Tree-level Unitarity and Renormalizability in Lifshitz Scalar Theory

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Purpose

Study **two quantum properties**

1. Renormalizability **2. Unitarity**

in Lifshitz scalar theory

Results

Obtained **1. Modified renormalizability conditions**

2. Modified tree-level unitarity conditions

The renormalizability conditions are equivalent to the tree-level unitairty conditions

Content

1.Introduction

1. Introduction

Conjecture

C.H.Llewellyn Smith '73
J. M. Cornwall etal '73

tree unitarity ⟷ renormalizability
equivalent?

Tree unitarity in Lorentz inv. theory

an scattering amplitude does not grow as $E \to \infty$

$$\mathcal{M} \sim E^{\varepsilon} \ (\varepsilon \leq 0) \qquad E \to \infty$$

$\mathcal{M}$ amplitude $\quad E$ Energy in center of mass

if $\epsilon \leq 0$ → a theory has tree unitarity

this condition is modified in Lifshitz theories

1. Introduction

No counterexample is known

tree unitarity ⟷ renormalizability

example

	tree unitarity	renormalizability
QED	○	○
Y-M theory	○	○
Weinberg-Salam model	○	○
Massive-vector theory	×	×
4-Fermi theory	×	×

No counter-example!

1. Introduction

The conjecture is also **valid** for **Einstein gravity**

Berends & Gastmans '74

tree unitarity ⟷ renormalizability

example

	tree unitarity	renormalizability
graviton-scalar system	×	×
graviton-photon system	×	×
graviton-graviton system	×	×

No counter-example!

1. Introduction

Equivalence between renormalizability and tree unitarity has been studied within framework Lorentz invariant field theory

⇩

It is worth checking whether the equivalence also holds true for more generic field theory

Lifshitz-type field theory, Non-Commutative field theory…..

1. Introduction

In our work, we have studied Lifshitz-type field theory in which Lorentz symmetry is violated

⇩

- **The equivalence is originated from quantum theory?**
- **Tree unitarity can be tool for checking the renormalizability of less symmetric field theory instead of loop calculation?**

(e.g.)

Horava-Lifshitz gravity

2. Lifshitz Scalar Theory

Lifshitz Scalar Theory

Lifshitz scaling $[x] = -1 \quad [t] = -z$ **in mass dim**

$\vec{x} \mapsto b\vec{x}$ **:** b **arbitrary number**

$t \mapsto b^z t$ **:** z **dynamical critical exponent**

z **:** degree of anisotropy between space and time

$$S_{LS} = S_{free} + S_{int}$$

$$S = \int dt d^d x \left[\frac{1}{2} \phi \left\{ \partial_t^2 - f(-\triangle) \right\} \phi + \mathcal{L}_{int} \right] \qquad \Delta := \partial^i \partial_i$$

$$f(-\triangle) = (-\triangle)^z + \cdots$$

dispersion relation

$$E = \sqrt{f(p^2)}$$

allow higher spatial derivative

$z = 3$

improve UV behavior $\sim \dfrac{1}{E^2 - p^6}$

$z = 1 \rightarrow$ **isotropic**

$z \neq 1 \rightarrow$ **~~Lorenz symmetry~~**

2. Lifshitz Scalar Theory

The dimension of the scalar field is depend on z and d

$$[x] = -1 \qquad [t] = -z$$ **in mass dim**

$$[\phi] = \frac{d-z}{2}$$

(i). $d = z$ $\quad [\phi] = 0$

(ii). $d < z$ $\quad [\phi] < 0$

(iii). $d > z$ $\quad [\phi] > 0$

Coupling constant

$$[\lambda] = 0$$

In the case of (i). and (ii).,

conventional Power-Counting Renormalizable (PCR) condition is **not enough**

to check the renormalizability of the Lifshitz scalar (-type) field theory

We show that

To Judge Renormalizability of Lifshitz-type theory needs modified PCR condition

↓

Extended PCR condition

3.Renormalizability

(i). $d = z$ $[\phi] = 0$

Investigate **one-loop** structure

but, I will skip this

(ii). $d < z$ $[\phi] < 0$

studied by using **superficial degree of divergence**

3.Renormalizability

Conventional PCR

$$[x] = -1 \qquad [t] = -z$$

$$[p] = 1 \qquad [E] = z$$

second order action

$$S_2 = \int dt d^d x \phi(-\partial_t^2 - (-\triangle)^z)\phi$$

$$[p] = 1, \quad [E] = z, \quad [\phi] = (d-z)/2$$

$$[dt] + d[dx] + 2[\partial_t] + 2[\phi] = 0$$

interaction term

$$S_{int} = \lambda \int dt d^d x \partial_x^a \phi^b$$

$$\begin{aligned}[][\lambda] &= -[dt] - d[dx] - a[\partial_x] - b[\phi] \\ &= z + d - a - b(d-z)/2 \qquad \geq 0\end{aligned}$$

conventional PCR condition

3.Renormalizability

Nonrenormalizable term with conventional PCR

Example d=3, z=5

$$S_2 = \int dt d^3x \phi(-\partial_t^2 - (-\triangle)^5)\phi \qquad [\phi] = -1$$

$[\lambda] = 0$ (satisfies the conventional PCR)

$$S_{int} = \lambda \int dt d^3x \, \phi^2 (\triangle^3 \phi)^2$$

$$[\lambda] = -[dt] - 3[dx] - 12[\partial_x] - 4[\phi] = 0$$

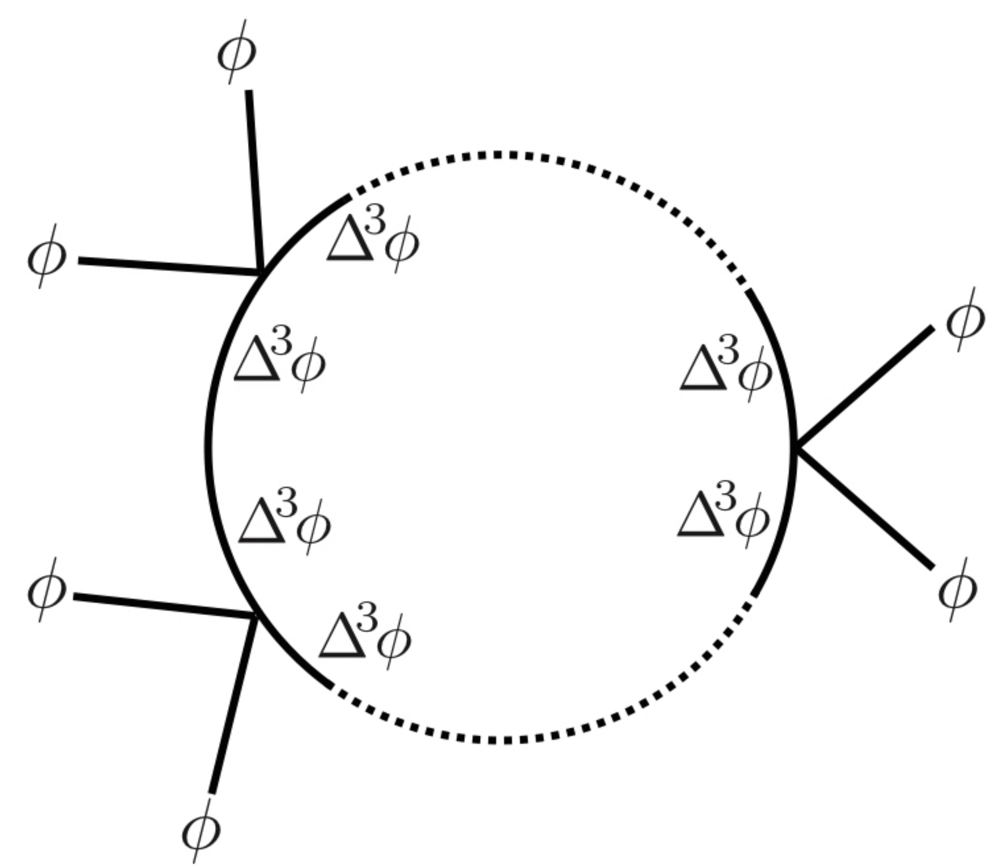
-5 -1 1 -1

1-loop 2n-point function

$$\int d\omega d^3k (\frac{1}{\omega^3 - p^{10}})^n (p^{12})^n \sim \Lambda^{8+2n}$$

For any n, this diverges

infinite number of counter terms are required

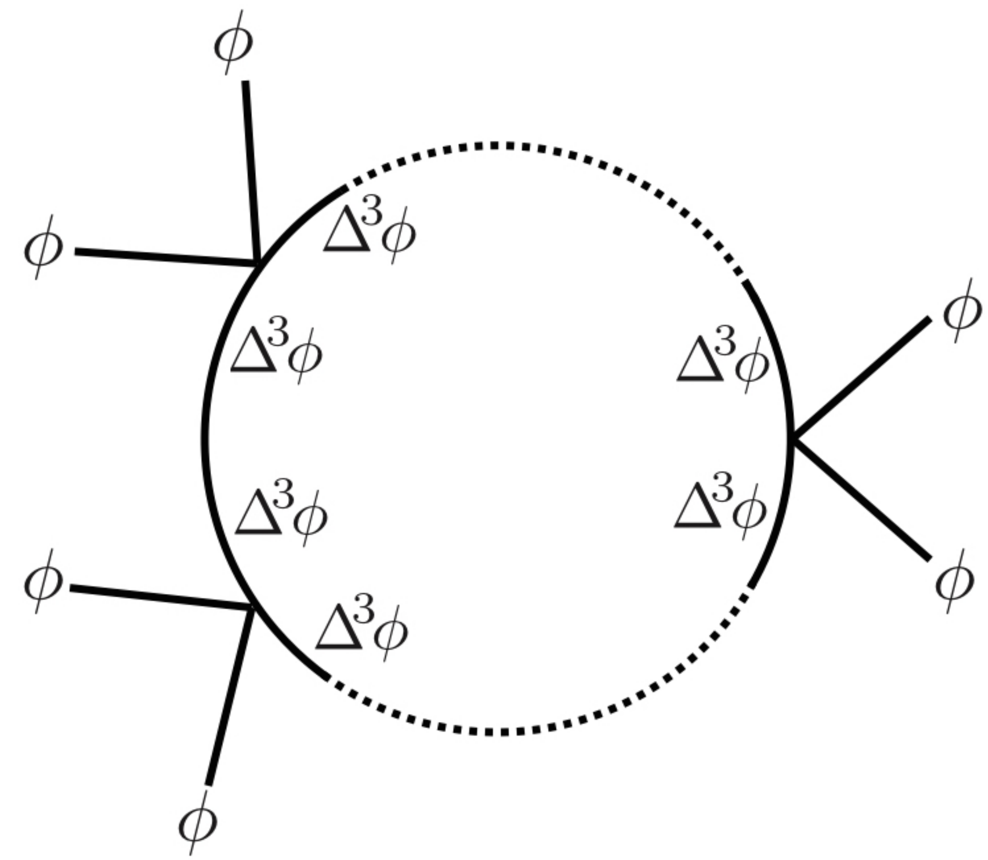


3.Renormalizability

Extended PCR

Check the divergence structure of the loop diagram

Only internal lines are important



$$S_{int} = \lambda \int dtd^3x \; \phi^2 (\triangle^3 \phi)^2$$

-8 rerated to loop

with d=3, z=5, if dimension of operator became **8**
it is marginal

However, $(\triangle^3\phi)^2$, which is the part contributes to loop calculation is dimension **10**

Therefore, this interaction term lead to **non-renormalizable**

Any part must be less than **dimension 8**

inverse of $[dtd^3x]$

3.Renormalizability

Extended PCR

From the following interaction term

$$S_{int} = \lambda \int dt d^3x \, \phi^2 (\triangle^3 \phi)^2 \qquad -[dt d^3 x] > (a_3 + [\phi] + a_4 + [\phi])$$

we can describe the Extended PCR condition for quartic interaction

1. whole

$$S_4 = \int dt d^d x \, (\partial_x^{a_1} \phi) \, (\partial_x^{a_2} \phi) \, (\partial_x^{a_3} \phi) \, (\partial_x^{a_4} \phi)$$

2. a portion (i)

3. a portion (ii)

$a_1 \leq a_2 \leq a_3 \leq a_4$

1. whole: $z + d \geq a_1 + a_2 + a_3 + a_4 + 4[\phi]$ → $a_1 + a_2 + a_3 + a_4 \leq 3z - d$

2. a portion (i): $z + d > a_2 + a_3 + a_4 + 3[\phi]$ → $a_2 + a_3 + a_4 \leq (5z - d - 1)/2$

3. a portion (ii): $z + d > a_3 + a_4 + 2[\phi]$ → $a_3 + a_4 \leq 2z - 1$

→ **Extended PCR**

a_1, a_2, a_3, a_4, z, d are integers

3.Renormalizability

Extended PCR

$$S_3 = \int dt d^d x \left(\partial_x^{a_1}\phi\right)\left(\partial_x^{a_2}\phi\right)\left(\partial_x^{a_3}\phi\right), \quad a_1 \le a_2 \le a_3$$

we can describe the Extended PCR condition for cubic interaction

Extended PCR

$$a_1 + a_2 + a_3 \le (5z - d)/2$$

$$a_2 + a_3 \le 2z - 1$$

a_1, a_2, a_3, z integers

4. Tree-level Unitarity

4. Tree-level unitarity

Optical theorem originated from unitarity of S-matrix

(a) Unitarity of S-matrix $\mathrm{S}^\dagger \mathrm{S} = 1$

(b) Optical theorem $\mathrm{Im}\mathcal{M}_{\mathrm{nn}} = -\pi \Sigma_{\mathrm{n}'} \mid \mathcal{M}_{\mathrm{n}'\mathrm{n}} \mid^2$

cross section

Unitarity bound is obtained from optical theorem

$$\mid \mathcal{M}_{\mathrm{nn}} \mid \geq \mid \mathrm{Im}\mathcal{M}_{\mathrm{nn}} \mid \geq \pi \mid \mathcal{M}_{\mathrm{nn}} \mid^2 \rightarrow \mid \mathcal{M}_{\mathrm{nn}} \mid \leq \frac{1}{\pi}$$

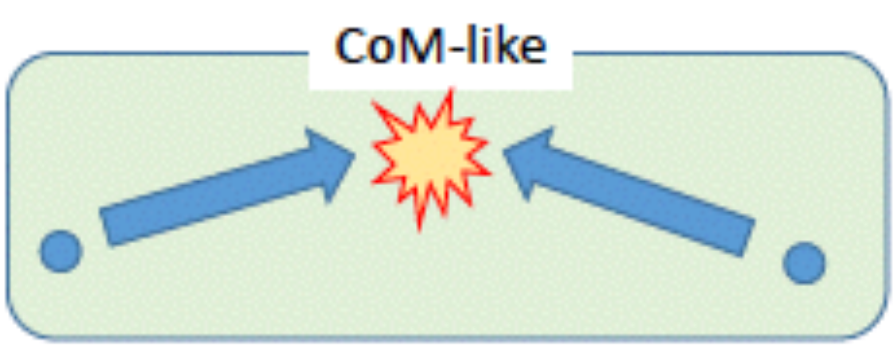
Scattering amplitudes is bounded by using optical theorem originated unitarity condition of S-matrix

4. Tree-level unitarity

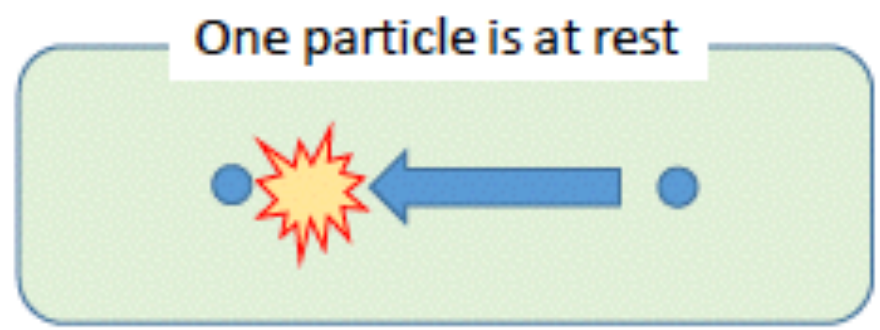
Two scattering states are considered in Lifshitz-type theory

All scattering process are independent due to broken Lorentz symmetry

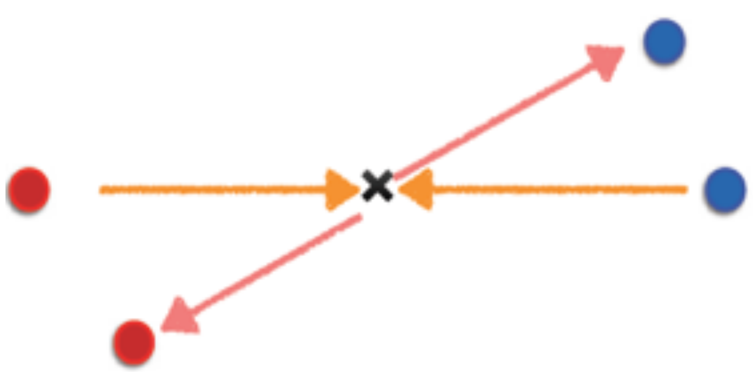
Need to consider even laboratory-like system



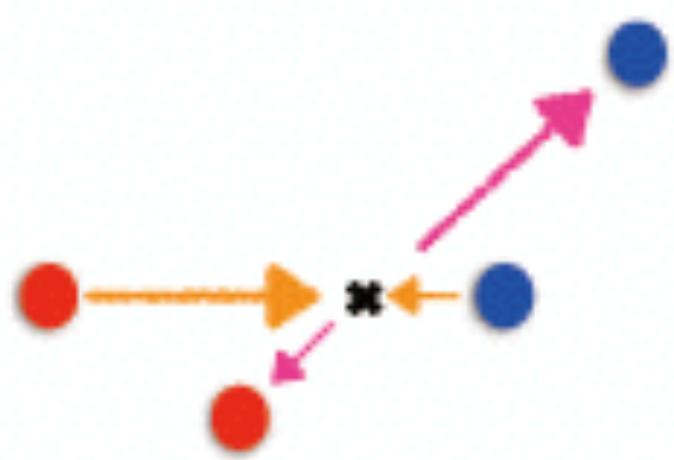
$|\alpha\rangle$



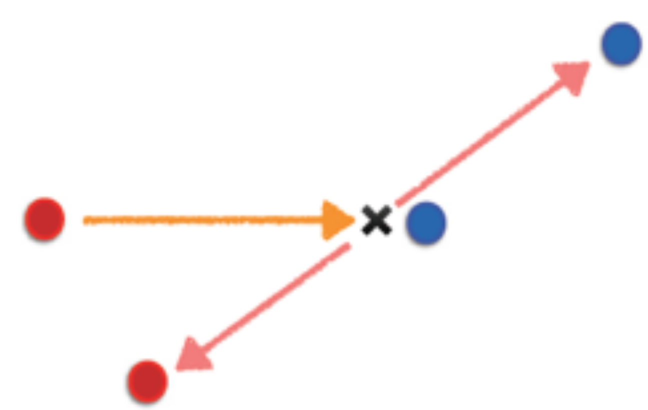
$|\beta\rangle$



$\mathcal{M}(\alpha \to \alpha)$



$\mathcal{M}(\beta \to \beta)$



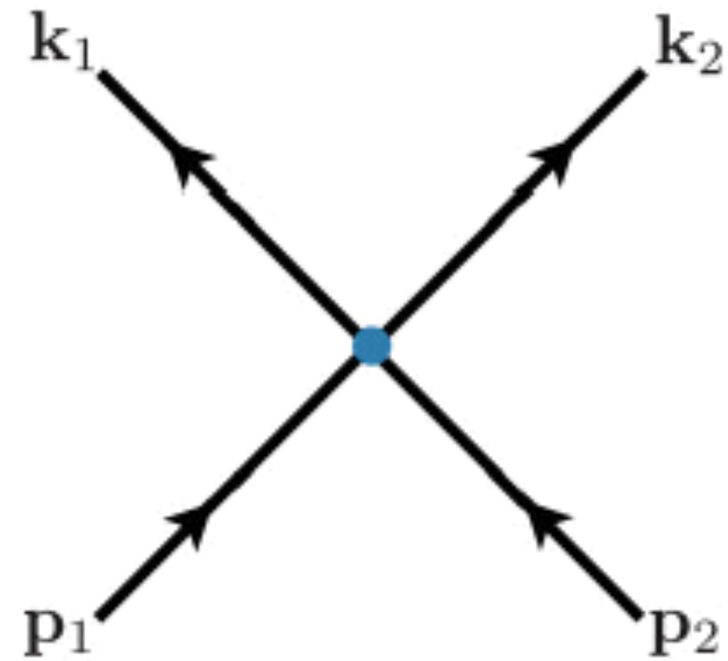
$\mathcal{M}(\alpha \to \beta)$

4. Tree-level unitarity $([\lambda]=0,[\phi]=0)$

Quartic Interactions

$$S_4 = \int dt d^d x \,(\partial_x^{a_1}\phi)(\partial_x^{a_2}\phi)(\partial_x^{a_3}\phi)(\partial_x^{a_4}\phi)$$

$$a_1 \le a_2 \le a_3 \le a_4$$



$\mathcal{M}(\alpha \to \alpha)$

$$\mathcal{M}(\mathbf{p}_1,\mathbf{p}_2 \to \mathbf{k}_1,\mathbf{k}_2) \simeq P^{a_1+a_2+a_3+a_4}.$$

$$a_1+a_2+a_3+a_4 \le 3z-d$$

$\mathcal{M}(\beta \to \beta)$

$$\mathcal{M}(\mathbf{p}_1,\mathbf{p}_2 \to \mathbf{k}_1,\mathbf{k}_2) \simeq P^{a_3+a_4}.$$

$$a_3+a_4 \le 2z-1$$

$$\mathcal{M}(\mathbf{p}_1,\mathbf{p}_2 \to \mathbf{k}_1,\mathbf{k}_2)| = p_1^{a_1} p_2^{a_2} k_1^{a_3} k_2^{a_4} + [\text{ perm. }]$$

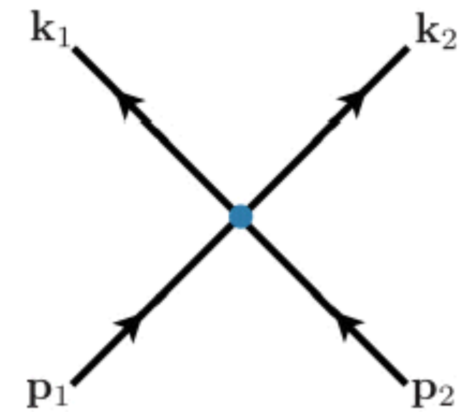
$\mathcal{M}(\alpha \to \beta)$

$$\mathcal{M}(\mathbf{p}_1,\mathbf{p}_2 \to \mathbf{k}_1,\mathbf{k}_2) \approx P^{a_2+a_3+a_4}$$

$$a_2+a_3+a_4 \le (5z-d-1)/2$$

4. Tree-level unitarity $([\lambda]=0,[\phi]=0)$

Examples of quartic interaction $z=d=3$



$$S_4=\int dtd^dx\,(\partial_x^{a_1}\phi)\,(\partial_x^{a_2}\phi)\,(\partial_x^{a_3}\phi)\,(\partial_x^{a_4}\phi)$$

$$a_1\le a_2\le a_3\le a_4$$

$\mathcal{M}(\alpha\to\alpha)$

$$\mathcal{M}(\mathbf{p}_1,\mathbf{p}_2\to\mathbf{k}_1,\mathbf{k}_2)\ \simeq\ P^{a_1+a_2+a_3+a_4}.$$

$$a_1+a_2+a_3+a_4\le 6\quad(\le 3z-d)$$

$\mathcal{M}(\beta\to\beta)$

$$\mathcal{M}(\mathbf{p}_1,\mathbf{p}_2\to\mathbf{k}_1,\mathbf{k}_2)\ \simeq\ P^{a_3+a_4}.$$

$$a_3+a_4\le 5\qquad(\le 2z-1)$$

examples

$a_1=a_2=a_3=0,a_4=6$

○ $\phi^3(\triangle^3\phi)$

more strict

$a_1=a_2=a_3=0,a_4=6$

× $\phi^3(\triangle^3\phi)$

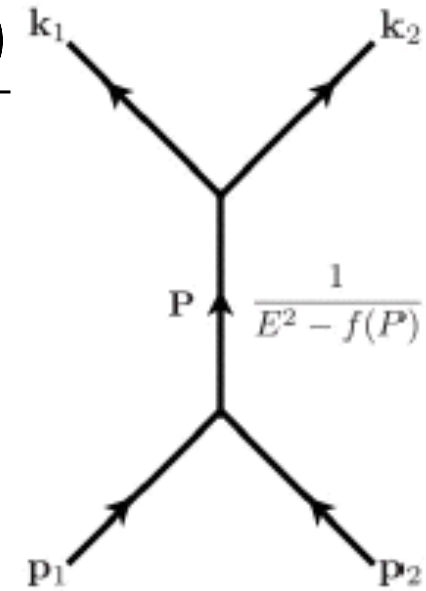
4. Tree-level unitarity $([\lambda]=0,[\phi]=0)$

Cubic Interaction

$$S_3=\int dtd^dx\,(\partial_x^{a_1}\phi)\,(\partial_x^{a_2}\phi)\,(\partial_x^{a_3}\phi)\,,\qquad a_1\le a_2\le a_3$$

$$\mathcal{M}(\mathbf{p}_1,\mathbf{p}_2\to\mathbf{k}_1,\mathbf{k}_2)\ \propto\ \frac{V(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1+\mathbf{p}_2)V(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_2)}{E^2-f(P^2)}$$

$$\sim(p_1^{a_1}p_2^{a_2}P^{a_3}k_1^{a_1}k_2^{a_2}P^{a_3}+\ \text{perm.}\ )/P^{2z}$$



s-channel

$\mathcal{M}(\alpha\to\alpha)$

$$\mathcal{M}(\mathbf{p}_1,\mathbf{p}_2\to\mathbf{k}_1,\mathbf{k}_2)\simeq P^{2a_1+2a_2+2a_3-2z}.$$

$$a_1+a_2+a_3\le(5z-d)/2$$

$\mathcal{M}(\beta\to\beta)$

$$M(\mathbf{p}_1,\mathbf{p}_2\to\mathbf{k}_1,\mathbf{k}_2)\approx P^{2a_2+2a_3-2z}$$

$$a_2+a_3\le 2z-1$$

$\mathcal{M}(\alpha\to\beta)$

$$M(\mathbf{p}_1,\mathbf{p}_2\to\mathbf{k}_1,\mathbf{k}_2)\approx P^{a_1+2a_2+2a_3-2z}$$

$$a_1+2a_2+2a_3\le(9z-d-1)/2.$$

4. Tree-level unitarity

Quartic Interactions

$$S_4 = \int dt d^d x \, (\partial_x^{a_1} \phi) \, (\partial_x^{a_2} \phi) \, (\partial_x^{a_3} \phi) \, (\partial_x^{a_4} \phi)$$

$$a_1 \leq a_2 \leq a_3 \leq a_4$$

Tree unitarity constraints ⟷ **Extended PCR**

$\mathcal{M}(\alpha \to \alpha)$

$a_1 + a_2 + a_3 + a_4 \leq 3z - d$ ⟷ $a_1 + a_2 + a_3 + a_4 \leq 3z - d$

$\mathcal{M}(\beta \to \beta)$

$a_3 + a_4 \leq 2z - 1$ ⟷ $a_3 + a_4 \leq 2z - 1$

$\mathcal{M}(\alpha \to \beta)$

$a_2 + a_3 + a_4 \leq (5z - d - 1)/2$ ⟷ $a_2 + a_3 + a_4 \leq (5z - d - 1)/2$

4. Tree-level unitarity

Cubic Interaction

$$S_3 = \int dt d^d x \, (\partial_x^{a_1} \phi) \, (\partial_x^{a_2} \phi) \, (\partial_x^{a_3} \phi) \, , \qquad a_1 \leq a_2 \leq a_3$$

Tree unitarity constraints ⟷ **Extended PCR**

$\mathcal{M}(\alpha \to \alpha)$

$a_1 + a_2 + a_3 \leq (5z - d)/2$ ⟷ $a_1 + a_2 + a_3 \leq (5z - d)/2$

$\mathcal{M}(\beta \to \beta)$

$a_2 + a_3 \leq 2z - 1$ ⟷ $a_2 + a_3 \leq 2z - 1$

$\mathcal{M}(\alpha \to \beta)$

$a_1 + 2a_2 + 2a_3 \leq (9z - d - 1)/2.$ **no additional condition**

Outlook

Tree unitarity conditions and Extended PCR condition
are identical in Lifshitz (non-relativistic) field theory

↓

It inferred that the equivalence between both the two
hold true for **more large class of field theory**

Thus,

the results of our work can be applicable to study in various field theory,
such as **Horava-Lifshitz gravity, non-commutative field theory…**

Outlook

- **Lifshitz scalar (LS) theory is renormalizable if LS theory is invariant under certain symmetry such as a shift symmetry**

allow finite counterterms

- **tree unitarity is useful to study in proof of the renormalizability even in Lifshitz-type field theory**

↓

- **Horava-Lifshitz gravity is a power-counting renormalizable gravity theory with foliation-preserving diffeomorphisms**

which is more strict than shift symmetry

↓

- **HL gravity is expected to be a renormalizable gravity theory**

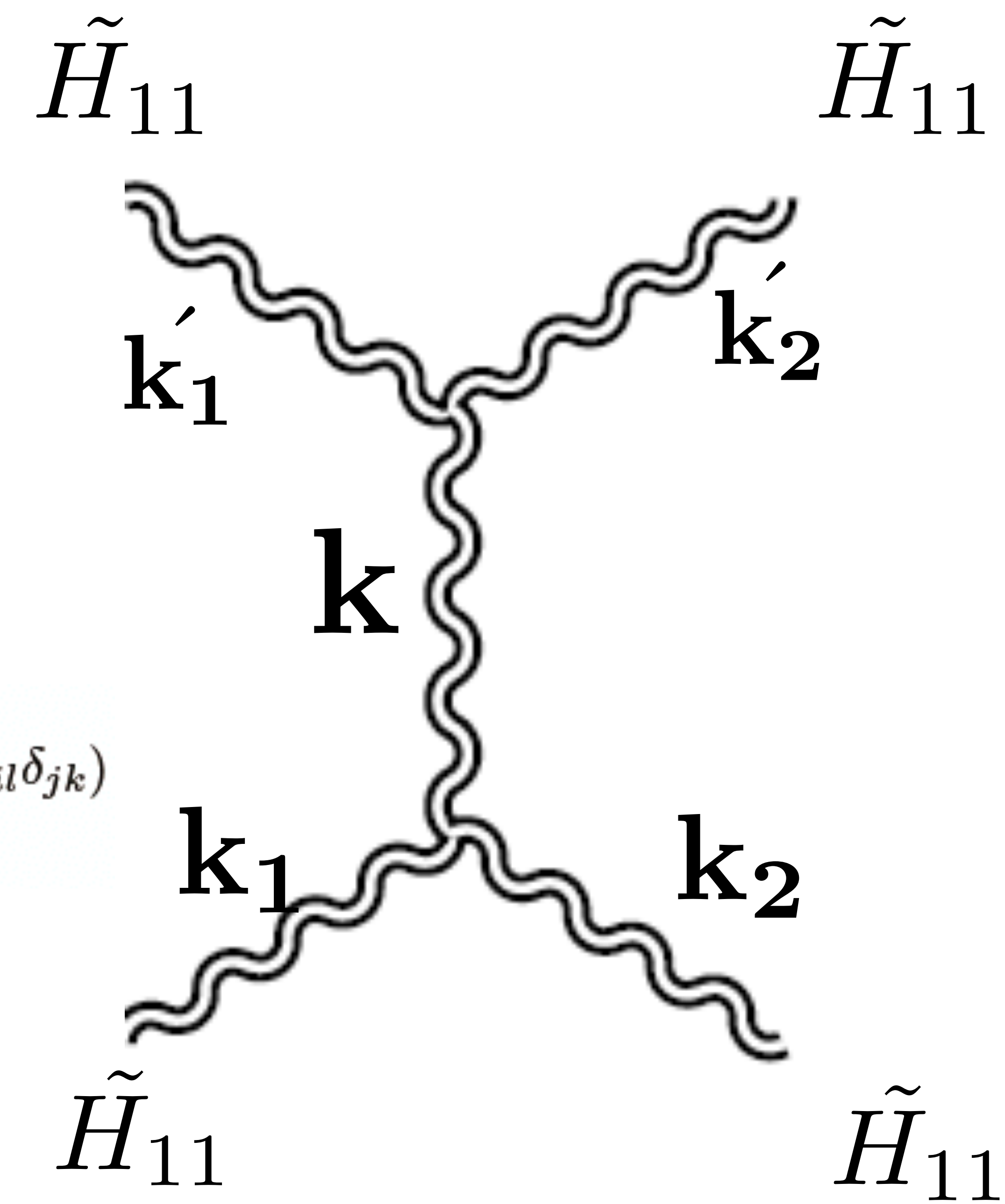
Outlook

$\mathcal{M}(\alpha \to \alpha)$

$$\mathcal{M}(\mathbf{p}_1, \mathbf{p}_2 \to \mathbf{k}_1, \mathbf{k}_2) \simeq P^{2a_1+2a_2+2a_3-2z}.$$

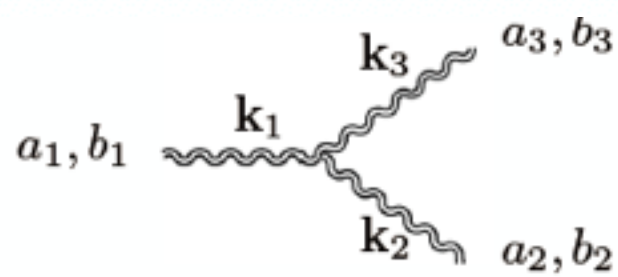
$$a_1 + a_2 + a_3 \leq 6$$

$$\mathcal{M} \sim \mathbf{k}^6$$



$\tilde{H}_{ij}$  $\tilde{H}_{kl}$ $\quad \mathrm{P}^{(\mathrm{G})}_{ij;kl} = \dfrac{\gamma^2}{\omega_{\mathrm{E}}{}^2 - \frac{\gamma^2}{64}\mathbf{k}^6}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$\partial_m \tilde{H}_{ij}\,(\partial^2)\tilde{H}_{jk}\partial_n \tilde{H}_{kl}$



$$v_{1\,(\mathrm{V})}^{HHH} \sim -(\mathbf{k}_1{}^2\mathbf{k}_3{}^2(\mathbf{k}_1)_l(\mathbf{k}_3)_l(\delta_{i(a_1}\delta_{b_1)(a_2}\delta_{b_2)(a_3}\delta_{b_3)i}) + (\mathrm{sym})$$

Summary

1 . Investigated the renoralizability of Lifshitz scalar theory

1-1. formulated Extended PCR condition

2. Investigated the unitarity of Lifshitz scalar theory

2-1. derived the tree-unitarity conditions for
the scattering amplitudes of Lifshitz scalar theory

2-2. constraint on the quartic and cubic interaction
terms from the tree-unitarity conditions

Equivalence between tree unitarity and renormalizability

hold true for Lifshitz (non-relativistic) field theory!!

Thank you!!

Appendices

Unitarity bound for scattering amplitude

$$\mathcal{M}(\mathbf{p}_1, \mathbf{p}_2 \to \mathbf{k}_1, \mathbf{k}_2)| = P^a$$

$$\mathcal{M}(\alpha \longrightarrow \alpha) \qquad a \leq 3z - d$$

$$\mathcal{M}(\beta \longrightarrow \beta) \qquad a \leq 2z - 1$$

$$\mathcal{M}(\alpha \longrightarrow \beta) \qquad a \leq (5z - d - 1)/2$$

Appendices

(i). $d = z \quad [\phi] = 0$

z=3 (1+3)dim

$$\mathcal{L}_{LS} = \mathcal{L}_{free} + \mathcal{L}_{int} \qquad \mathcal{L}_{free} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi\triangle^3\phi$$

$$[dx] = -1 \quad [dt] = -3 \quad \rightarrow \quad [\phi] = 0$$

marginal interaction terms z=3 marginal → up to 6th order ∂_i

$$(\triangle\phi)^3, \phi^2(\triangle\phi)^3, etc \qquad (\triangle^3\phi)\phi^n$$

lead to non-renormalizable?

with shift symmetry ($\phi \longrightarrow \phi + c$)

$$(\triangle\phi)^3, (\triangle^2\phi)(\partial_i\phi)^2, \cdots$$

without shift symmetry

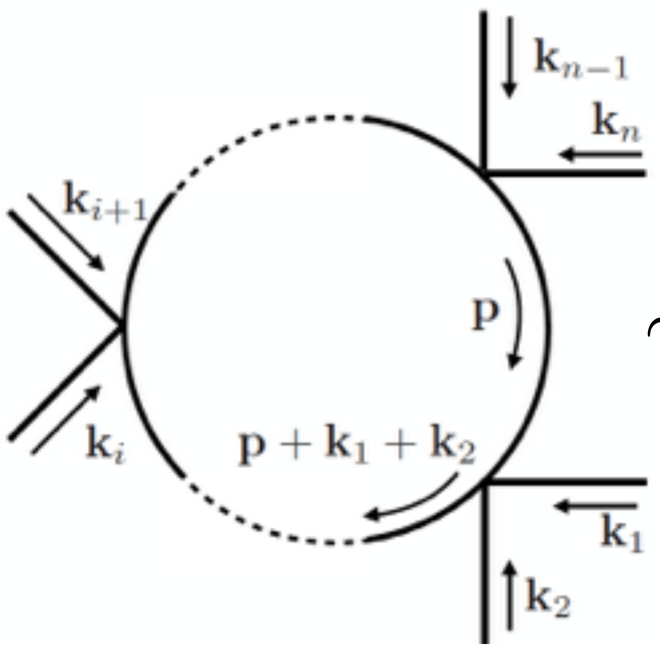
$$\phi^2(\triangle^3\phi), \phi^2(\partial_i\phi)(\partial_i\triangle^2\phi), \cdots$$

Appendices

(i). $d = z$

One loop graph with shift symmetric case

(e.g) **Vertex** $\alpha_4(\triangle^2\phi)^2(\partial_i\phi)^2$



$$\sim \alpha_4{}^{n/2}\left(\prod_{i=1}^{n/2}\mathbf{k}_{2i}\cdot\mathbf{k}_{2i-1}\right)\int d\rho d\theta\rho^{5-n}\left[\sin^{\frac{n}{3}}\theta + O\left(\frac{k}{\rho}\right)\right]$$

$$0 \le \rho \le \infty$$

if $n > 6$**,** **no divergence**

even if $n \le 6$**,** **there are divergence** → **renormalizable**

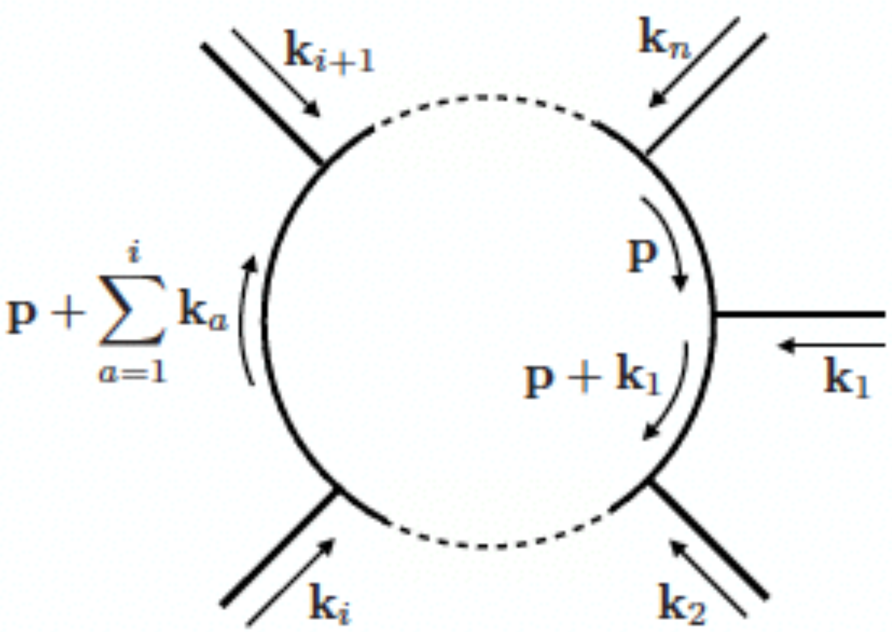
but we can renormalize by using counter term

Appendices

(i). $d = z$

One loop graph without shift sym

(e.g) **Vertex** $\alpha_1 \phi^2 (\triangle^3 \phi)$ ← I will show that this term is forbade by unitarity



$$\sim {\alpha_1}^n \int d\rho d\theta \rho^5 \left[\sin^{2n} \theta + O\left(\frac{k}{\rho}\right) \right]$$

$$\sim {\alpha_1}^n \Lambda^6 \qquad 0 \leq \rho \leq \infty$$

$$S_{counter} \sim \sum_{n=0}^{\infty} {\alpha_1}^n \int dt d^3x (\Lambda^6 \phi^n + \text{ term with derivatives})$$

$$\phi^2(\triangle^3\phi), \phi(\partial_i\phi)(\partial_i\triangle^2\phi), \cdots \quad (\triangle^3\phi)\phi^n$$

need infinite number of counter terms → **not renormalizable**

Appendices

with shift symmetry ($\phi \longrightarrow \phi + c$)

$(\triangle\phi)^3, (\triangle^2\phi)(\partial_i\phi)^2, \cdots$ → **renormalizable**

without shift symmetry

$\phi^2(\triangle^3\phi), \phi(\partial_i\phi)(\partial_i\triangle^2\phi), \cdots$ → **allow non-renormalizable terms**

- **Lifshitz scalar (LS) theory is a power-counting renormalizable, but LS theory needs certain symmetry to be described finite number of counter terms**

Appendices

Example of the correspondence
in the case with/without shift symmetry

	with shift sym	without shift sym
One-loop	○	✕
Tree unitarity	○	✕ (Lab-like system)

Equivalence between tree unitarity and renormalizability hold true for Lifshitz (non-relativistic) field theory!!

Appendices

Unitarity Bound (general discussion)

Unitarity

$$SS^\dagger = 1 \quad \xrightarrow{S = 1 + iT} \quad -i(T - T^\dagger) = TT^\dagger$$

Scattering amplitude $\mathcal{M}(i \to f)$

$$\langle f|T|i\rangle = \delta(E_i - E_f)\delta^d(\mathbf{p}_i - \mathbf{p}_f)\mathcal{M}(i \to f)$$

Orthonomal basis $|X\rangle$ $\quad \sum_X |X\rangle\langle X| = 1$

$$-i[\mathcal{M}(i \to f) - \mathcal{M}(f \to i)^*]$$

$$= \sum_X \delta(E - E_X)\delta^d(\mathbf{p} - \mathbf{p}_X)\mathcal{M}(i \to X)\mathcal{M}(f \to X)^*$$

$$i = f$$

$$2 \ \mathrm{Im} \ \mathcal{M}(i \to i) = \sum_X \delta(E - E_X)\delta^d(\mathbf{p} - \mathbf{p}_X)|\mathcal{M}(i \to X)|^2$$

Appendices

Unitarity bound (perturbative)

Normalized n-particle state $|\mathbf{p}_1 \cdots \mathbf{p}_n\rangle$

$$\int \prod_{j=1}^{n} \frac{d^d p}{2E_{p_j}} |\mathbf{p}_1 \cdots \mathbf{p}_n\rangle \langle \mathbf{p}_1 \cdots \mathbf{p}_n| = 1$$

Normalized state with discrete parameter l

Orthonormal func. on constant E and P $h_l(\mathbf{p}_j)$

$$|E, \mathbf{P}, l\rangle = \int d\Pi_n h_l(\mathbf{p}_j) |\mathbf{p}_1 \cdots \mathbf{p}_n\rangle$$

$$d\Pi_n := \prod_{j=1}^{n} \frac{d^d p_j}{2E_j} \delta(E_1 + \ldots + E_n - E) \delta^d(\mathbf{p}_1 + \ldots + \mathbf{p}_n - \mathbf{P})$$

Scattering amplitude on constant E and P sub-space

$$\mathcal{M}(E, \mathbf{P}; l \to l')$$

$$\langle E, \mathbf{P}, l | T | E', \mathbf{P}', l' \rangle = \delta(E - E') \delta^d(\mathbf{P} - \mathbf{P}') \mathcal{M}(E, \mathbf{P}; l \to l')$$

Appendices

Unitarity bound (l-basis)

$$\mathrm{Im}\ \mathcal{M}(i \to i) = \sum_X \delta(E - E_X)\delta^d(\mathbf{p} - \mathbf{p}_X)|\mathcal{M}(i \to X)|^2$$

$$\langle E, \mathbf{P}, l|T|E', \mathbf{P}', l'\rangle = \delta(E - E')\delta^d(\mathbf{P} - \mathbf{P}')\mathcal{M}(E, \mathbf{P}; l \to l')$$

$$|\ \mathrm{Im}\ \mathcal{M}(E, \mathbf{P}; l \to l)| = \sum_{l'} |\mathcal{M}(E, \mathbf{P}; l \to l')|^2$$

$$\geq \qquad\qquad \geq$$

$$|\mathcal{M}(E, \mathbf{P}; l \to l)| \qquad\qquad |\mathcal{M}(E, \mathbf{P}; l \to l')|^2$$

$l' = l$

$$|\mathcal{M}(E, \mathbf{P}; l \to l) \geq |\mathcal{M}(E, \mathbf{P}; l \to l)|^2$$

$$\Rightarrow \quad |\mathcal{M}(E, \mathbf{P}; l \to l)| \leq const.$$

Appendices

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$$\mathrm{Im}\ \mathcal{M}(i \to i) = \sum_X \delta(E - E_X)\delta^d(\mathbf{p} - \mathbf{p}_X)|\mathcal{M}(i \to X)|^2$$

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$\|\wedge$ $\vee\|$

$|\mathcal{M}(E, \mathbf{P}; l \to l)|$ $|\mathcal{M}(E, \mathbf{P}; l \to l')|^2$

$\|\wedge$

$const.$

$l' = l$

$$|\mathcal{M}(E, \mathbf{P}; l \to l) \geq |\mathcal{M}(E, \mathbf{P}; l \to l)|^2$$

$$|\mathcal{M}(E, \mathbf{P}; l \to l)| \leq const.$$

Appendices

Unitarity bound (l-basis)

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$$\geq \qquad\qquad \leq$$

$$|\mathcal{M}(E, \mathbf{P}; l \to l)| \qquad |\mathcal{M}(E, \mathbf{P}; l \to l')|^2$$

$$\geq$$

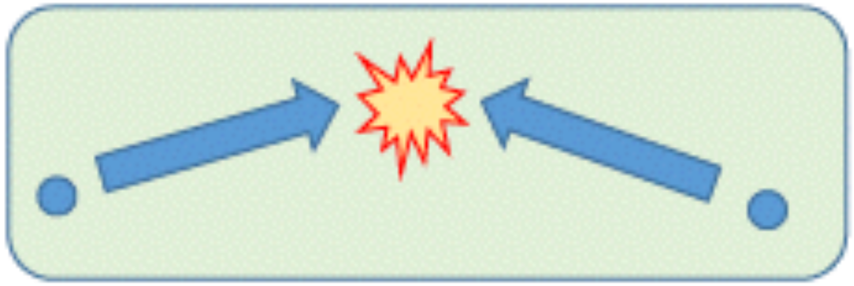
$$const.$$

$$|\mathcal{M}(E, \mathbf{P}; l \to l')| \leq const.$$

Unitarity Bound

Appendices

CoM-like state $|\alpha\rangle$



Orthonormal func. on constant E and P $h_l(\mathbf{p}_j)$

$$|\alpha\rangle = \int d\Pi \;\; h_\alpha(\mathbf{p}_j)|\mathbf{p_1}, \mathbf{p_2}\rangle$$

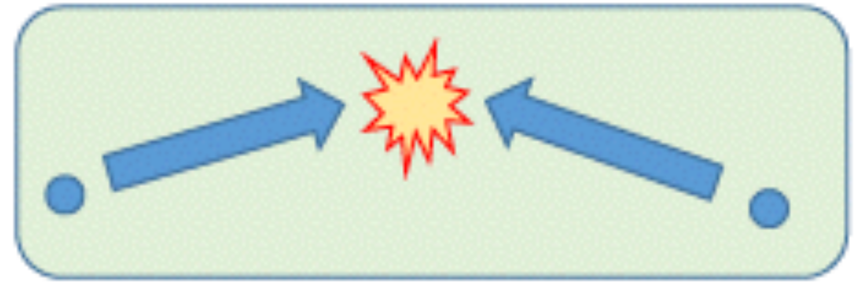
$$d\Pi := \frac{d^d p_1}{2E_1}\frac{d^d p_2}{2E_2}\delta(E_1 + E_2 - E)\delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P})$$

$$h_\alpha(p_j) = \frac{1}{\sqrt{N_\alpha(P)}} \times \begin{cases} 1 & (|\mathbf{p}_1| - |\mathbf{p}_2| \leq P/2) \\ 0 & (|\mathbf{p}_1| - |\mathbf{p}_2| > P/2) \end{cases}$$

$$N_\alpha = \int_{I_\alpha} \frac{d^d p_1}{2E_1}\frac{d^d p_2}{2E_2}\delta(E_1 + E_2 - E)\delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P})$$

Appendices

CoM-like state $|\alpha\rangle$



Orthonormal func. on constant E and P $h_l(\mathbf{p}_j)$

$$|\alpha\rangle = \int d\Pi \;\; h_\alpha(\mathbf{p}_j)|\mathbf{p_1}, \mathbf{p_2}\rangle$$

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$$N_\alpha = \int_{I_\alpha} \frac{d^d p_1}{2E_1}\frac{d^d p_2}{2E_2}\delta(E_1 + E_2 - E)\delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P})$$

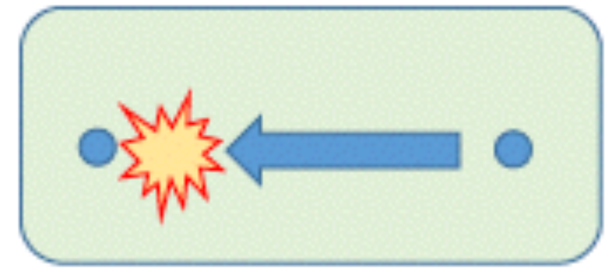
P^d P^d P^{-z} P^{-d}

P^{-z} P^{-z}

$$\propto P^{d-3z}$$

Appendices

One particle is at rest $|\beta\rangle$



$$h_\beta(p_j) = \frac{1}{\sqrt{N_\beta(P)}} \times \begin{cases} 1 & (|\mathbf{p}_1| \le \epsilon) \\ 0 & (|\mathbf{p}_1| > \epsilon) \end{cases}$$

$$p_1 = \mathcal{O}(1)$$

$$p_2 = \mathcal{O}(P)$$

P^0 P^d P^{-d}

$$N_\beta = \int_{I_\beta} \frac{d^d p_1}{2E_1} \frac{d^d p_2}{2E_2} \delta(E_1 + E_2 - E) \delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P})$$

P^{-z}

$E_2 \simeq E \simeq P^z$ ➡ $E_1 + E_2 - E = \mathcal{O}(P^{z-1})$

$$N_\beta \propto P^{-2z+1}$$

Appendices

$$\mathcal{M}(\alpha \to \alpha)$$

Unitarity bound

$$const. \geq |\mathcal{M}(E, \mathbf{P}; \alpha \to \alpha)|$$

$$= |\int d\Pi(p) d\Pi(k) h_\alpha(p) h_\alpha(k) \mathcal{M}(\mathbf{p}_1, \mathbf{p}_2 \to \mathbf{k}_1, \mathbf{k}_2)|$$

$$\frac{1}{\sqrt{N_\alpha}}$$

$$d\Pi(p) := \frac{d^d p_1}{2E_1} \frac{d^d p_2}{2E_2} \delta(E_1 + E_2 - E) \delta^d(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P})$$

$$\sim N_\alpha \propto P^{d-3z}$$

$$\sim N_\alpha P^a \propto P^{d-3z+a}$$

Suppose $\mathcal{M}(\mathbf{p}_1, \mathbf{p}_2 \to \mathbf{k}_1, \mathbf{k}_2)| = P^a$

$$a \leq 3z - d$$

4. Lifshitz Scalar Theory

第二論文の話

Tree-level unitarity and Renormalizability in Lifshitz Scalar Theory

with Fujimori, Inami, Izumi, Kitamura

accepted by PTEP

Lifshitz Scalar理論において

1. 場の次元が負になる場合までの繰り込み可能性の条件を導出

2. Tree unitarityの条件を導出

例

繰り込み可能性とTree unitarityの条件が等価である事を示した

with shift symmetry ($\phi \to \phi + c$ **)**

第一論文の話 ↓

$(\triangle\phi)^3, (\triangle^2\phi)(\partial_i\phi)^2, \cdots$ → Tree unitarity⭕ → 繰り込み可能

without shift symmetry

$\phi^2(\triangle^3\phi), \phi(\partial_i\phi)(\partial_i\triangle^2\phi), \cdots$ → Tree unitarity❌ → 繰り込み不可能

Appendices

Optical theorem originated from unitarity of S-matrix

(a) Unitarity of S-matrix $S^{\dagger}S = 1$

(b) Optical theorem $\mathrm{Im}\mathcal{M}_{nn} = -\pi\Sigma_{n'} \mid \mathcal{M}_{n'n} \mid^2$

cross section

Remark; n: information of external line, n': information of internal line

Unitarity bound is obtained from optical theorem

(1) $\mid \mathcal{M}_{nn} \mid \geq \mid \mathrm{Im}\mathcal{M}_{nn} \mid \geq \pi \mid \mathcal{M}_{nn} \mid^2 \rightarrow \mid \mathcal{M}_{nn} \mid \leq \frac{1}{\pi}$

(2) $$\mathrm{Im}\langle f \mid T \mid i\rangle = \sum_{n} \int \frac{d^{D-1}k_1}{\omega_1} \cdots \frac{d^{D-1}k_n}{\omega_2} \delta(\Sigma\omega_n - E)\delta^{D-1}(\Sigma_i k_i - \mathbf{p})$$

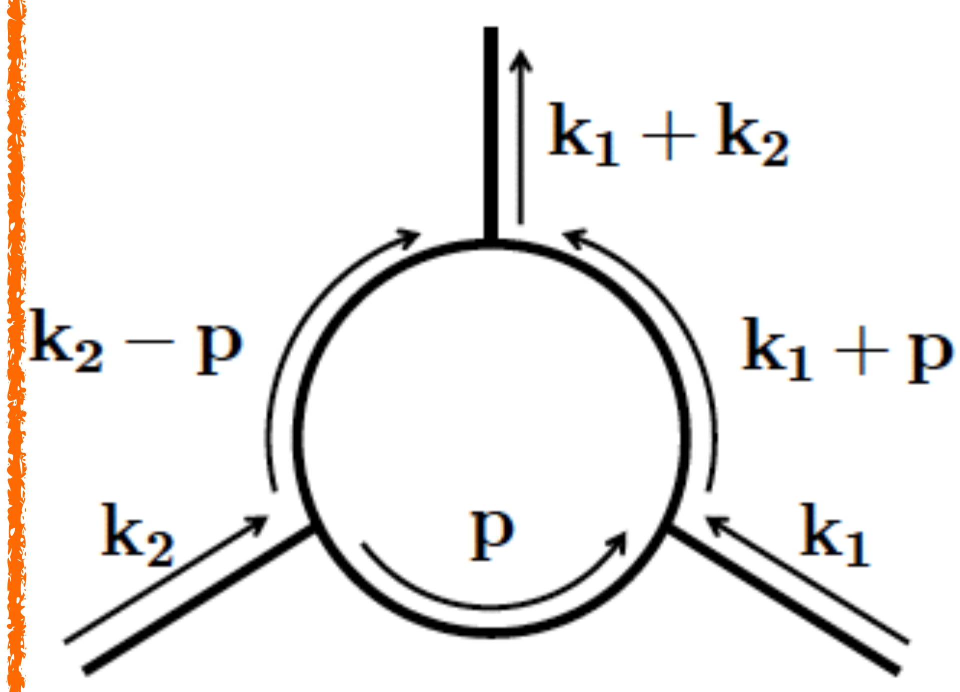
$$\times \langle k_1 \cdots k_n \mid T \mid i\rangle^* \langle k_1 \cdots k_n \mid T \mid i\rangle$$

By using (1)and (2), the value of unitarity bound for scattering amplitudes satisfying tree unitarity is determined

Appendices

One loop graph with shift symmetric case

(e.g) $\alpha_3(\triangle\phi)^3$



$$\sim \alpha_3{}^3 \mid \mathbf{k}_1{}^2 \mid\mid \mathbf{k}_2{}^2 \mid\mid \mathbf{k}_1{}^2 + \mathbf{k}_2{}^2 \mid \log\Lambda$$

we can use

$(\triangle^3\phi), (\triangle^2\phi)(\partial_i\phi)^2, \cdots$ **as counter terms**

→ **renormalizable**

Appendices

Tree-level Unitarity condition:Lorentz invariant theory

$$|M_{(l_1,\ldots,l_{D-2})(l_1,\ldots,l_{D-2})}| \geq \pi \int \frac{d^{D-1}k_1}{2\omega_1}\frac{d^{D-1}k_2}{2\omega_2}\delta(\omega_1+\omega_2-2E)\delta^{D-1}(\mathbf{k_1}+\mathbf{k_2})$$
$$\times \sum_{(l'_1,\ldots,l'_{D-2})} Y_{(l'_1,\ldots,l'_{D-2})}(\hat{\mathbf{k}}_1) M_{(l'_1,\ldots,l'_{D-2})(l_1,\ldots,l_{D-2})}$$
$$\times \sum_{(l''_1,\ldots,l''_{D-2})} Y^*_{(l''_1,\ldots,l''_{D-2})}(\hat{\mathbf{k}}_1) M^*_{(l''_1,\ldots,l''_{D-2})(l_1,\ldots,l_{D-2})}$$
$$= \frac{\pi}{8}E^{D-4}\sum_{(l'_1,\ldots,l'_{D-2})}\left|M_{(l'_1,\ldots,l'_{D-2})(l_1,\ldots,l_{D-2})}\right|^2,$$

$Y_{l_1,\ldots,l_{D-2}}(\hat{\mathbf{k}}_1)$ **spherical harmonic function**

direction to which 1 particle propagate $\hat{\mathrm{k}}_1$

2-2 scattering

energy in CM system $2E$

$|M_{2,2'}| \propto E^m$ with $m \leq 4-D$

Appendices

Tree-level Unitarity condition:Lifshitz-type theory

CM system

$$|M_{(l_1,\dots,l_{D-2})(l_1,\dots,l_{D-2})}| \geq \pi \int \frac{d^{D-1}k_1}{2\omega_1}\frac{d^{D-1}k_2}{2\omega_2}\delta(\omega_1+\omega_2-2E)\delta^{D-1}(\mathbf{k_1}+\mathbf{k_2})$$
$$\times \sum_{(l'_1,\dots,l'_{D-2})} Y_{(l'_1,\dots,l'_{D-2})}(\hat{\mathbf{k}}_1) M_{(l'_1,\dots,l'_{D-2})(l_1,\dots,l_{D-2})}$$
$$\times \sum_{(l''_1,\dots,l''_{D-2})} Y^*_{(l''_1,\dots,l''_{D-2})}(\hat{\mathbf{k}}_1) M^*_{(l''_1,\dots,l''_{D-2})(l_1,\dots,l_{D-2})}$$
$$= \frac{\pi}{8z} k_1^{D-3z-1} \sum_{(l'_1,\dots,l'_{D-2})} \left|M_{(l'_1,\dots,l'_{D-2})(l_1,\dots,l_{D-2})}\right|^2,$$

$Y_{l_1,\dots,l_{D-2}}(\hat{\mathbf{k}}_1)$ **spherical harmonic function**

2-2 scattering

direction to which 1 particle propagate $\hat{\mathrm{k}}_1$

energy in CM system $2E$

$|M_{2,2'}| \propto k_1^m$ with $m \leq 3z+1-D$

$\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}$

Appendices

Tree-level Unitarity condition:Lifshitz-type theory

Lab system

The total momentum $\mathbf{P}$ and energy E are $\mathbf{P} = \mathbf{k_1} + \mathbf{k_2}$ and $E(=: P^z + \epsilon) = k_1^z + k_2^z$

$$k_2^z = P^z - zP^{z-1}k_1(\hat{\mathbf{P}}\cdot\hat{\mathbf{k}}_1) + P^{z-2}k_1^2\left[\frac{z}{2} + \frac{z(z-1)}{2}(\hat{\mathbf{P}}\cdot\hat{\mathbf{k}}_1)^2\right] + O(P^{z-3})$$

$$\frac{\left(d^{D-1}k_1\right)}{2k_1^z}\frac{\left(d^{D-1}k_2\right)}{2k_2^z} = \frac{k_1^{D-2}\left(d^{D-2}\hat{k}_1\right)\left(d^{D-1}P\right)(d\epsilon)}{4zk_1^z|\mathbf{P}-\mathbf{k_1}|^z\left[k_1^{z-1} - |\mathbf{P}-\mathbf{k_1}|^{z-2}\left(P(\hat{\mathbf{P}}\cdot\hat{\mathbf{k}}_1) - k_1\right)\right]}$$

$$|M_{\alpha,\alpha}| \geq \pi\int\frac{k_1^{D-2}\left(d^{D-2}\hat{k}_1\right)}{4zk_1^z|\mathbf{P}-\mathbf{k_1}|^z\left[k_1^{z-1} - |\mathbf{P}-\mathbf{k_1}|^{z-2}\left(P(\hat{\mathbf{P}}\cdot\hat{\mathbf{k}}_1) - k_1\right)\right]}|M_{\hat{\mathbf{k}}_1,\alpha}|^2$$

$$= \frac{k_1^{D-z-2}k_2^{1-2z}}{4z}\sum_{l'}|M_{l',\alpha}|^2,$$

2-2 scattering

$$|M_{2,2'}| \propto k_1^m \qquad \text{with} \qquad m \leq 3z + 1 - D$$

Appendices

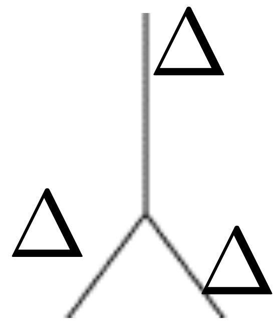
Concrete calculation

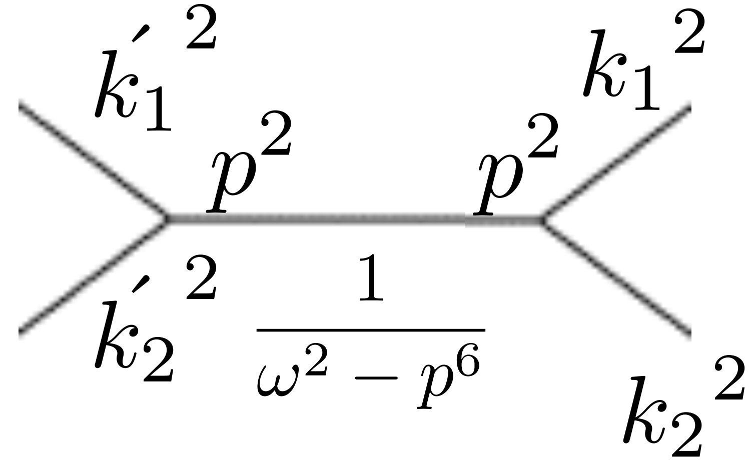
(1+3)dim $(z=3)$

$\beta \leq 6$

Vertex (with shift sym)

$$\alpha_3 (\triangle\phi)^3$$





$\mathcal{M} \sim p^{-2}$

with shift symmetry

$$\partial_i\phi\, \partial_i\phi(\triangle^2\phi)$$

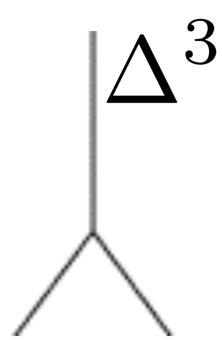
$\beta \leq 5$

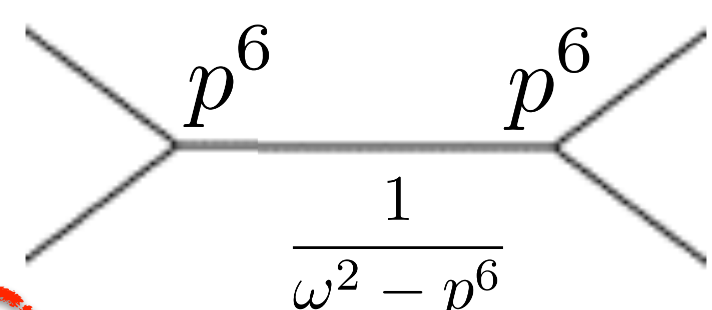
Vertex(without shift sym)

(1+3)dim $(z=3)$

$\beta \leq 5$

$$\alpha_1 \phi^2 (\triangle^3\phi)$$





$\mathcal{M} \sim p^{6}$

Unitarity Bound (general discussion)

Unitarity

$$SS^\dagger = 1 \quad \xrightarrow{S = 1 + iT} \quad -i(T - T^\dagger) = TT^\dagger$$

$$\langle E, \mathbf{P}, l | T | E', \mathbf{P}', l' \rangle = \delta(E - E')\delta^d(\mathbf{P} - \mathbf{P}')\mathcal{M}(E, \mathbf{P}; l \to l')$$

$$|\ \mathrm{Im}\ \mathcal{M}(E, \mathbf{P}; l \to l)| = \sum_{l'} |\mathcal{M}(E, \mathbf{P}; l \to l')|^2$$

$$\text{(left side)} \leq |\mathcal{M}(E, \mathbf{P}; l \to l)| \qquad \text{(right side)} \geq |\mathcal{M}(E, \mathbf{P}; l \to l')|^2$$

$$|\mathcal{M}(E, \mathbf{P}; l \to l')| \leq const.$$

Unitarity Bound

Appendices

Higher derivative QG

Higher derivative Quantum Gravity **K.S.Stelle '77**

$$S = \int d^4x \sqrt{-g}\,(a\mathrm{R}^2 + b\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \kappa^{-2}\gamma\mathrm{R})$$

$$\gamma = 2$$
$$\kappa^2 = 32\pi G$$

make the theory renormalizable

Propagator

$$\frac{1}{k^2} + \frac{1}{k^2}G(k^4)\frac{1}{k^2} + \frac{1}{k^2}G(k^4)\frac{1}{k^2}G(k^4)\frac{1}{k^2} + \cdots$$

$$= \frac{1}{k^2 - G(k^4)}$$

$$\frac{1}{k^2} - \frac{1}{k^2 - \frac{1}{\mathrm{G}}}$$

ghost term

renormalizable but ghost!

make the theory unstable

$$k = (E, \mathbf{k})$$

E from time derivative →

Ghost term results from including more than 2rd order time derivatives

Appendices

Hořava'idea

Lifshitz scaling

$[x] = -1 \quad [t] = -z$ **in mass dim**

$\vec{x} \mapsto b\vec{x}$ $\quad b$ **arbitrary number**

$t \mapsto b^z t$ $\quad z$ **dynamical critical exponent**

$$S = S_\mathrm{K} - S_\mathrm{V}$$

separate action into S_K **and** S_V

Kinetic term (S_K) **Potential term** (S_V)

S_K **including time derivatives**

S_V **including spatial derivatives without time derivatives**

modified Propagator

$$\frac{1}{\omega^2 - c^2\mathbf{k}^2 - G(\mathbf{k}^2)^z}$$

positive! (ω^2)

$z = 1 \quad \rightarrow$ **isotropic**

$z \neq 1 \quad \rightarrow$ **Lorenz ~~symmetry~~**

↓

evade the ghost problem