p_4|(\nabla^2\varphi^2)^2|p_1, p_2\rangle$$
$$= \frac{2a}{f^4}(s^2 + t^2 + u^2)$$

Other contributions to the effective action are less important or vanish on shell

At a fixed point the dilaton is formal and need not represent additional degrees of freedom, a physical dilaton represents SSB for scale invariance

Away from a fixed point such dilatons become massive

Assume $\mathcal{L}_{\text{dilaton}}$ is defined along RG flow from UV to IR

Crucial step in KS and LPR is in the effective dilaton action to take $a \to a_{UV} - a_{IR}$

Two arguments:

Anomaly matching of UV and IR fixed points,

Removes the singular high energy behaviour of the dilaton theory

Positivity of $a_{UV} - a_{IR}$ follows by assuming an unsubtracted dispersion relation for the forward dilaton-dilaton scattering amplitude $\mathcal{A}(s,0) = \mathcal{A}(-s,0)$

$$\mathcal{A}(s,0) \sim (a_{UV} - a_{IR})\,\frac{4}{f^4}\,s^2 + \mathrm{O}(s^4)$$

$$a_{UV} - a_{IR} = \frac{f^4}{2\pi}\int_0^\infty \frac{ds}{s^3}\; Im\,\mathcal{A}(s,0)$$

$$Im\,\mathcal{A}(s,0) = s\,\sigma(s) \geq 0$$

Positivity obtains from positivity of the absorptive part of dilaton-dilaton forward scattering (unitarity)

Perturbatively we expect

$$\sigma(s) \sim \frac{1}{f^4}\,\beta(g(s))^2\,s$$

or

$$\sigma(s) \sim \frac{1}{f^4}\,G_{IJ}\beta^I(g(s))\beta^J(g(s))\,s$$

but this is not straightforward to show
the analysis is significantly more intricate than in two dimensions

It is not clear how to to carry through perturbative calculations beyond lowest order

Generalisations to higher dimensions

Weyl anomalies are present in any even dimension

In six dimensions there are three terms constructed from the Weyl tensor, as well as the six dimensional Euler density with coefficient a

No good argument for $a > 0$

The KS argument requires analysis of 3 to 3 dilaton scattering and positivity of $a_{UV} - a_{IR}$ is problematic

Perturbatively the metric is negative in ϕ^3, but this is rather unphysical

AdS/CFT

This provides an alternative route to an a-theorem if a CFT has an AdS dual
Identify the radial direction away from the boundary with the RG scale
Construct a scalar from the metric which is monotonic under radial evolution subject to a positivity condition on the bulk energy momentum tensor
This works in any dimension

In odd dimensions an analogue of a can be found by considering contributions to the partition function on a sphere There is an analogue of a weak version of the a theorem, but no strong version in which a is stationary at a fixed point

The theorem is doubtless true for unitary theories, at least in even dimensions, but the derivation, and the necessary assumptions, is still rather murky

In general beta functions are not unique

$$\partial\bar{\phi}\, Z\, \partial\phi \to \partial\bar{\phi}_0\, \partial\phi_0$$

if

$$Z = \bar{\mathcal{Z}}\mathcal{Z} \qquad \bar{\phi}_0 = \bar{\phi}\bar{\mathcal{Z}} \quad \phi_0 = \mathcal{Z}\phi$$

but $\mathcal{Z}, \bar{\mathcal{Z}}$ are not unique

$$\delta\bar{\mathcal{Z}} = -\bar{\mathcal{Z}}\omega \quad \delta\mathcal{Z} = \omega\mathcal{Z}$$

leads to $\beta^I \sim \beta^I + (\omega g)^I \quad \gamma \sim \gamma + \omega$

γ is the anomalous dimension matrix for ϕ

Conventionally γ is hermitian but this is not essential

Fortin et al showed that at three loops there were solutions of

$$\beta^I = (Sg)^I$$

which appeared to generate limit cycles but

$$B^I = \beta^I - (Sg)^I = 0$$

gives $T_{\mu\mu} = 0$ and hence CFT

but the anomalous dimension matrix is then non hermitian and might have non real eigenvalues

a/c Theorem
Old and New Results

1987-1992 2012-2014

Hugh Osborn

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at DESY