Critical scaling in the large-N O(N) model in higher dimensions and its possible connection to quantum gravity

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2016

Functional renormalization group

$$Z_k[J] = \int \mathcal{D}\varphi e^{-S[\varphi]-\Delta S_k[\varphi]+\int J\varphi}$$

where $\Delta S_k[\varphi] = \frac{1}{2}\int_q \varphi(q) R_k(q) \varphi(-q)$ and

$$\lim_{k^2/p^2 \to 0} R_k = 0$$

$$\lim_{p^2/k^2 \to 0} R_k < \infty$$

$$\lim_{k^2 \to \Lambda} R_k = \text{large}$$

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$$W_k = \ln Z_k$$

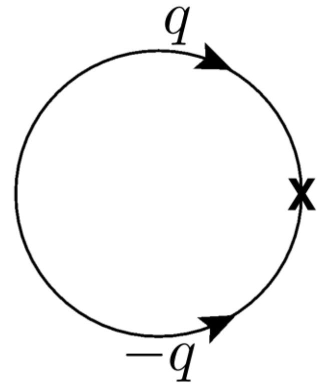
The average effective action at scale *k*

$$\Gamma_k[\phi] = \sup_J \left(\int J\phi - W_k[\phi]\right) - \frac{1}{2}\int \phi R_k \phi \qquad \langle\varphi\rangle = \phi$$

Functional renormalization group

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_t R_k$$

$$t = \log \frac{k}{\Lambda}$$

$\partial_t \Gamma_k =$ 

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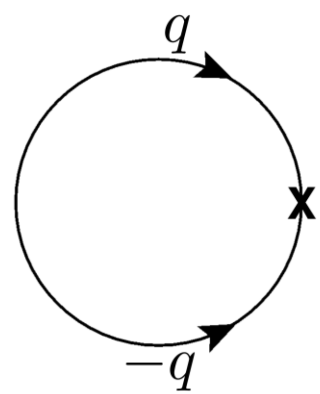

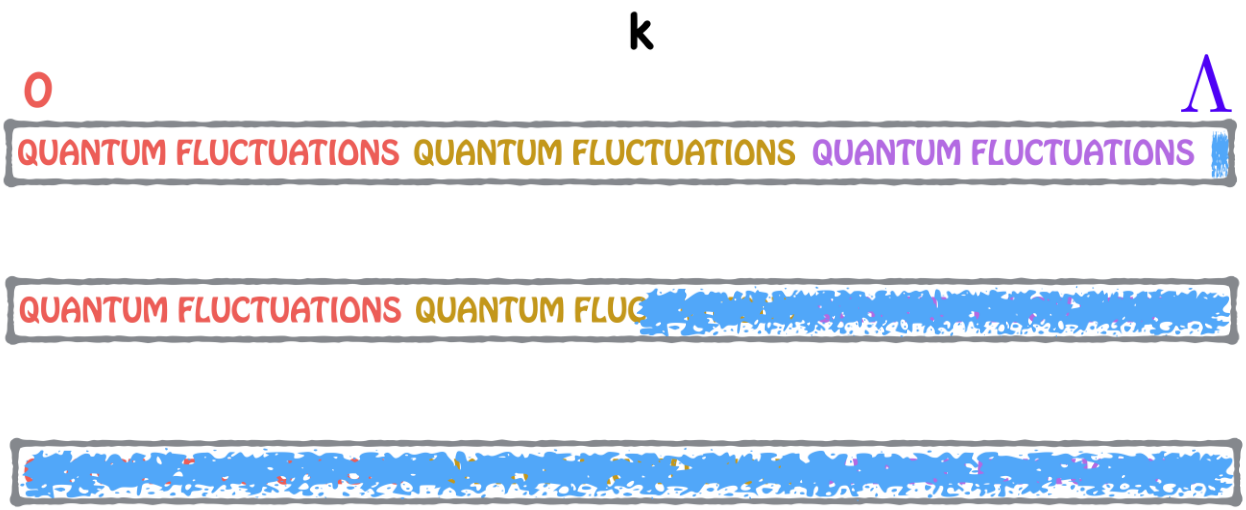

$$\lim_{k \to \Lambda} \Gamma_k = S_{\text{bare}}$$

$$\lim_{k \to k'} \Gamma_k = \Gamma_{k'}$$

$$\lim_{k \to 0} \Gamma_k = \Gamma$$

RG flow in the large-N O(N) model

The O(N) vector model = generalized Ising model for N spin dimension.

The flow

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$$\partial_t U_k = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \partial_t R_k(q) \left(\frac{N-1}{q^2 + R_k(q) + U_k'} + \frac{1}{q^2 + R_k(q) + U_k' + 2\bar{\rho} U_k''} \right)$$

and we use $R_k(q^2) = (k^2 - q^2)\, \theta(k^2 - q^2)$

$$\bar{\rho} \equiv \frac{\bar{\phi}^2}{2}$$

$$(.)' = \partial_{\bar{\rho}}$$

RG flow in the large-N O(N) model

Switching to dimensionless quantities and perform the integral

$$u = \frac{U}{k^D}, \qquad \phi = \frac{\bar{\phi}}{k^{\frac{D-2}{2}}} \Rightarrow \rho = \frac{\bar{\rho}}{k^{D-2}}$$

$$\partial_t u_k = -D u_k + (D-2)\rho u_k' + (N-1)\frac{A_D}{1+u_k'} + \frac{A_D}{1+u_k'+\rho u_k''} \qquad A_D = \frac{1}{2^{D-1}\pi^{D/2}\Gamma[D/2]D}$$

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In the large-N limit

$$\partial_t u = (D-2)\rho u' - Du + \frac{1}{1+u'}$$

The LPA is considered to be exact in the large-N limit.

Fixed points

Scale independent solutions $\partial_t u = 0$

Taylor expansion around vanishing field

$$u(\rho) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{g_i}{i!} \rho^i$$

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The coefficients are the couplings e.g.:

$$u'(0) = g_1 \equiv m^2, \qquad u''(0) = g_2 \equiv \lambda$$

Fixed points: beta functions of each couplings vanish

$$\partial_t u = 0 \quad \Longrightarrow \quad \begin{aligned} &\partial_t u'(0) = \partial_t m^2 = \beta_{m^2} = 0 \\ &\partial_t u''(0) = \partial_t \lambda = \beta_\lambda = 0 \\ &\quad\ldots \end{aligned}$$

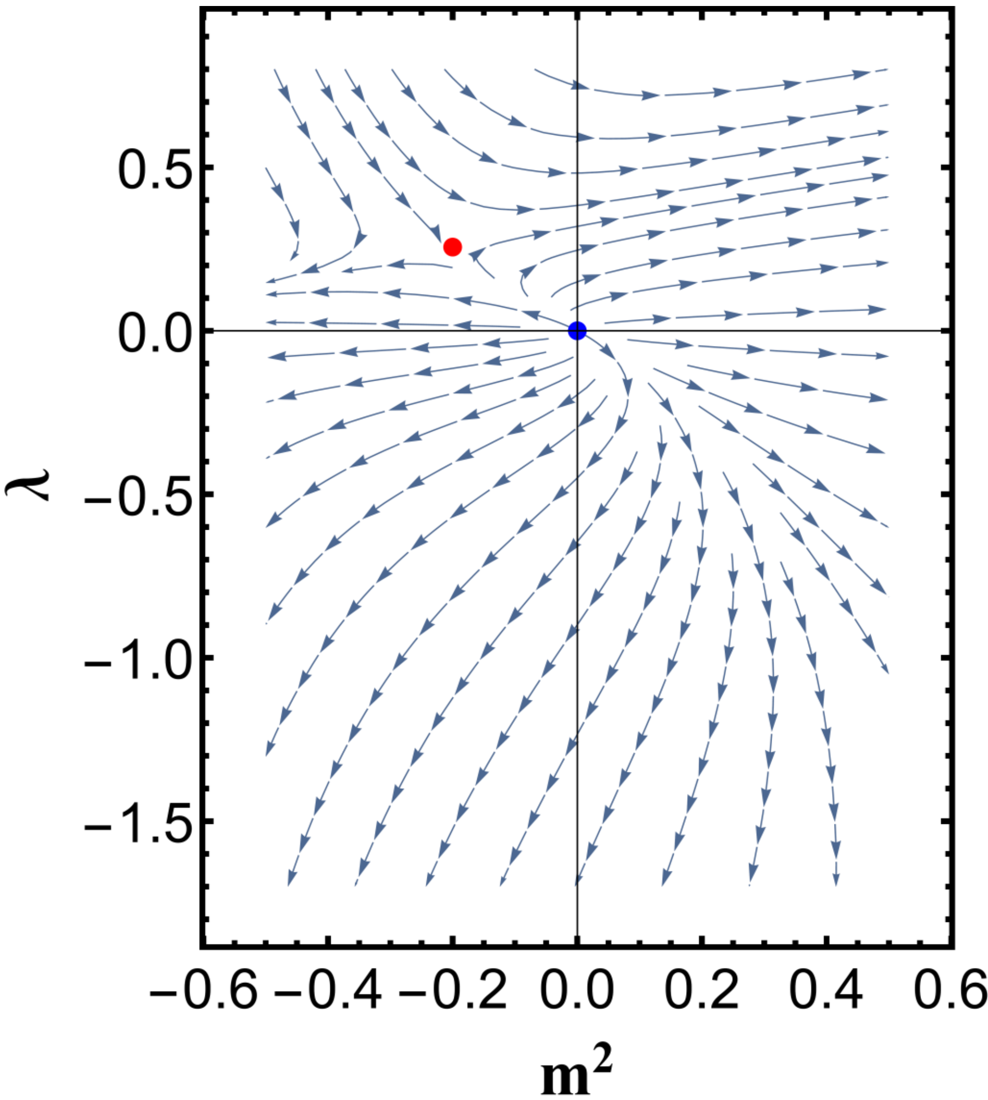
Fixed points

For truncation n=2

$$\partial_t m^2 = (D-2)m^2 - Dm^2 - \frac{\lambda}{(m^2+1)^2}$$

$$\partial_t \lambda = 2(D-2)\lambda - D\lambda + \frac{2\lambda^2}{(m^2+1)^3}$$

D=3



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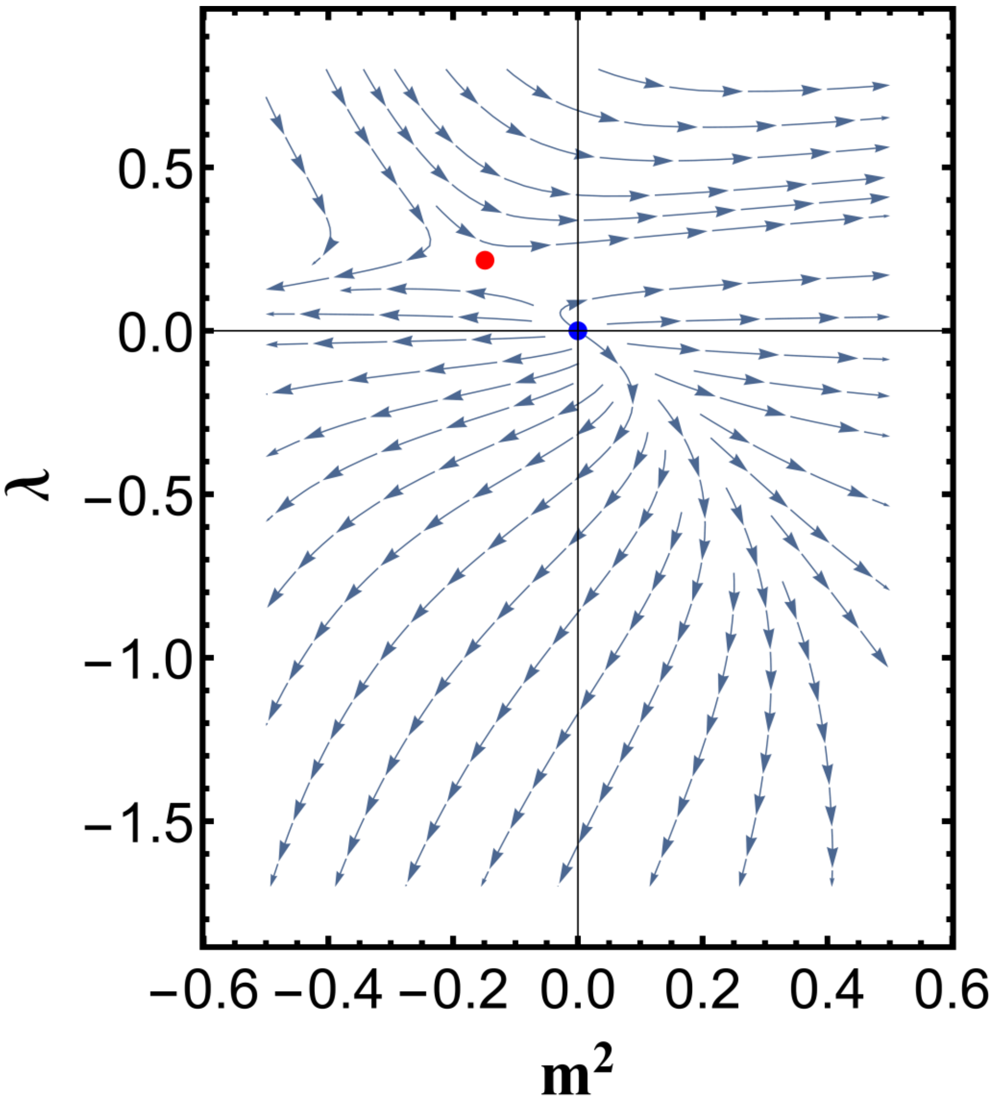
D=3.3



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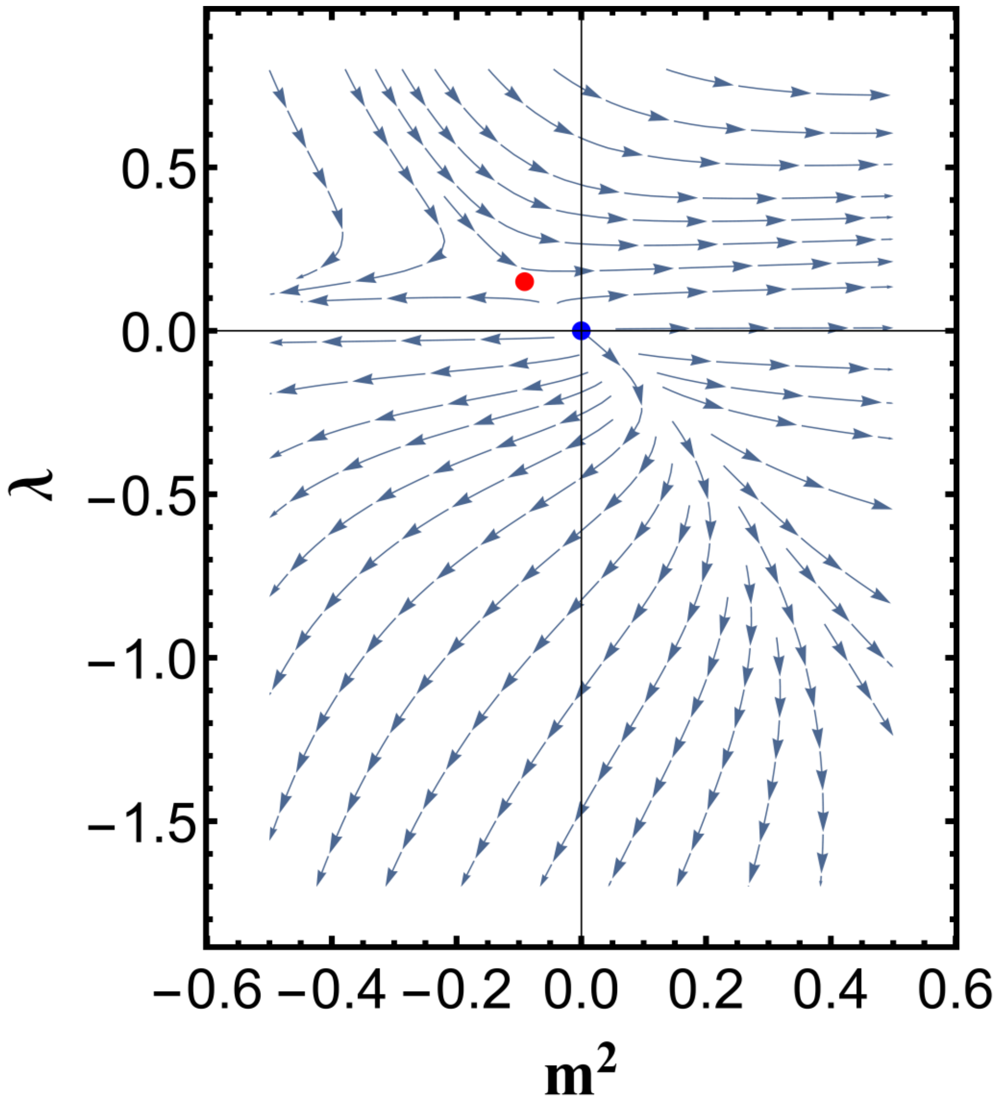

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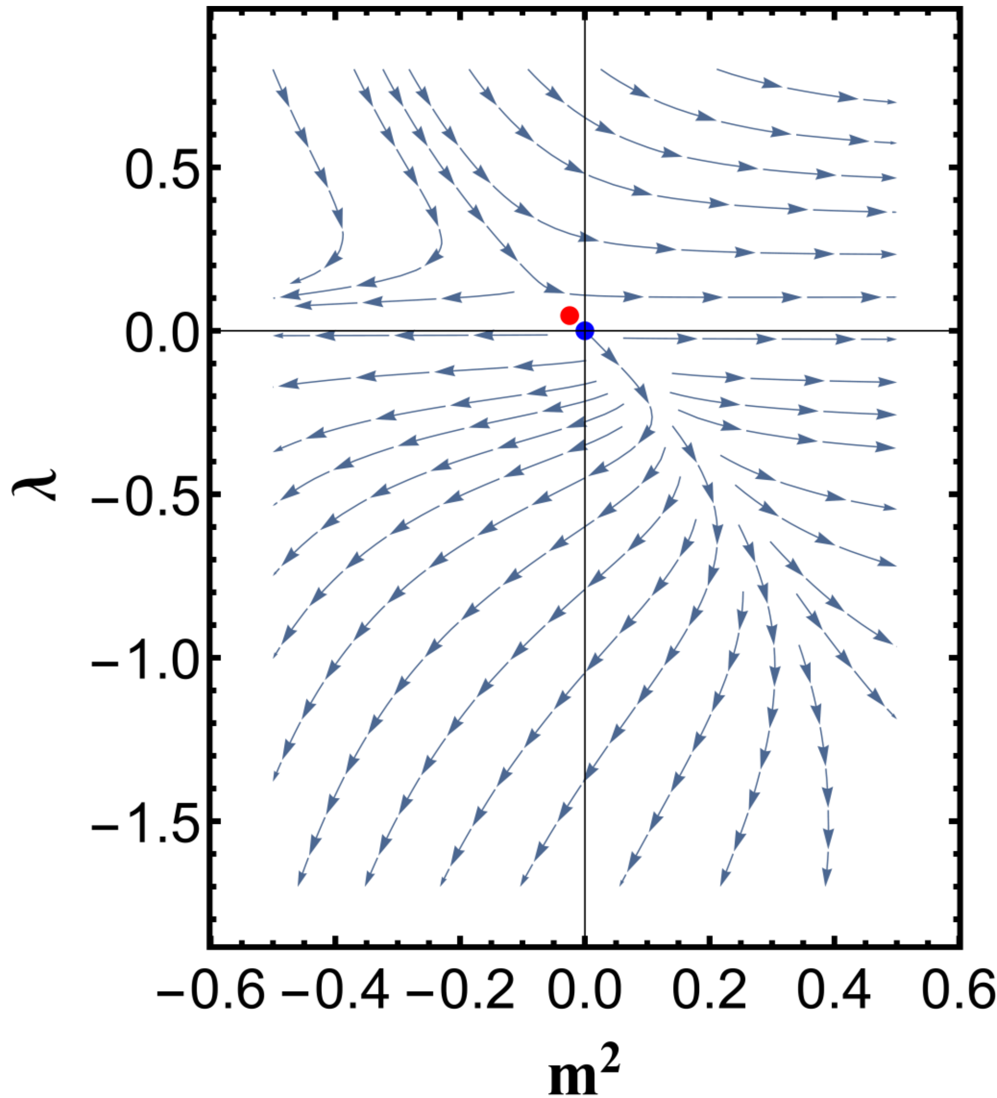
D=3.9



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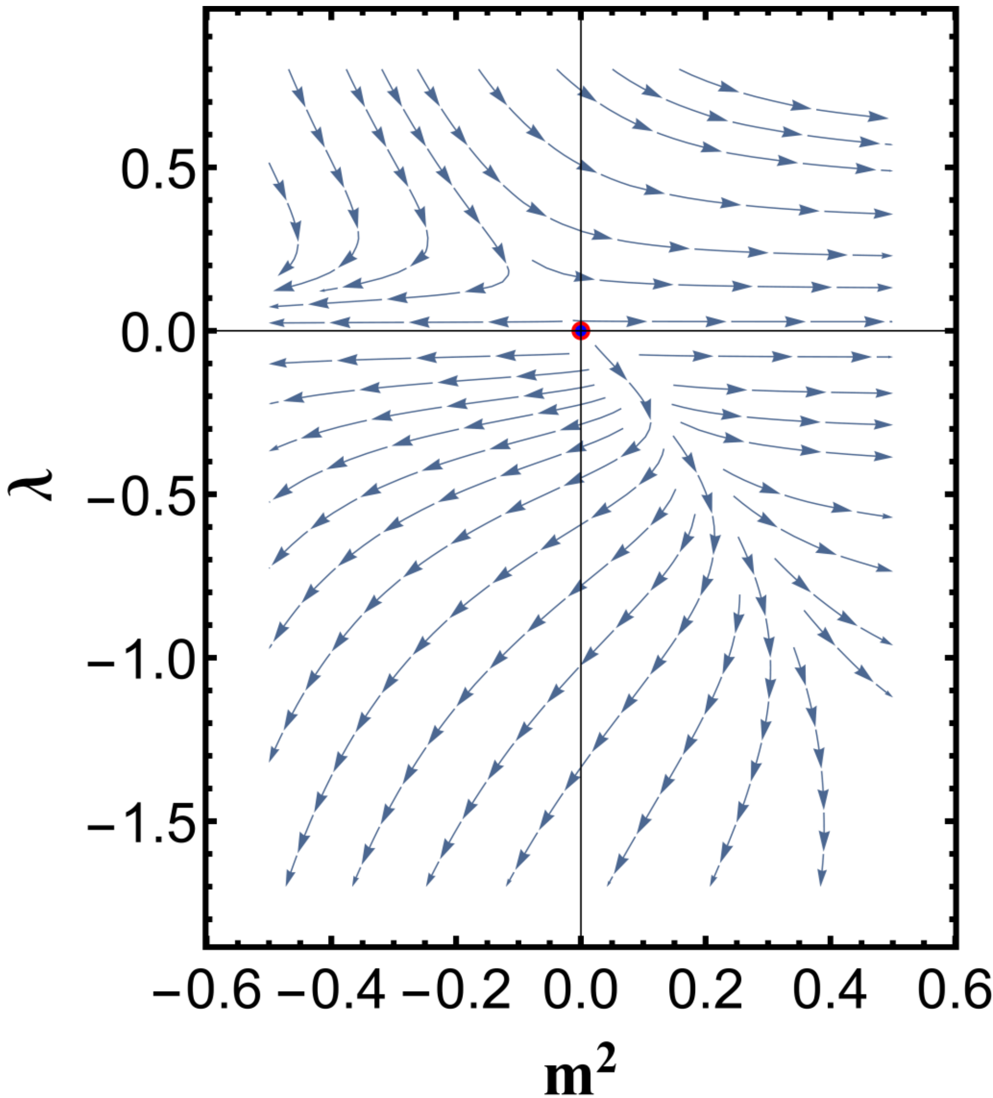
D=4



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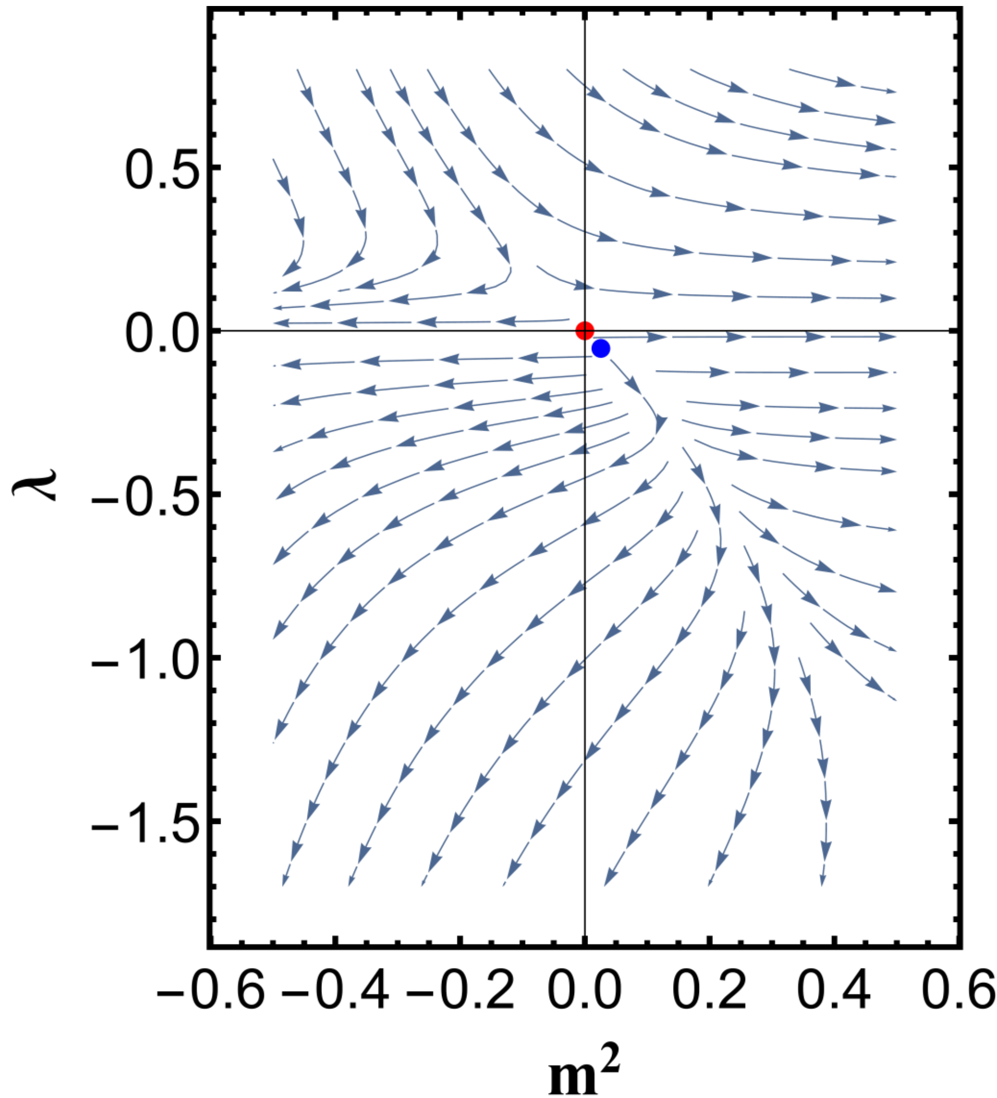
D=4.1



Fixed points

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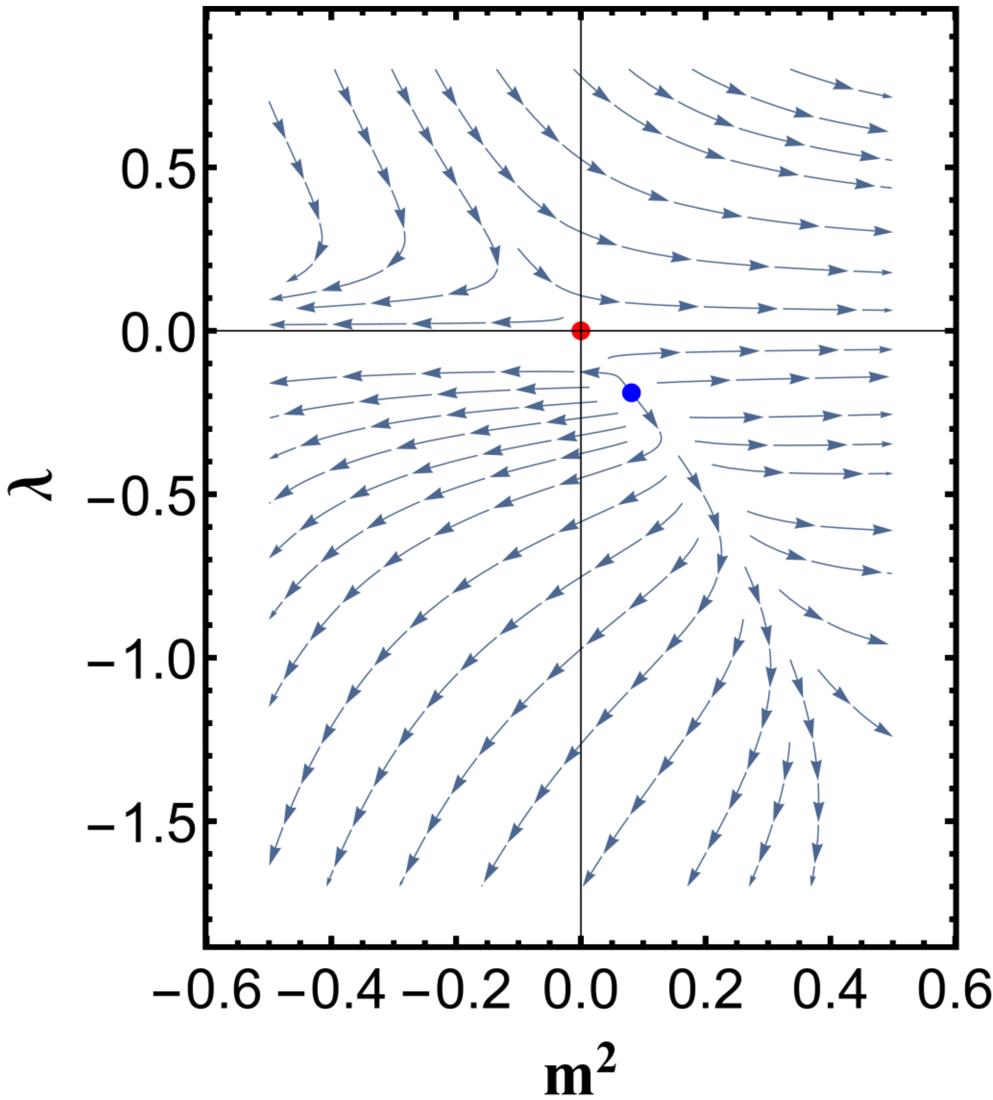

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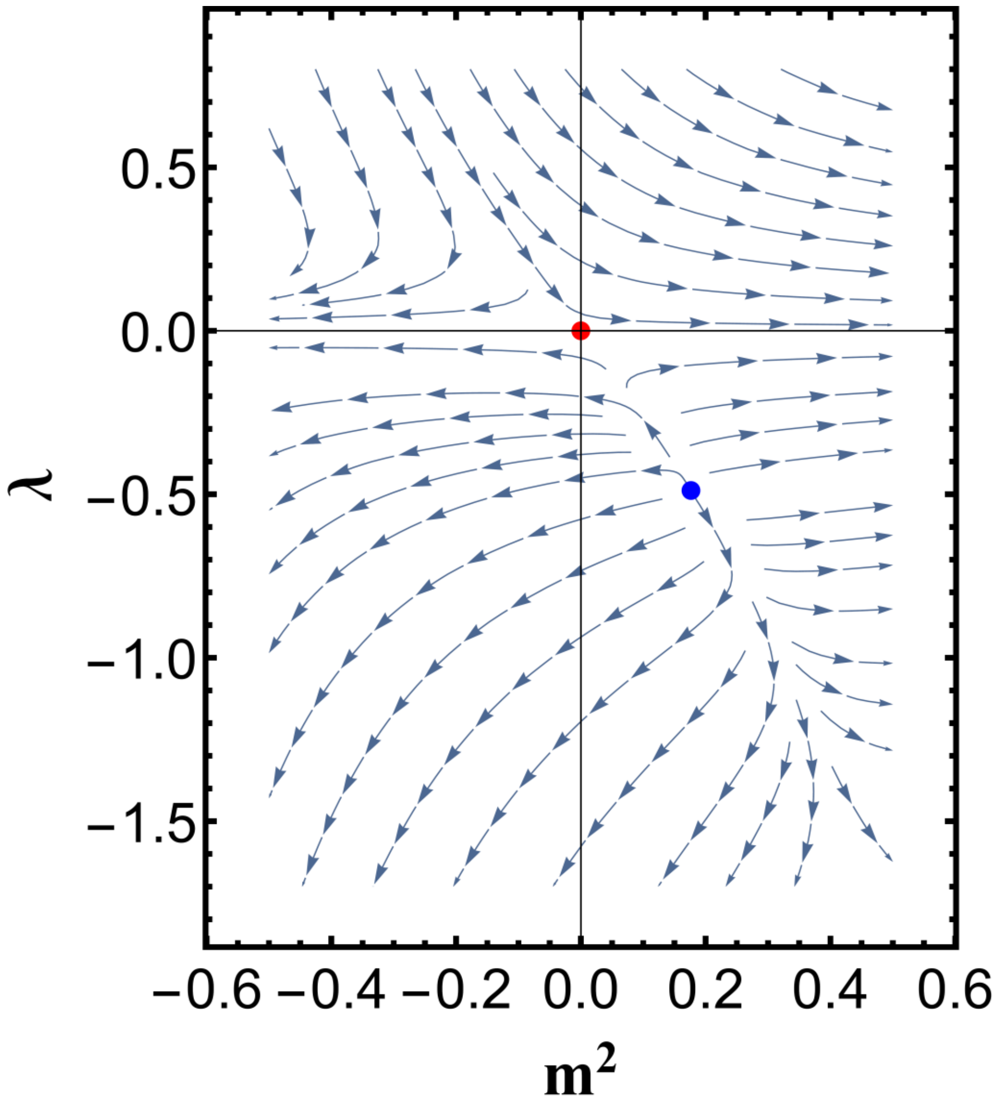

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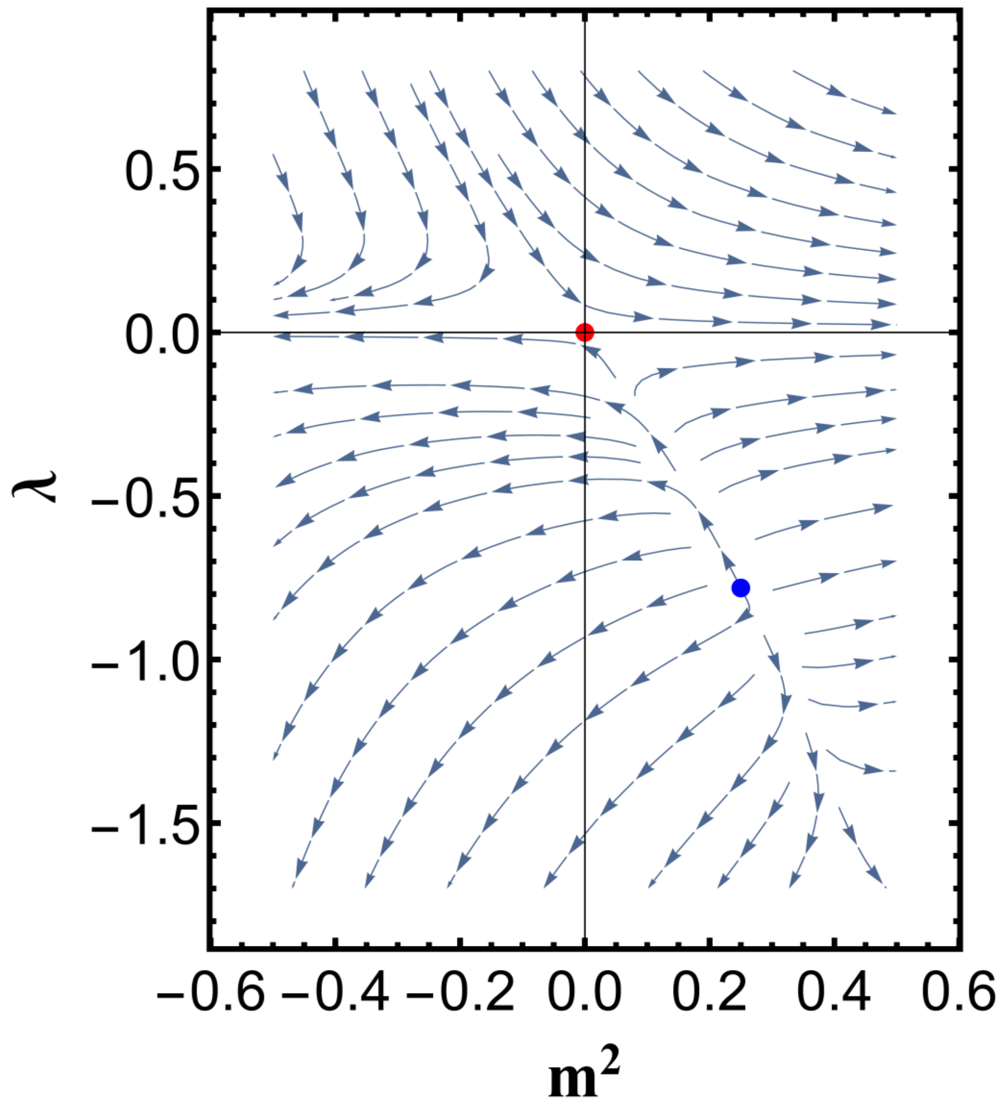
D=4.8



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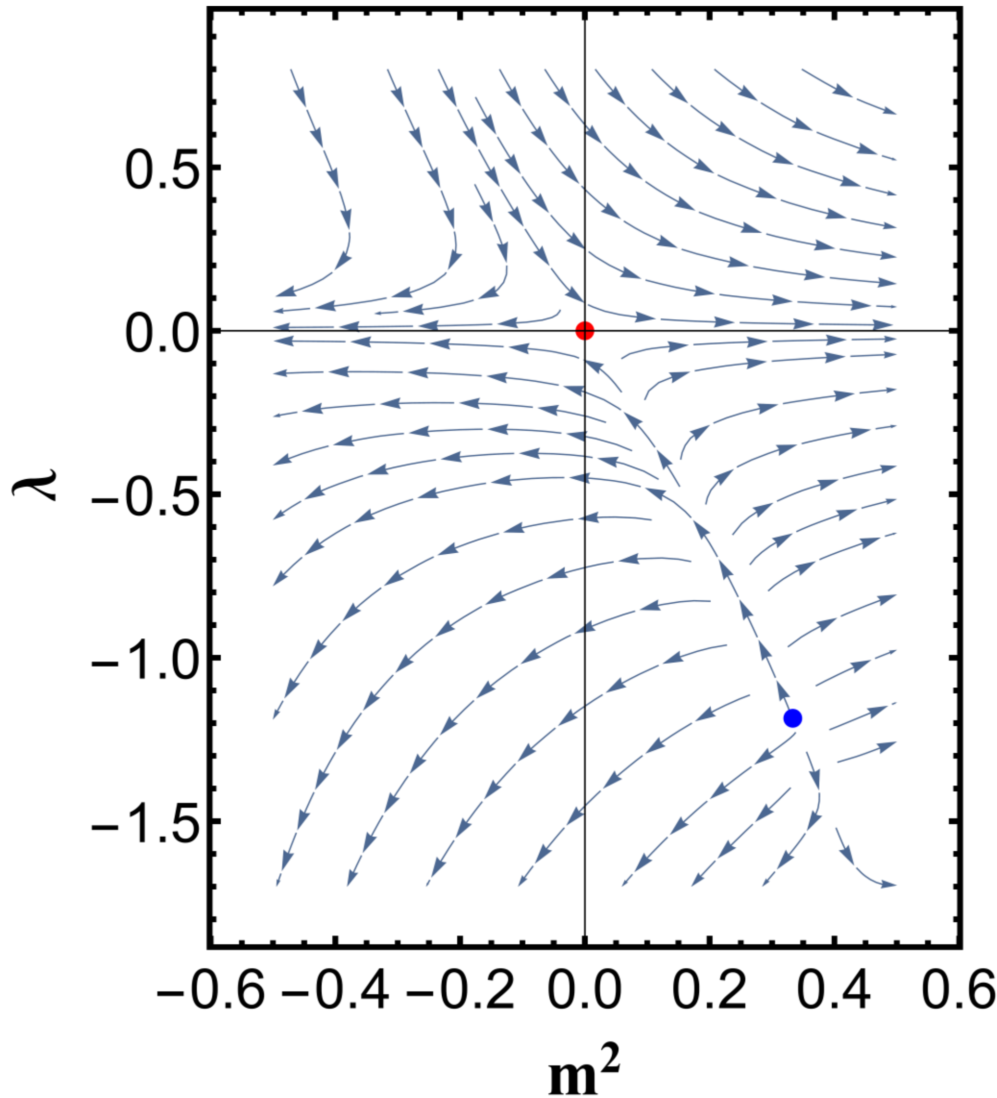
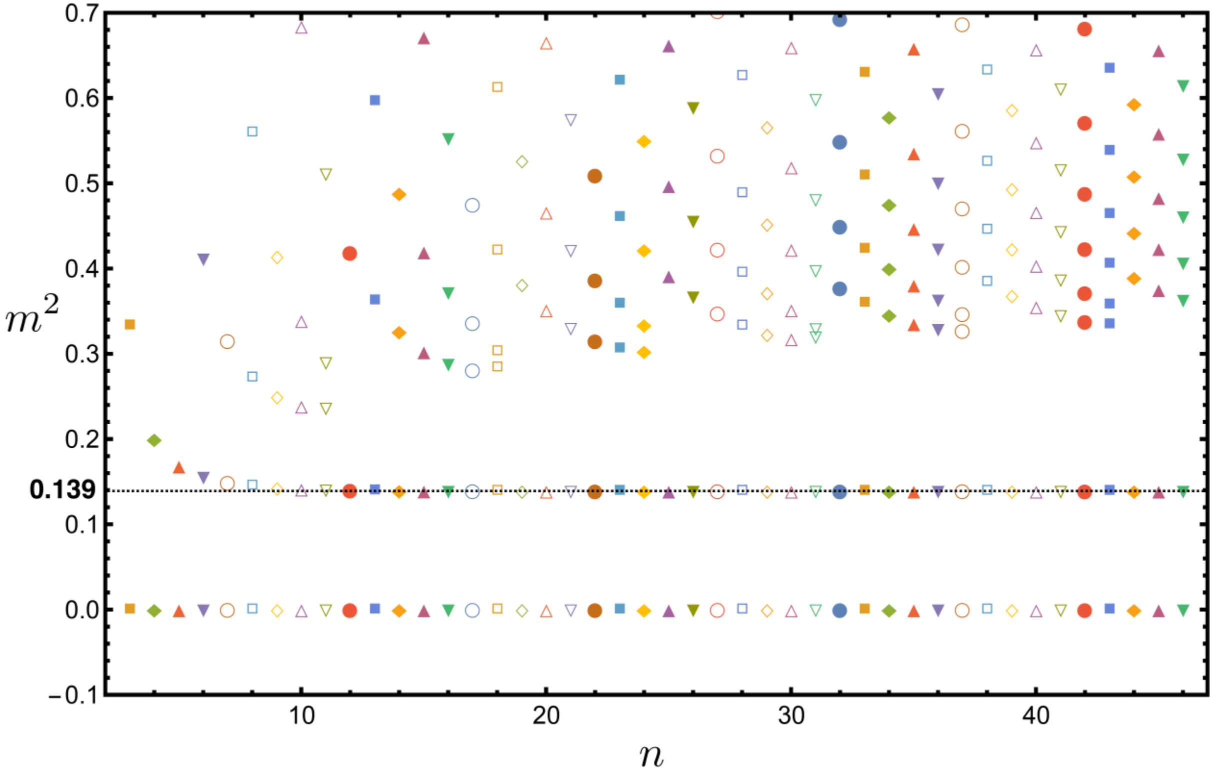
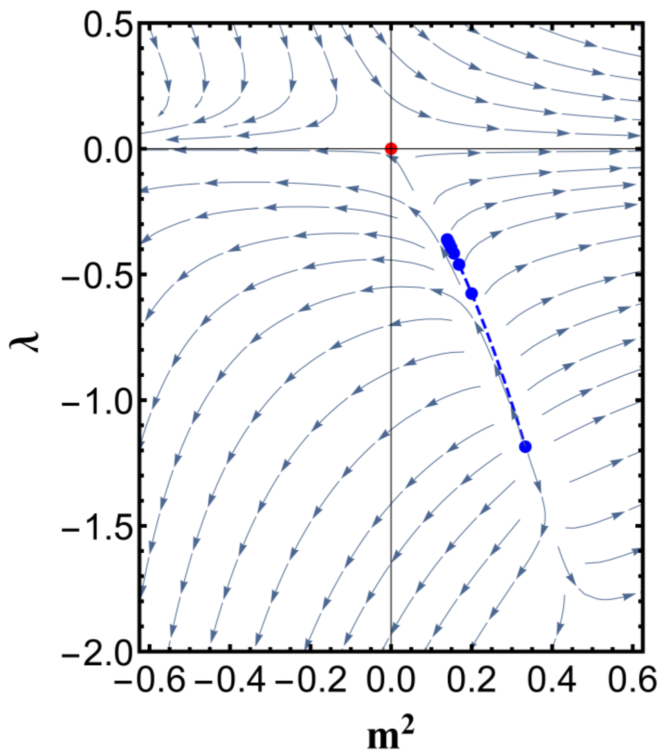
D=5



Fixed points

For truncation n=48 *D=5*





Converging to $\{m_*^2, \lambda_*\} \approx \{0.1392, -0.3613\}$

P. Mati, Phys.Rev. D 91, 125038 (2015)

Fixed points

Exact solution

R. Percacci, G. P. Vacca, Phys.Rev. D 90, 107702 (2014)

An exact solution can be worked out for the **derivative** the flow eq. using the method of characteristics

$$\partial_t u = (D-2)\rho u' - Du + \frac{1}{1+u'}$$

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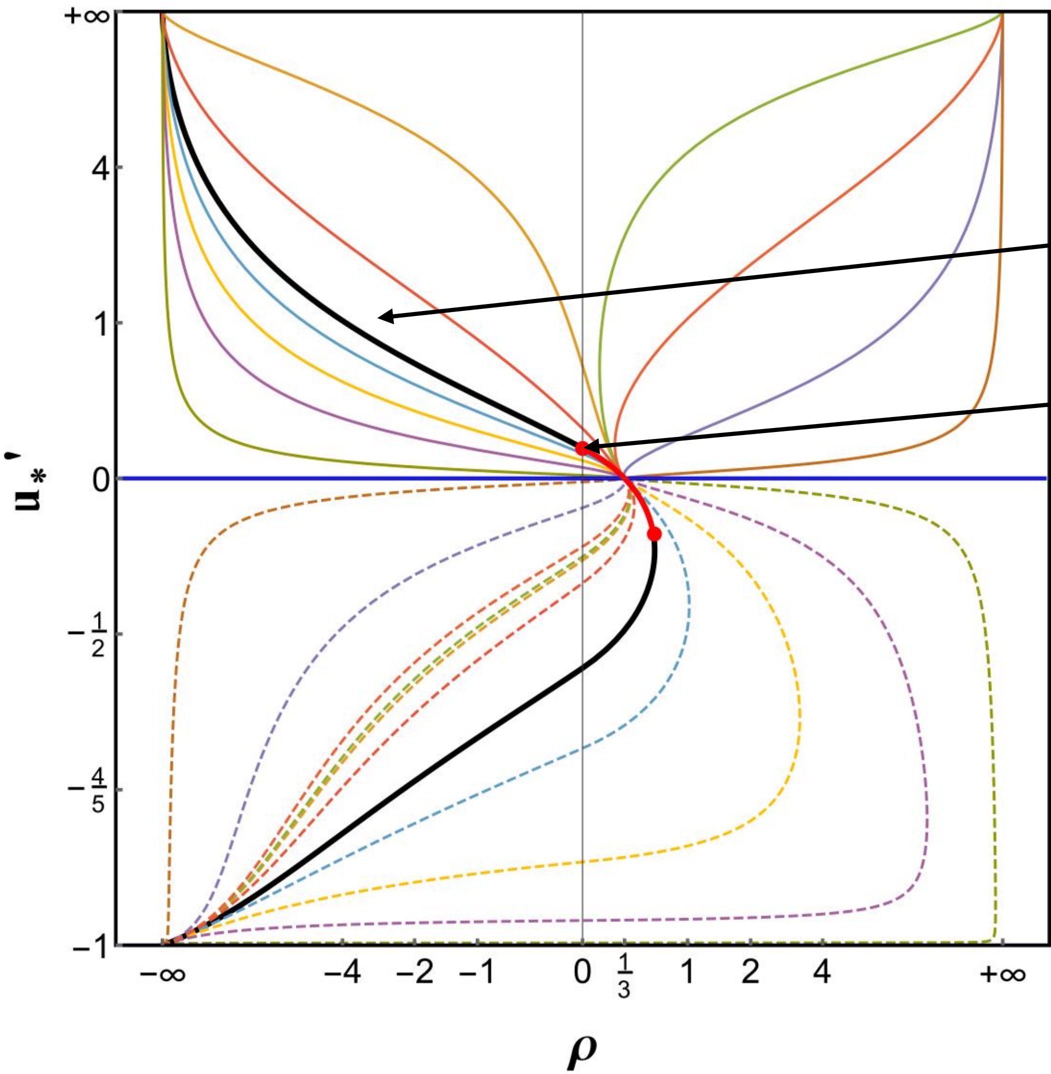
$$\rho = c {u'_*}^{\frac{D}{2}-1} + \frac{1}{(D-2)} \, {}_2F_1\left(2, 1-\frac{D}{2}; 2-\frac{D}{2}; -u'_*\right)$$ D = odd

$$\rho = \bar{c} {u'_*}^{\frac{D}{2}-1} + \frac{1}{(D+2)(1+u'_*)^2} \, {}_2F_1\left(1, 2; 2+\frac{D}{2}; \frac{1}{1+u'_*}\right)$$ D = even

$c, \bar{c}$ constants

Fixed points

D=5



$$\rho = c u_*'^{\frac{D}{2}-1} + \frac{1}{(D-2)}\,{}_2F_1\left(2, 1-\frac{D}{2}; 2-\frac{D}{2}; -u_*'\right)$$

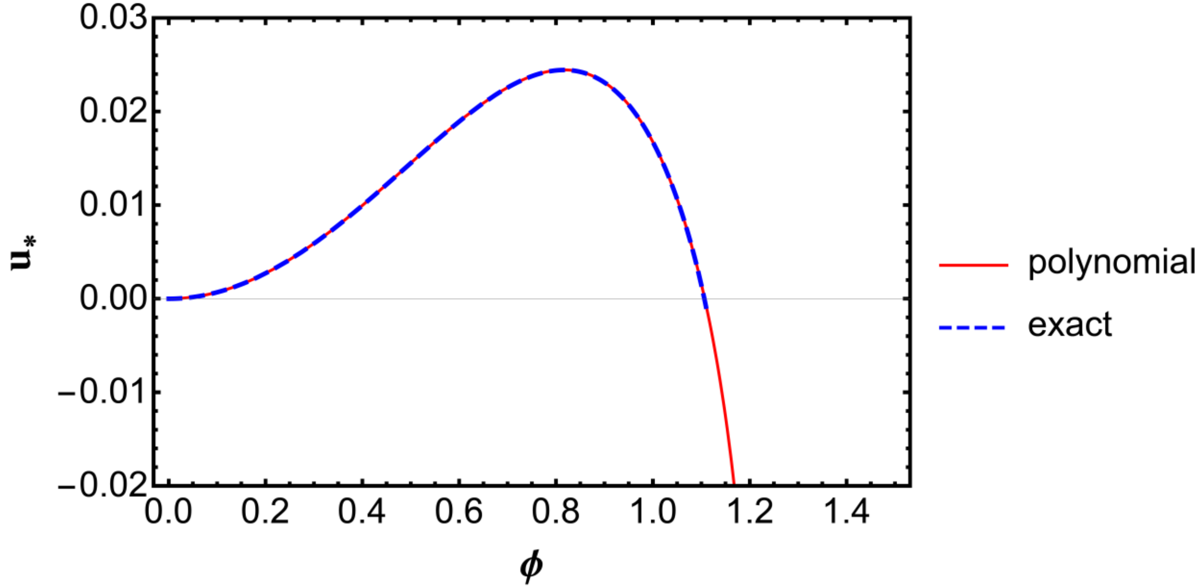
The solution corresponding to c=0

$$u'(0) = m^2 \approx 0.1392$$

This is the solution that is found from the polynomial expansion

Fixed points

The critical potential can be recovered from both the analytic and the polynomial solutions



Metastable / non-analytic

„UV complete" description & AdS/CFT

Klebanov et al. describes the RG flow from another theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3$$

O(N) symmetric theory with N+1 scalars and cubic interactions.
Using this Lagrangian the O(N) symmetric model appears as
an IR fixed point.

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The study was performed in $D = 6 - \epsilon$ and for a sufficient large N and an IR fixed point is found.

L. Fei, S. Giombi, I. R. Klebanov Phys. Rev. D 90, 025018 (2014). [4] — one-loop
L. Fei, S. Giombi, I. R. Klebanov, G. Tarnopolsky Phys. Rev. D 91, 045011 (2015). — three-loop
J. A. Gracey, Phys. Rev. D 92, 025012 (2015) — four-loop

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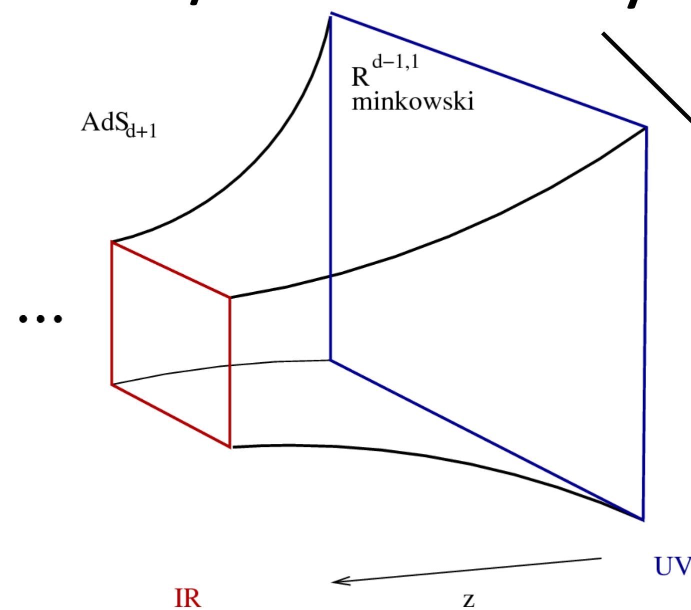
The **existence of interacting CFTs for the O(N) model in** $4 < D < 6$
was proposed.

„UV complete” description & AdS/CFT

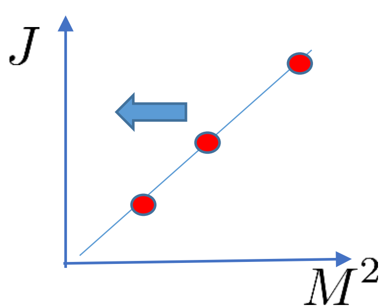
AdS_{D+1} / CFT_D duality

Bosonic Vasiliev theory: classical gravity with massless higher-spin fields in the spectrum



(Fig: J. McGreevy Adv.High Energy Phys. 2010 723105)

Interacting large-N O(N) model at criticality



Conjectured by Polyakov & Klebanov in 2002

Universality & critical exponents

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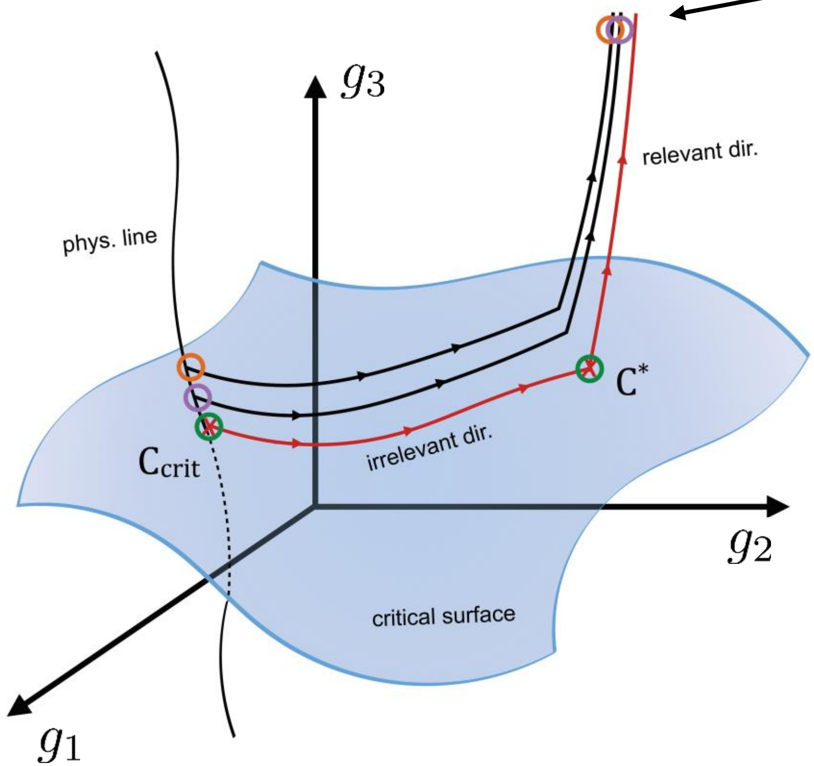

In the vicinity of the fixed point: critical scaling.

The critical scaling exponents can be obtained from the eigenvalues of the stability matrix at the fixed point.

$$B_{ij} = \left. \frac{\partial \beta_i}{\partial g_j} \right|_{g=g^*}$$

Universality & critical exponents

The critical exponent for the correlation length can also be computed via eigenperturbation: we linearize the flow around the fixed point solution

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$$\partial_t \delta u' = 2\frac{u'_*}{u''_*}\left(\partial_\rho - \frac{(u'_* u''_*)'}{u'_* u''_*} - \frac{D-4}{2}\frac{u''_*}{u'_*}\right)\delta u' \qquad (\partial_t \delta u' = \theta \delta u')$$

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This can be solved via the separation of variables.
And the solution close to the node

$$\delta u' \propto e^{t\theta}\left(\rho - \frac{1}{D-2}\right)^{\frac{1}{2}(\theta + D - 2)}$$

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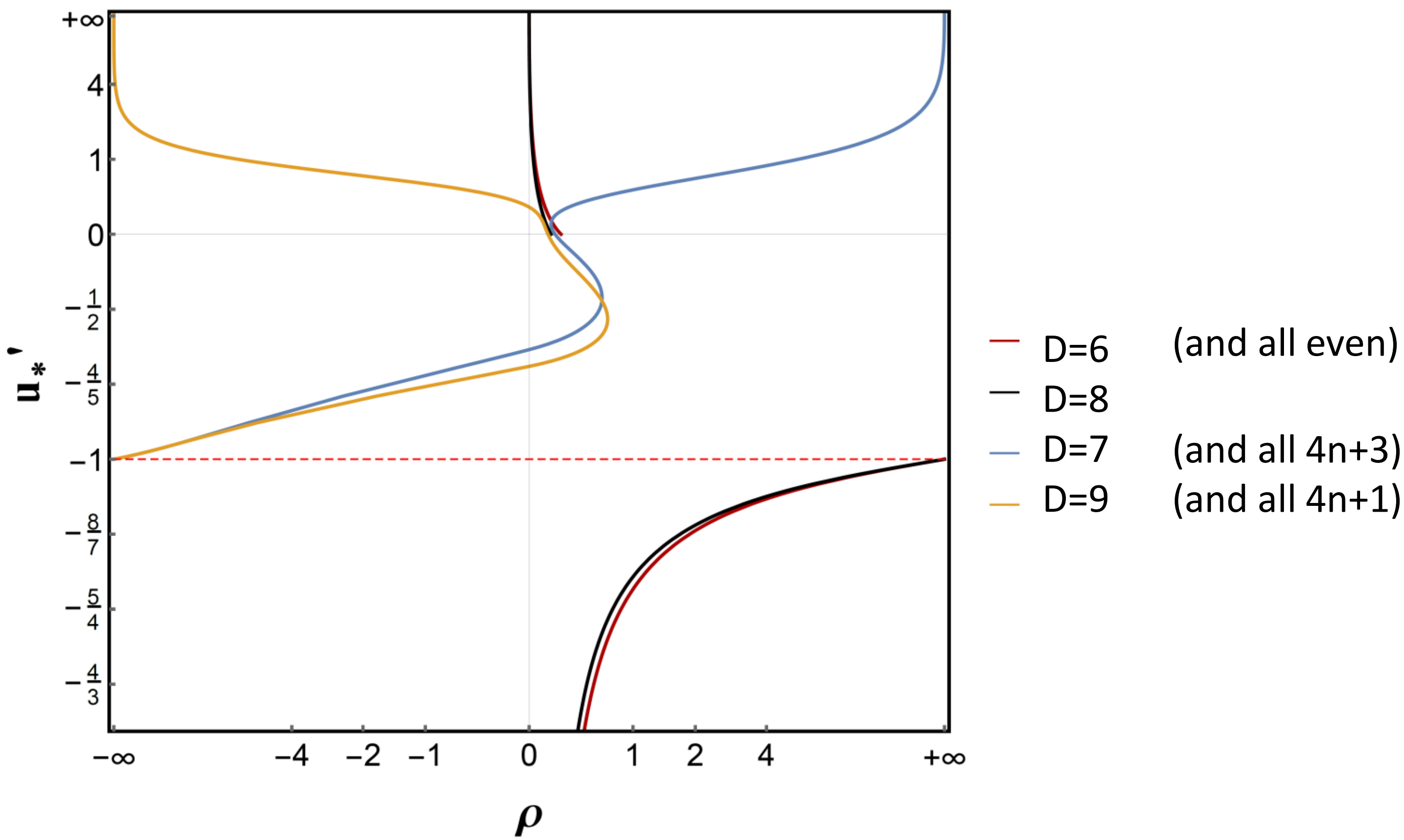
$$\delta u' \propto e^{t\theta}\left(\rho - \frac{1}{D-2}\right)^{\frac{1}{2}(\theta + D - 2)} \xrightarrow{\text{regularity}} \theta = 2(l + 1 - D/2) \xrightarrow{l=0} \nu = \frac{1}{D-2}$$

For D=5 $\nu = \frac{1}{3}$

Higher dimensions



$$\nu(D) = \frac{1}{D-1} \qquad D \geq 4$$

Quantum gravity

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FRG: $\Gamma_k = \int d^d x \frac{\sqrt{\det g_{\mu\nu}}}{16\pi G_k} \left[2\mathbf{\Lambda}_k - R(g_{\mu\nu})\right]$

$$\nu^{-1} = -6 + 4/D + 2D$$

$$\nu(D=4) = 1/3, \quad \nu(D=5) = 0.208, \quad \nu(D=6) = 0.15$$

(K. Falls)

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Lattice: Regge lattice action $\nu(D=4) \simeq 1/3$ higher dim.: no data

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Geometric argument: $\nu(D) = \frac{1}{D-1}$ $\quad D \geq 4$ $\quad\quad \nu(D=4) \simeq 1/3$

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Assuming $\nu_G(D) \simeq \dfrac{1}{D-1}$ $\quad D \geq 4$

And knowing $\nu_O = \dfrac{1}{D-2}$

$$\nu_O(D) \simeq \nu_G(D-1)$$

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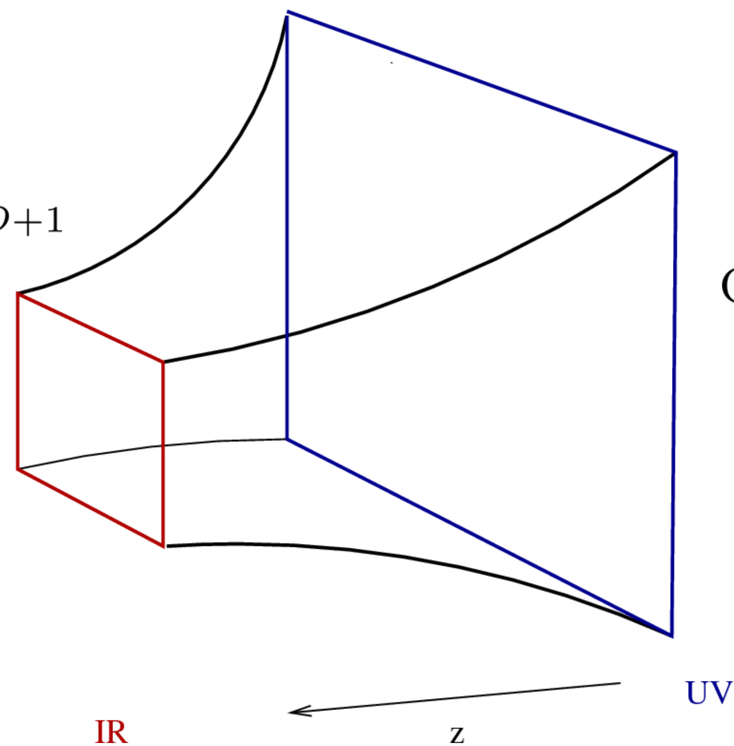
And knowing $\nu_O = \dfrac{1}{D-2}$

Same universality class?
Parisi-Sourlas dimensional reduction: relates a classical field theory in D to a corresponding QFT in D-2.

AdS_{D+1}

Classical gravity theory with higher-spin fields ...



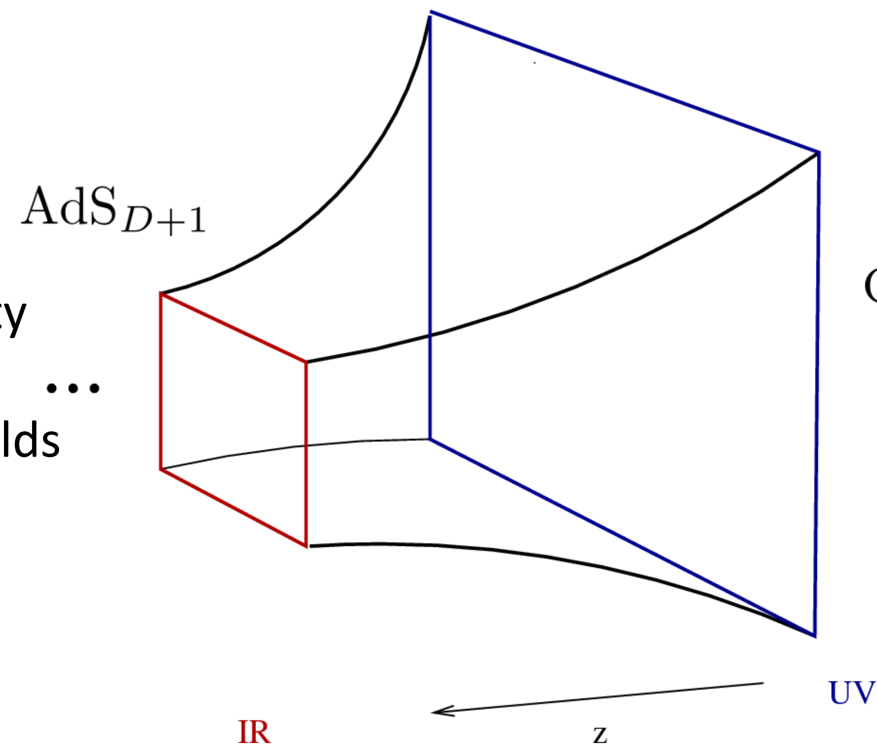
$$\mathrm{CFT}_D = O_D^{\mathrm{crit}}(N) \simeq \mathrm{QEG}_{D-1}$$

Assuming $\nu_G(D) \simeq \frac{1}{D-1} \qquad D \geq 4$

And knowing $\nu_O = \frac{1}{D-2}$

$$\nu_O(D) \simeq \nu_G(D-1)$$

Same universality class?
Parisi-Sourlas dimensional reduction: relates a classical field theory in D to a corresponding QFT in D-2.



Classical gravity theory with higher-spin fields …

$$\mathrm{CFT}_D = O_D^{\mathrm{crit}}(N) \simeq \mathrm{QEG}_{D-1}$$


THANK YOUFOR YOUR ATTENTION