Renormalisation group flow of perturbative gauge-Yukawa theories

Andrew Bond (University of Sussex)

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Outline

Goal: describe a quantum field theory which is predictive to arbitrarily high energies.

Particularly interested in field theories which are perturbative.

- Have use of the extensive tools of perturbation theory.
- Relevance of operators governed by canonical mass dimension.

Renormalisation group equations (RGEs)

Flow of couplings with energy is described by system of coupled non-linear differential equations

$$\frac{d\alpha_i}{dt} = \beta_i(\alpha_j)\,,$$
$$\text{with } t = \log(\tfrac{k}{k_c})\,.$$

β-functions can be calculated perturbatively in a loop expansion:

$$\beta(\alpha) = -B\alpha^2 + C\alpha^3 + \dots$$

UV behaviour in perturbation theory

To remain within perturbation theory up to high energies, we must have it so that our couplings do not become large in the UV.

Two options for our UV fixed point:

- Asymptotic freedom.
  Couplings are all zero — theory becomes free (Gaussian) at high energies.
- Asymptotic safety.
  Some couplings small but non-zero — theory remains (weakly) coupled at high energies.

General perturbatively renormalisable theories

The general Lagrangian for perturbatively renormalisable theories in four dimensions is

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{2}\left(D_\mu \Phi^A\right)^2 + i\overline{\Psi}_J \sigma^\mu D_\mu \Psi_J$$
$$- \frac{1}{2}\left(Y^A_{JK}\Phi^A \Psi_J \Psi_K + (Y^A_{JK})^* \Phi^A \overline{\Psi}_J \overline{\Psi}_K\right) +$$
$$- \frac{1}{4!}\lambda_{ABCD}\Phi_A\Phi_B\Phi_C\Phi_D + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{mass}}$$

Three types of marginal couplings:

- Gauge coupling g
- Yukawa couplings Y^A_{JK}
- Scalar quartic couplings λ_{ABCD}

Form of β-functions - no gauge

The schematic form of the beta functions with no gauge fields is

$$\beta_y = \alpha_y(E\alpha_y \qquad) \qquad + \dots$$
$$\beta_\lambda = H\alpha_\lambda^2 + I\alpha_\lambda\alpha_y - J\alpha_y^2 \qquad + \dots$$

Only fixed point is Gaussian — β-functions positive so no asymptotic freedom!

Form of β-functions

The schematic form of the beta functions with gauge fields is

$$
\begin{aligned}
\beta_g &= -B\alpha_g^2 && + \alpha_g^2(C\alpha_g - D\alpha_y) + \ldots \\
\beta_y &= \alpha_y(E\alpha_y - F\alpha_g) && + \ldots \\
\beta_\lambda &= H\alpha_\lambda^2 + I\alpha_\lambda\alpha_y - J\alpha_y^2 && + \ldots \\
&\quad - K\alpha_\lambda\alpha_g + L\alpha_g^2
\end{aligned}
$$

$B \sim (\text{gauge fields}) - (\text{matter})$
Crucial to have (non-Abelian) gauge fields!

Example theory

- Gauge group $SU(N_C)$.
- N_F Dirac fermions in the fundamental representation.
- $N_F \times N_F$ matrix of uncharged scalars.
- $SU(N_F) \times SU(N_F)$ global flavour symmetry.

$$\mathcal{L} = \mathcal{L}_{kin} + y\,\mathrm{Tr}(\overline{Q}_L H Q_R + \mathrm{h.\,c.}) + u\,\mathrm{Tr}(H^\dagger H H^\dagger H) + v(\mathrm{Tr}(H^\dagger H))^2\,.$$

Makes sense to work with rescaled couplings

$$\alpha_g = \frac{g^2 N_C}{(4\pi)^2}\,, \qquad \alpha_y = \frac{y^2 N_C}{(4\pi)^2}\,,$$

$$\alpha_h = \frac{u N_F}{(4\pi)^2}\,, \qquad \alpha_v = \frac{v N_F^2}{(4\pi)^2}\,.$$

Venziano limit: $N_C, N_F \to \infty$ such that

$$\epsilon = \frac{N_F}{N_C} - \frac{11}{2}$$

finite.

Gauge-Yukawa β-functions

For theories with gauge and Yukawa couplings β-functions have the form

$$\beta_g = \alpha_g^2(-B + C\alpha_g - D\alpha_y)\,,$$
$$\beta_y = \alpha_y(E\alpha_y - F\alpha_g)\,.$$

Have a potential fixed point

$$\begin{pmatrix} \alpha_g^* \\ \alpha_y^* \end{pmatrix} = \frac{B}{CE - DF} \begin{pmatrix} E \\ F \end{pmatrix}\,.$$

Is it physical?

Gauge-Yukawa fixed point

Interacting fixed point

$$\begin{pmatrix} \alpha_g^* \\ \alpha_y^* \end{pmatrix} = \frac{B}{CE - DF} \begin{pmatrix} E \\ F \end{pmatrix} .$$

- Need B small, so fixed point is perturbative
- $CE - DF > 0 \rightarrow$ physical when we have asymptotic freedom — fixed point is infrared.
- $CE - DF < 0 \rightarrow$ physical when we have no asymptotic freedom — fixed point is ultraviolet. Asymptotic safety!

Asymptotic safety

For asymptotic safety need

$$CE - DF < 0\,,$$

exactly the case in our example theory.

Fixed point exists when we lose asymptotic freedom $B < 0$. Theory is 'QED-like'.

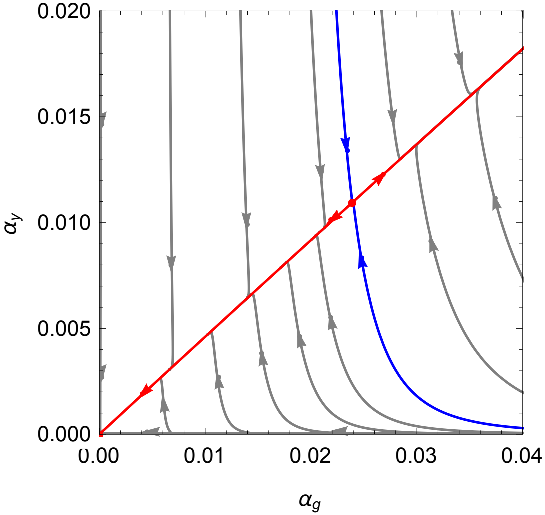
Have a single UV relevant direction.

Find fixed point systematically as power series in ϵ,

$$\alpha_g = c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3 \dots$$

c_1 completely fixed by $(\beta_g^{(2)}, \beta_y^{(1)})$.
c_2 completely fixed by $(\beta_g^{(3)}, \beta_y^{(2)}, \beta_\lambda^{(1)})$.
c_3 completely fixed by $(\beta_g^{(4)}, \beta_y^{(3)}, \beta_\lambda^{(2)})$.
...
Fixed point is perturbatively reliable!

Phase diagram — asymptotic safety



One-loop asymptotic freedom in gauge-Yukawa theories

Gaussian fixed point — linearised system vanishes!
Have one-loop equations

$$\begin{aligned}\partial_t\alpha_g &= -B\alpha_g^2\,,\\ \partial_t\alpha_y &= \alpha_y(E\alpha_y - F\alpha_g)\\ \partial_t\alpha_\lambda &= H\alpha_\lambda^2 + I\alpha_\lambda\alpha_y - J\alpha_y^2\,.\end{aligned}$$

Can in fact solve explicitly!

Scaling solution

Always have scaling solution — all couplings run in proportion to one another:

$$\alpha_\lambda(t) \propto \alpha_y(t) \propto \alpha_g(t)\,.$$

Can find proportionality coefficients algebraically from β-functions.

Family of solutions

Solutions which reach UV safely:

$$\alpha_g(t) = \frac{\alpha_g^0}{1 + B\alpha_g^0 t}\,,$$

$$\alpha_y(t) = c_1 \frac{\alpha_g(t)}{1 + x(t)^{-1}}\,,$$

$$\alpha_\lambda(t) = c_2 x(t)^{s+1} \left(\frac{{}_2F_1(1+S_1, 1+\tilde{S}_1; 2+s; -x(t))}{{}_2F_1(S_1, \tilde{S}_1; 1+s; -x(t))} - \frac{1}{1+x(t)} \right)\,,$$

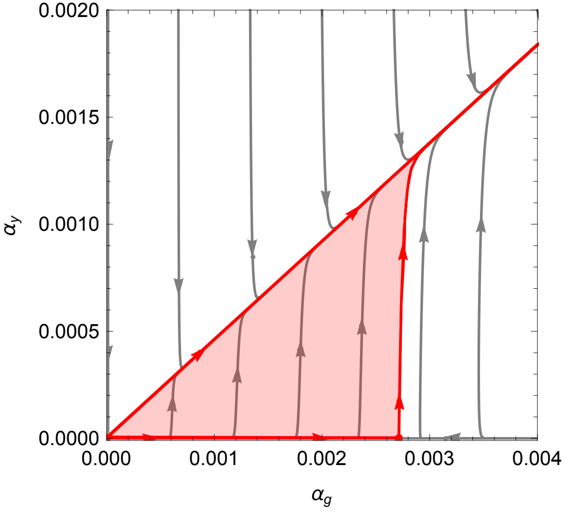
$$x(t) \equiv \left(\frac{\alpha_g(t)}{\alpha_g^c} \right)^{\frac{1}{s}}\,.$$

Solutions to one-loop β-functions

- Have integration constants from gauge and Yukawa β-functions, but from quartic it is fixed $\rightarrow$ 2-dimensional critical surface.
- Have hierarchy of couplings in UV:

$$\alpha_g \gg \alpha_y \gg \alpha_\lambda$$

Gauge-Yukawa one-loop asymptotic freedom



Two-loop asymptotic freedom

Boundary between two cases — $B = 0$

Now have lowest order equations

$$\begin{aligned} \partial_t \alpha_g &= \alpha_g^2 (C\alpha_g - D\alpha_y)\,, \\ \partial_t \alpha_y &= \alpha_y (E\alpha_y - F\alpha_g)\,, \\ \partial_t \alpha_\lambda &= H\alpha_\lambda^2 + I\alpha_\lambda\alpha_y - J\alpha_y^2\,. \end{aligned}$$

Can no longer solve explicitly!

Can construct a unique implicit power series solution in e.g. the gauge coupling

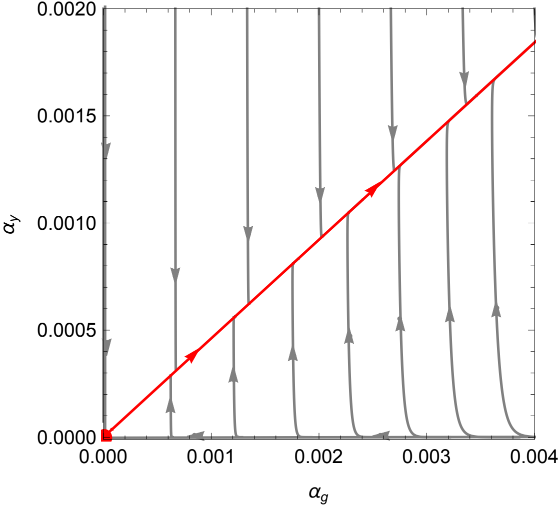
$$\alpha_y(t) = y_1\alpha_g(t) + y_2\alpha_g(t)^2 + \dots\,,$$
$$\alpha_\lambda(t) = \lambda_1\alpha_g(t) + \lambda_2\alpha_g(t)^2 + \dots\,,$$

- Asymptotically free exactly when it is infrared stable ($\Longrightarrow$ unique asymptotically free solution), and this happens when

$$CE - DF < 0\,.$$

- Same regime as where interacting fixed point is asymptotically safe!

Phase diagram — two-loop asymptotic freedom



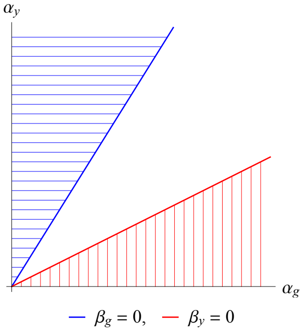
Interpretation of two-loop AF condition

We can understand the condition for two-loop asymptotic freedom geometrically.

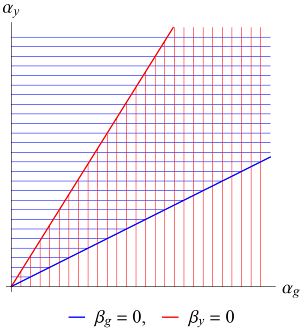
$$\beta_g = \alpha_g^2(C\alpha_g - D\alpha_y)\,,$$
$$\beta_y = \alpha_y(E\alpha_y - F\alpha_g)\,.$$

To have any hope of reaching free fixed point in UV, need a region where both β-functions are negative.

$$\beta_g = \alpha_g^2 (C\alpha_g - D\alpha_y)\,, \qquad \beta_y = \alpha_y (E\alpha_y - F\alpha_g)\,.$$



$$\frac{C}{D} > \frac{F}{E}$$



$$\frac{C}{D} < \frac{F}{E}$$

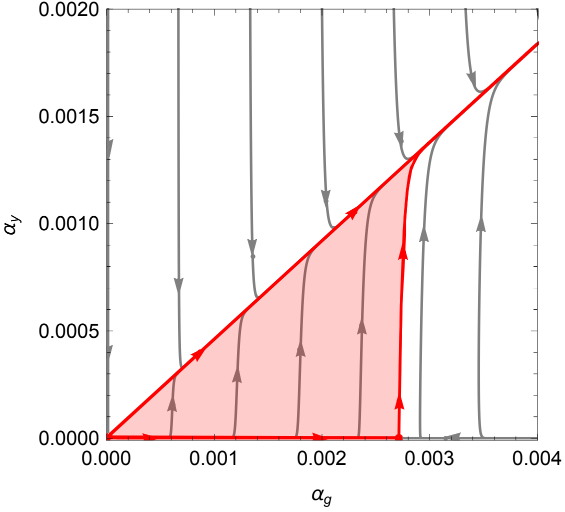
Two-loop asymptotic freedom

- Gauge fields are not asymptotically free by themselves — only through a balance between gauge and Yukawa couplings.
- More predictive than one-loop asymptotic freedom — lower-dimensional critical surface.
- Works at finite N.
- Closely linked to existence of ultraviolet fixed point in nearby parameter range.

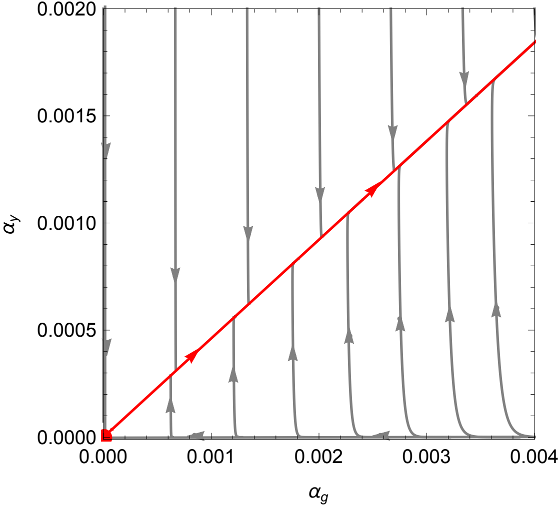
Can see how ultraviolet fixed points emerge from two-loop asymptotic freedom by considering how the theory varies as we change the hard parameter ϵ.

- $\epsilon < 0$ we have asymptotic freedom, 2d critical surface, and a Banks-Zaks type infrared fixed point.
- $\epsilon = 0$ we have two-loop asymptotic freedom as balance between gauge and Yukawa couplings, and 1d critical surface.
- $\epsilon > 0$ we have asymptotic safety with a 1d critical surface — two trajectories leading to strong/weak coupling in the infrared.

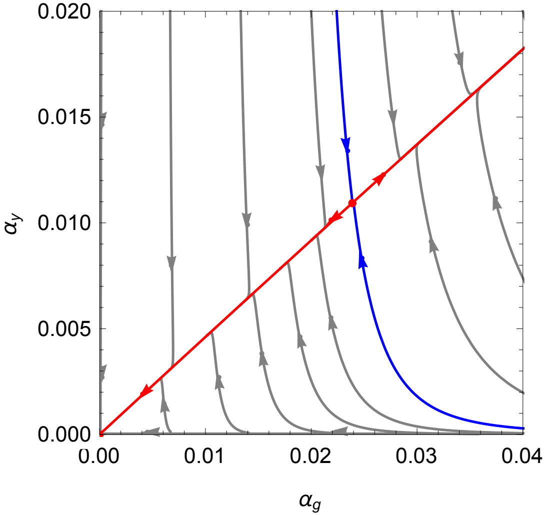
Phase diagram — $\epsilon < 0$



Phase diagram — $\epsilon = 0$



Phase diagram — $\epsilon > 0$



Conclusions

- Asymptotic freedom generated at one-loop is not the only way to define a UV complete perturbative field theory.
- Possible to have asymptotically safe theories within perturbation theory, need large N to be strictly reliable.
- Can have asymptotic freedom generated at two-loop through balance of gauge and Yukawa couplings — more predictive than one-loop case, and achievable at finite N.

Thanks for listening