PDF evolution in Mellin space, technical aspects and applications for PDF fits

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Outline

- Motivation
- Generalities of Mellin moments transforms
- A numerical approach to Mellin moments of PDFs
- Applications to PDF fits

Motivation

- Usage of Mellin-moments predictions for PDF fits is not new, fits of polarised PDFs, as well as NNPDF fits, make extensive use of this technology
- In the recent years, the development of the APPLGRID and FASTNLO technologies has provided an extremely fast framework for obtaining very fast predictions with arbitrary input PDFs, which are extremely well suited for PDF fits
- So, why bother about the old technology of Mellin moments?
- Predictions based on Mellin moments could overcome two intrinsic limitations of APPLGRID and FASTNLO
  - They work well not only for fixed order predictions, but also with all-order resummed predictions
  - With few exceptions like α_s and CKM matrix elements, APPLGRID and FASTNLO cannot be used to vary fundamental parameters of the theory. Predictions based on Mellin moments, are slower but could be still fast enough to perform combined fits of PDFs and fundamental parameters
- Not excluded that the two technologies can be combined

Mellin transforms

The N-Mellin moment of f(x) is defined as

$$F(N) = \int_0^1 x^{N-1} f(x) dx$$

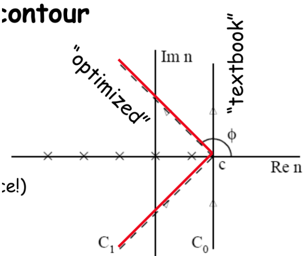
The inverse mellin transform allows to transform F(N) back to f(x)

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-N} F(N) dx$$

The important advantage of Mellin transforms, is that they transform convolutions in simple products. In Mellin space the DGLAP evolution takes a very simple form.

Contours of integration in the complex plane

- The contour of integration in the complex plane is traditionally chosen in Math text books as $[c - i\infty, c + i\infty]$, with c a real number which lays on the right of all singularities in the $x^{N-1}f(x)$ function
- In some cases, it can be beneficial to deform the contour of integration in the complex plane, to improve the convergence of the inverse trasform integral
- For instance, Stratmann and Vogelsang (PRD 64 (2001) 114007), suggest to use a fish tail shape, with an angle $\phi > \pi/2$ with respect to the real axis



Evaluation of PDF Mellin moments

- Provided that DGLAP evolution in Mellin space allows to calculate PDFs in Mellin space at all scales, we still need to calculate a set of Mellin moments at the starting scale
- If we consider $f(x)$ as a PDF, functional forms of the type $f(x) = x^a(1-x)^b P(x)$, where P(x) is a polynomial, allows to compute arbitrary complex N moments by using the Γ function

$$\int_0^1 x^a(1-x)^b dx = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$$

- This form allows complete freedom on the choice of the integration contour, but imposes strong contraints on the PDF flexibility

Numerical evaluation of PDF Mellin moments

- However, if we restrict the integration contour to the Math text book shape $[c - i\infty, c + i\infty]$, the numerical integration of

$$F(N) = \int_0^1 x^{N-1} f(x) dx$$

- can be performed very efficiently by using gaussian quadrature rules
- Roughly speaking, for PDF which are not this holds as far as $Re(N-1) > 0$, while for $Re(N-1) < 0$

Gaussian quadrature rule

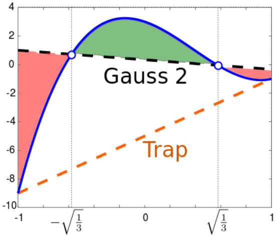
- In numerical analysis, a quadrature rule is an approximation of the definite integral of a function, usually acheived with a weighted sum of function values at specified points
- An n-point Gaussian quadrature rule (key n) is a quadrature rule constructed to yield an exact result for polynomials of degree $2n - 1$ or less by a suitable choice of the points x_i and weights w_i for $i = 1, ..., n$
- The rule is stated as

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

- Expected to be accurate if the function $f(x)$ is well approximated by a polynomial function

- The Gaussian quadrature rules are associated to Legendre polynomials, $P_n(x)$, and the method is usually known as Gauss-Legendre quadrature
- The i-th Gauss node, x_i, is the i-th root of P_n

$$w_i = \frac{2}{(1 - x_i^2) P_n(x_i)^2}$$



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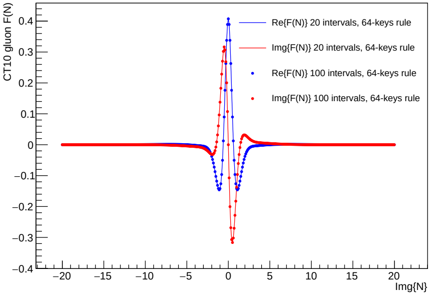
Application to Mellin moments of PDFs

- Check the procedure by evaluating N moments of the CT10 gluon PDF
- To increase the accuracy of the integration, the $[0, 1]$ interval is divided in 20 or 100 subintervals, and a 64-points gaussian rule is used in each interval
- About 100 moments for all PDFs can be evaluated in $< 1s$

Test of accuracy

- Tested the accuracy of the integrals by comparing two number of intervals subdivision, 20 and 100
- The results agree at the level of 10^{-6}



- As a side remark, the same technique can be used to compute efficiently sum rules for arbitrary PDF form

Phenomenological applications for PDF fits

Possible applications of fast predictions based on Mellin moments

- Threshold resummation and qt-resummation (Jets, DY, diboson)
- Simultaneous fits of PDF and weak-mixing angle in DY productions
- Simultaneous fits of α_s, top mass and PDFs in $t\bar{t}$ production

Summary

- Predictions based on Mellin moments is a well known and established technique
- The long-term project is to implement a general framework of Mellin transform in xFitter, which provides PDFs evolved in Mellin space, and perform the Mellin inversion
- Theorists would need only to provide coefficient functions in Mellin space
- xFitter should perform also the phase space integration afterwards
- The Mellin moments implementation could allow a broader spectrum of phenomenological applications of xFitter

References

For the interested reader

- *Towards a global analysis of polarized parton distributions*
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- http://arxiv.org/abs/hep-ph/9506333
- http://arxiv.org/abs/hep-ph/9708392
- *Threshold Resummation and Rapidity Dependence*
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