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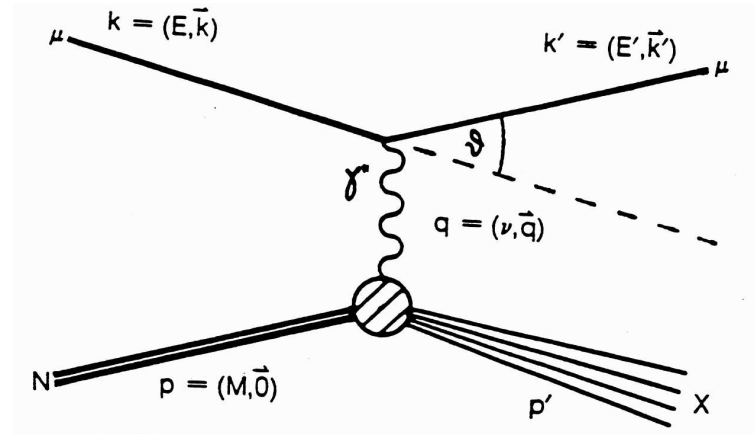
Parton densities at small x values. Combined H1 and ZEUS F_2 data. F_2^C data. BFKL corrections

OUTLINE

1. Introduction
2. Results: flat initial conditions for PDFs at small x values
(i.e. *min* information about initial conditions, or
min contribution from initial conditions.)
3. Conclusions and Prospects

1. Introduction to DIS

A. Deep-inelastic lepton-hadron scattering



Deep-inelastic scattering cross-section:

$$\sigma \sim L^{\mu\nu} F^{\mu\nu}$$

Hadron part $F^{\mu\nu}$ ($Q^2 = -q^2 > 0$, $x = Q^2/[2(pq)]$):

$$F^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x, Q^2) \\ - \left(p^\mu - \frac{(pq)}{q^2} q^\mu\right) \left(p^\nu - \frac{(pq)}{q^2} q^\nu\right) \frac{2x}{q^2} F_2(x, Q^2) + \dots,$$

where $F_k(x, Q^2)$ ($k = 1, 2, 3, L$) - are DIS SF and q and p are photon and hadron (parton) momentums.

B. Wilson operator expansion: Mellin moments $M_k(n, Q^2)$ of DIS SF $F_k(x, Q^2)$ can be represented as sum

$$M_k(n, Q^2) = \sum_{a=N, S, SI, g} \underbrace{C_k^a(n, Q^2/\mu^2)}_{\text{Coeff. function}} A_a(n, \mu^2),$$

where $A_a(n, \mu^2) = \langle N | \mathcal{O}_{\mu_1, \dots, \mu_n}^a | N \rangle$ are matrix elements of the Wilson operators $\mathcal{O}_{\mu_1, \dots, \mu_n}^a$.

!!! There is a factorization in the Mellin-moment space \rightarrow there are Mellin convolutions in Bjorken- x space **!!!**

C. The matrix elements $A_a(n, \mu^2)$ are Mellin moments of the unpolarized PDF $f_a(n, \mu^2)$.

DGLAP [= Renormgroup] equations:

$$\frac{d}{d \ln Q^2} f_a(x, Q^2) = \int_x^1 \frac{dy}{y} \sum_b W_{b \rightarrow a}(x/y) f_b(y, Q^2). \quad (1)$$

The anomalous dimensions (ADs) $\gamma_{ab}(n)$ of the twist-2 Wilson operators $\mathcal{O}_{\mu_1, \dots, \mu_n}^a$ (hereafter $a_s = \alpha_s/(4\pi)$)

$$\gamma_{ab}(n) = \int_0^1 dx x^{n-1} W_{b \rightarrow a}(x) = \sum_{m=0}^{\infty} \gamma_{ab}^{(m)}(n) a_s^m,$$

All parton densities are multiplied by x , t.e.

structure function = combination of parton densities.

3. Method

(C.Lopez and F.J.Yndurain, 1980,1981), (A.V.K., 1994)

Here I present briefly the method, which leads to the possibility to replace the Mellin convolution of two functions

$$f_1(x) \otimes f_2(x) \equiv \int_x^1 \frac{dy}{y} f_1(y) f_2(x/y)$$

by a simple products at small x .

So, if $f_1(x) = B_k(x, Q^2)$ is perturbatively calculated Wilson kernel and $f_2(x) = x f_a(x, Q^2) \sim x^{-\delta}$ at $x \rightarrow 0$, then

$$f_1(x) \otimes f_2(x) \approx M_k(1 + \delta, Q^2) f_2(x) \quad (2)$$

where $M_k(1 + \delta, Q^2)$ is the analytical continuation to non-integer arguments of the Mellin moment $M_k(n, Q^2)$ of $B_k(x, Q^2)$:

$$M_k(n, Q^2) = \int_0^1 x^{n-2} B_k(x, Q^2) \quad (3)$$

The equation (2) is correct if the moment $M_k(n, Q^2)$ has no singularity at $n \rightarrow 1$.

Remember !!! The Mellin convolution of two functions

$$f_1(x) \otimes f_2(x) \equiv \int_x^1 \frac{dy}{y} f_1(y) f_2(x/y)$$

3. Generalized double-logarithmic approach

(A.V.K. and G.Parente, 1998),

(A.Yu.Illarionov, A.V.K. and G.Parente, 2004) are an extension of the previous studies in

(A.D.Rujula, S.L.Glashow, H.D.Politzer, S.B.Treiman, F.Wilczek and A.Zee, 1974),

(R.D.Ball and S.Forte, 1994),

(L. Mankiewicz, A. Saalfeld and T. Weigl , 1994)

(Generalized double-logarithmic approach) \equiv (Generalized double asymptotic scaling)

1 Leading order (LO) without quarks (a pedagogical example)

At the momentum space, the solution of the DGLAP equation is

$$M_g(n, Q^2) = M_g(n, Q_0^2) e^{-d_{gg}(n)s},$$

where $M_g(n, Q^2)$ are the moments of the gluon distribution,

$$s = \ln \left(\frac{a_s(Q_0^2)}{a_s(Q^2)} \right), \quad a_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi} \quad \text{and} \quad d_{gg} = \frac{\gamma_{gg}^{(0)}(n)}{2\beta_0}$$

The terms $\gamma_{gg}^{(0)}(n)$ and β_0 are respectively the LO coefficients of the gluon-gluon AD and the QCD β -function.

For any perturbatively calculable variable $Q(n)$, it is very convenient to separate the singular part when $n \rightarrow 1$ (denoted by “ \widehat{Q} ”) and the regular part (marked as “ \overline{Q} ”):

$$Q(n) = \frac{\widehat{Q}}{n-1} + \overline{Q}(n)$$

Then, the above equation can be represented by the form

$$M_g(n, Q^2) = M_g(n, Q_0^2) e^{-\hat{d}_{gg} s_{LO}/(n-1)} e^{-\bar{d}_{gg}(n) s_{LO}},$$

with $\hat{\gamma}_{gg} = -8C_A$ and $C_A = N$ for $SU(N)$ group.

Finally, if one takes the flat boundary conditions (i.e. *min* information about initial conditions, or *min* contribution from initial conditions.)

$$x f_a(x, Q_0^2) = A_a, \quad \rightarrow \quad M_a(n, Q_0^2) = \frac{A_a}{n-1} \quad (4)$$

!!! Modern experimental data (HERA I+II, LHC):
 $xf_a(x, Q^2 = 1.9\text{GeV}^2)$ starts to decrease at $x \leq 10^{-3}$. !!!

So, in a sense, the above equation

$$xf_g(x, Q_0^2) = A_g,$$

should be wrong (in our approach $Q_0^2 \ll 1.9 \text{ GeV}^2$).

Possible solution: there are higher twist corrections at low x :

$$xf_g^{\tau 2}(x, Q_0^2) = A_g, \quad \text{but} \quad xf_g^{\text{full}}(x, Q_0^2) \text{ decreases at } x \leq 10^{-3}$$

Usually the higher twist corrections (at least its Q^2 -dependence) are not well studied.

In infrared renormalon model

(E. Stein, M. Maul, L. Mankiewicz, A. Schafer, 1998)

it is possible to show:

(A.Yu.Illarionov, A.V.K. and G.Parente, 2004)

$$x f_a^{\text{full}}(x, Q_0^2) = A_a + \frac{1}{Q_0^2} x f_q^{R\tau^4}(x, Q_0^2) + \frac{1}{Q_0^4} x f_q^{R\tau^5}(x, Q_0^2),$$

where

$$x f_q^{R\tau^4}(x, Q_0^2) = \frac{32c_F f}{15\beta_0} a_q^{\tau^4} \left[A_q \ln^2 \frac{1}{x} + \dots \right],$$

$$x f_g^{R\tau^4}(x, Q_0^2) = \frac{16a_g^{\tau^4}}{5\beta_0 a_s(Q_0^2)} \left[A_g \ln \frac{1}{x} + \dots \right],$$

$$x f_b^{R\tau^6}(x, Q_0^2) = -\frac{8}{7} \left[x f_b^{R\tau^4}(x, Q_0^2) \text{ with } a_b^{\tau^4} \rightarrow a_q^{\tau^6} \right], \quad (b = q, g).$$

1.1 Classical double-logarithmic case ($\bar{d}_{gg}(n) = 0$)

(A.D.Rujula, S.L.Glashow, H.D.Politzer, S.B.Treiman, F.Wilczek and A.Zee, 1974)

Then, expanding the second exponential in the above equation

$$M_g^{cdl}(n, Q^2) = A_g \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(-\hat{d}_{gg} s_{LO})^k}{(n-1)^{k+1}}$$

and using the Mellin transformation for $(\ln(1/x))^k$:

$$\int_0^1 dx x^{n-2} (\ln(1/x))^k = \frac{k!}{(n-1)^{k+1}}$$

we immediately obtain the well known double-logarithmic behavior

$$f_g^{cdl}(x, Q^2) = A_g \sum_{k=0}^{\infty} \frac{1}{(k!)^2} (-\hat{d}_{gg} s_{LO})^k (\ln(1/x))^k = A_g I_0(\sigma_{LO}),$$

where $I_0(\sigma_{LO})$ is the modified Bessel function with argument $\sigma_{LO} = 2\sqrt{\hat{d}_{gg} s_{LO} \ln(x)}$. (R.D.Ball and S.Forte, 1994),

1.2 The more general case

For a regular kernel $\tilde{K}(x)$, having Mellin moment
(nonsingular at $n \rightarrow 1$)

$$K(n) = \int_0^1 dx x^{n-2} \tilde{K}(x)$$

and the PD $f_a(x)$ in the form $I_\nu(\sqrt{\hat{d} \ln(1/x)})$ we have the following equation

$$\tilde{K}(x) \otimes f_a(x) = K(1) f_a(x) + O\left(\sqrt{\frac{\hat{d}}{\ln(1/x)}}\right)$$

So, one can find the general solution for the LO gluon density without the influence of quarks

$$f_g(x, Q^2) = A_g I_0(\sigma_{LO}) e^{-\bar{d}_{gg}(1) s_{LO}} + O(\rho_{LO}),$$

where (R.D.Ball and S.Forte, 1994)

$$\rho_{LO} = \sqrt{\frac{\hat{d}_{gg} s_{LO}}{\ln(x)}} = \frac{\sigma_{LO}}{2 \ln(1/x)}, \quad \bar{\gamma}_{gg}^{(0)}(1) = 22 + \frac{4}{3}f$$

and

$$\bar{d}_{gg}(1) = 1 + \frac{4f}{3\beta_0}$$

with f as the number of active quarks.

2 Leading order (complete)

At the momentum space, the solution of the DGLAP equation at LO has the form (*after diagonalization*)

$$M_a(n, Q^2) = M_a^+(n, Q^2) + M_a^-(n, Q^2) \quad \text{and}$$
$$M_a^\pm(n, Q^2) = M_a^\pm(n, Q_0^2) e^{-d_\pm(n)s} = M_a^\pm e^{-\hat{d}_\pm s / (n-1)} e^{-\bar{d}_\pm(n)s},$$

where $(\varepsilon_{ab}^\pm(n))$ are projectors)

$$M_a^\pm(n, Q^2) = \varepsilon_{ab}^\pm(n) M_b(n, Q^2), \quad d_{ab} = \frac{\gamma_{ab}^{(0)}(n)}{2\beta_0}, \quad (5)$$

As the singular (when $n \rightarrow 1$) part of the + component of the anomalous dimension is **!!!** $\hat{\gamma}_+ = \hat{\gamma}_{gg} = -8C_A$ **!!!** while the – component does not exist: **!!!** $(\hat{\gamma}_- = 0)$ **!!!**, we consider below both cases separately.

2.1 The “+” component

The analysis of the “+” component is practically identical to the case studied before. The only difference lies in the appearance of new terms $\varepsilon_{ab}^+(n)$!!! . If they are expanded in the vicinity of $n = 1$ in the form $\varepsilon_{ab}^+(n) = \bar{\varepsilon}_{ab}^+ + (n - 1)\tilde{\varepsilon}_{ab}^+$, !!! then for the terms $\bar{\varepsilon}_{ab}^+$ multiplying $M_b(n, Q^2)$, we have the same results as in previous section:

$$\bar{\varepsilon}_{ab}^+ M_b(n, Q^2) \xrightarrow{\mathcal{M}^{-1}} \bar{\varepsilon}_{ab}^+ A_b I_0(\sigma_{LO}) e^{-\bar{d}_+(1)s_{LO}} + O(\rho_{LO}),$$

where the symbol $\xrightarrow{\mathcal{M}^{-1}}$ denotes the inverse Mellin transformation.

The values of σ and ρ coincide with those defined in the previous section because $\hat{d}_+ = \hat{d}_{gg}$.

The terms $\tilde{\varepsilon}_{ab}^+$ that come with the additional factor $(n - 1)$ in front, lead to the following results

$$(n - 1)\tilde{\varepsilon}_{ab}^+ \frac{A_b}{(n - 1)} e^{-\hat{d}_+ s_{LO}/(n-1)} = \tilde{\varepsilon}_{ab}^+ A_b \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(-\hat{d}_+ s_{LO})^k}{(n - 1)^k}$$

$$\xrightarrow{\mathcal{M}^{-1}} \tilde{\varepsilon}_{ab}^+ A_b \left[\delta(1 - x) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{(k - 1)!} (-\hat{d}_+ s_{LO})^k (\ln(1/x))^{k-1} \right]$$

$$= \tilde{\varepsilon}_{ab}^+ A_b [\delta(1 - x) + \rho_{LO} I_1(\sigma_{LO})] \approx \tilde{\varepsilon}_{ab}^+ A_b \rho_{LO} I_1(\sigma_{LO}),$$

i.e. the additional factor $(n - 1)$ in momentum space leads to replacing the Bessel function $I_0(\sigma_{LO})$ by $\rho_{LO} I_1(\sigma_{LO})$ in x -space.

Thus, we obtain that the term $\varepsilon_{ab}^+(n) M_b(n, Q^2)$ leads to the following contribution in x space **!!!** :

$$(\bar{\varepsilon}_{ab}^+ I_0(\sigma_{LO}) + \tilde{\varepsilon}_{ab}^+ \rho_{LO} I_1(\sigma_{LO})) A_b e^{-\bar{d}_+(1) s_{LO}} + O(\rho_{LO})$$

Because the Bessel function $I_\nu(\sigma)$ has the ν -independent asymptotic behavior **!!!** $e^\sigma/\sqrt{\sigma}$ at $\sigma \rightarrow \infty$ (i.e. $x \rightarrow 0$), the second term is $O(\rho)$ and must be kept only **!!!** when $\bar{\varepsilon}_{ab}^+ = 0$. This is the case for the quark distribution at the LO approximation.

Using the concrete AD values, one has

$$f_g^+(x, Q^2) = (A_g + \frac{4}{9}A_q)I_0(\sigma_{LO})e^{-\bar{d}_+(1)s_{LO}} + O(\rho_{LO}) \quad \text{and}$$

$$f_q^+(x, Q^2) = \frac{f}{9}(A_g + \frac{4}{9}A_q)\rho_{LO}I_1(\sigma_{LO})e^{-\bar{d}_+(1)s_{LO}} + O(\rho_{LO})$$

where $\bar{d}_+(1) = 1 + 20f/(27\beta_0)$.

2.2 the “-” component

In this case the anomalous dimension is regular !!! and one has

$$\varepsilon_{ab}^-(n) A_b e^{-d_-(n)s} \xrightarrow{\mathcal{M}^{-1}} \bar{\varepsilon}_{ab}^-(1) A_b e^{-d_-(1)s_{LO}} + O(x)$$

Using the concrete AD values !!! , we have

$$f_g^-(x, Q^2) = -\frac{4}{9} A_q e^{-d_-(1)s_{LO}} + O(x) \text{ and}$$

$$f_q^-(x, Q^2) = A_q e^{-d_-(1)s_{LO}} + O(x),$$

where $d_-(1) = 16f/(27\beta_0)$.

Finally we present the full small x asymptotic results for PD and F_2 structure function at LO of perturbation theory:

$$f_a(x, Q^2) = f_a^+(x, Q^2) + f_a^-(x, Q^2) \quad \text{and}$$
$$F_2(x, Q^2) = e \cdot f_q(z, Q^2)$$

where f_q^+, f_g^+, f_q^- and f_g^- were already given before and $e = \frac{\sum_1^f e_i^2}{f}$ is the average charge square of the f active quarks.

Extension to NLO is trivial and can be found in (A.V.K. and G.Parente, 1998)

Extension to an arbitrary form of the boundary conditions

$f_a^\pm(x, Q_0^2)$, leads to the following changes in our analysis:

$$\begin{aligned} A_a^\pm &\rightarrow f_a^\pm(x, Q_0^2), \quad \text{then} \\ f_a^\pm(x, Q^2) &\rightarrow f_a^\pm(x, Q_0^2) \otimes F_a^\pm(x, Q^2), \end{aligned} \quad (6)$$

where $F_a^\pm(x, Q^2)$ are same as $f_a^\pm(x, Q^2)$ (shown before) but

1. no A_a^\pm , because the constants A_a^\pm are changed to the more complicated initial conditions $f_a^\pm(x, Q_0^2)$.

2. with the replacement:

$$\rho_{LO}^m I_m(\sigma_{LO}) \rightarrow \rho_{LO}^{m+1} I_{m+1}(\sigma_{LO}).$$

4. Fits of HERA data

At low x , the structure function $F_2(x, Q^2)$ is related to parton densities as (A.V.K. and G.Parente, 1998)

at LO

$$F_2(x, Q^2) = \frac{5}{18} f_q(x, Q^2)$$

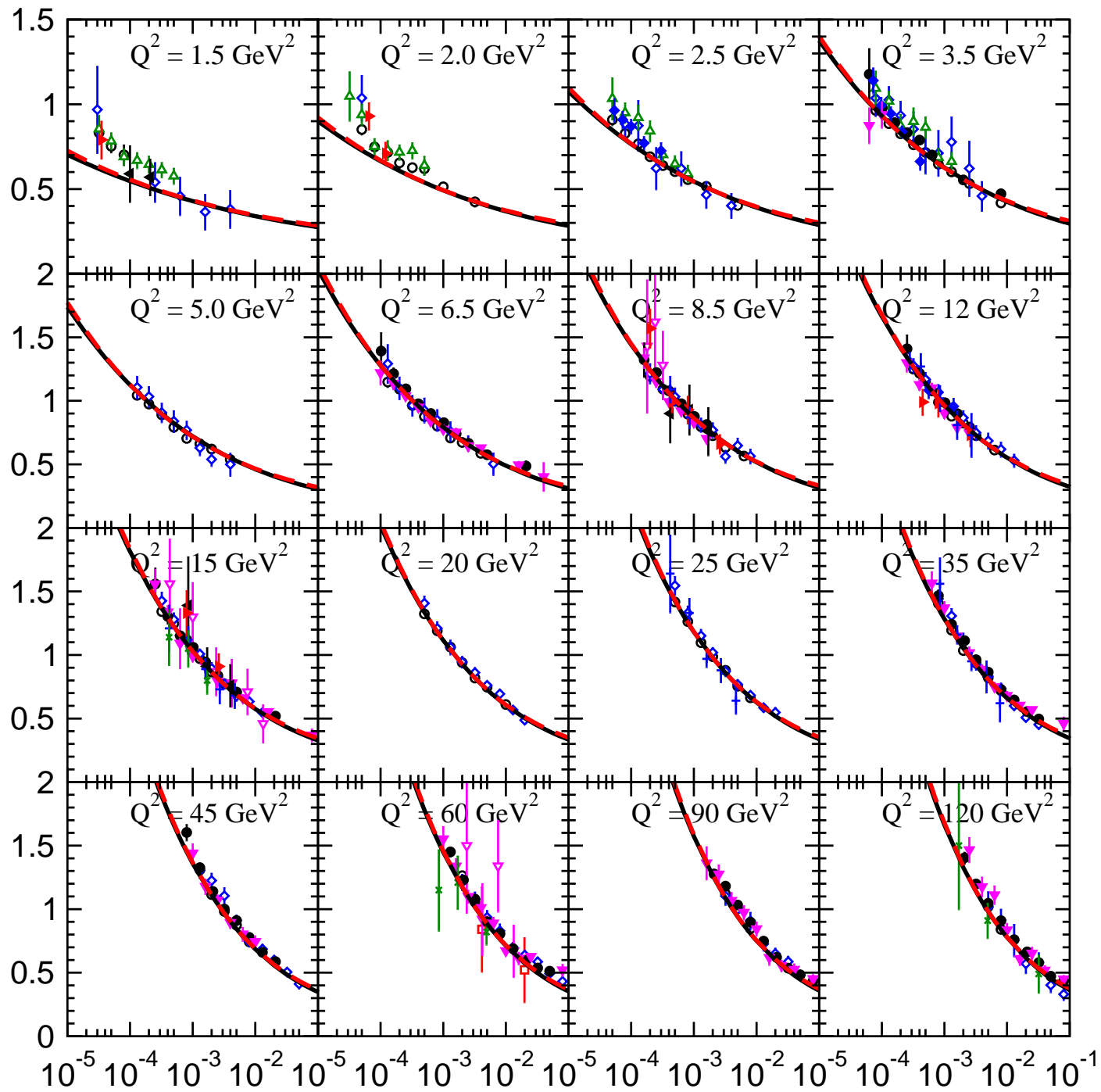
at NLO

$$F_2(x, Q^2) = \frac{5}{18} \left[f_q(x, Q^2) + \frac{2f}{3} a_s(Q^2) f_g(x, Q^2) \right].$$

Fits of HERA experimental data of the structure function $F_2(x, Q^2)$ (A.Yu.Illarionov, A.V.K. and G.Parente, 2004)

!!! Only three parameters: Q_0^2 , A_q and A_g

Λ_{QCD} cannot be extract in small x Physics.



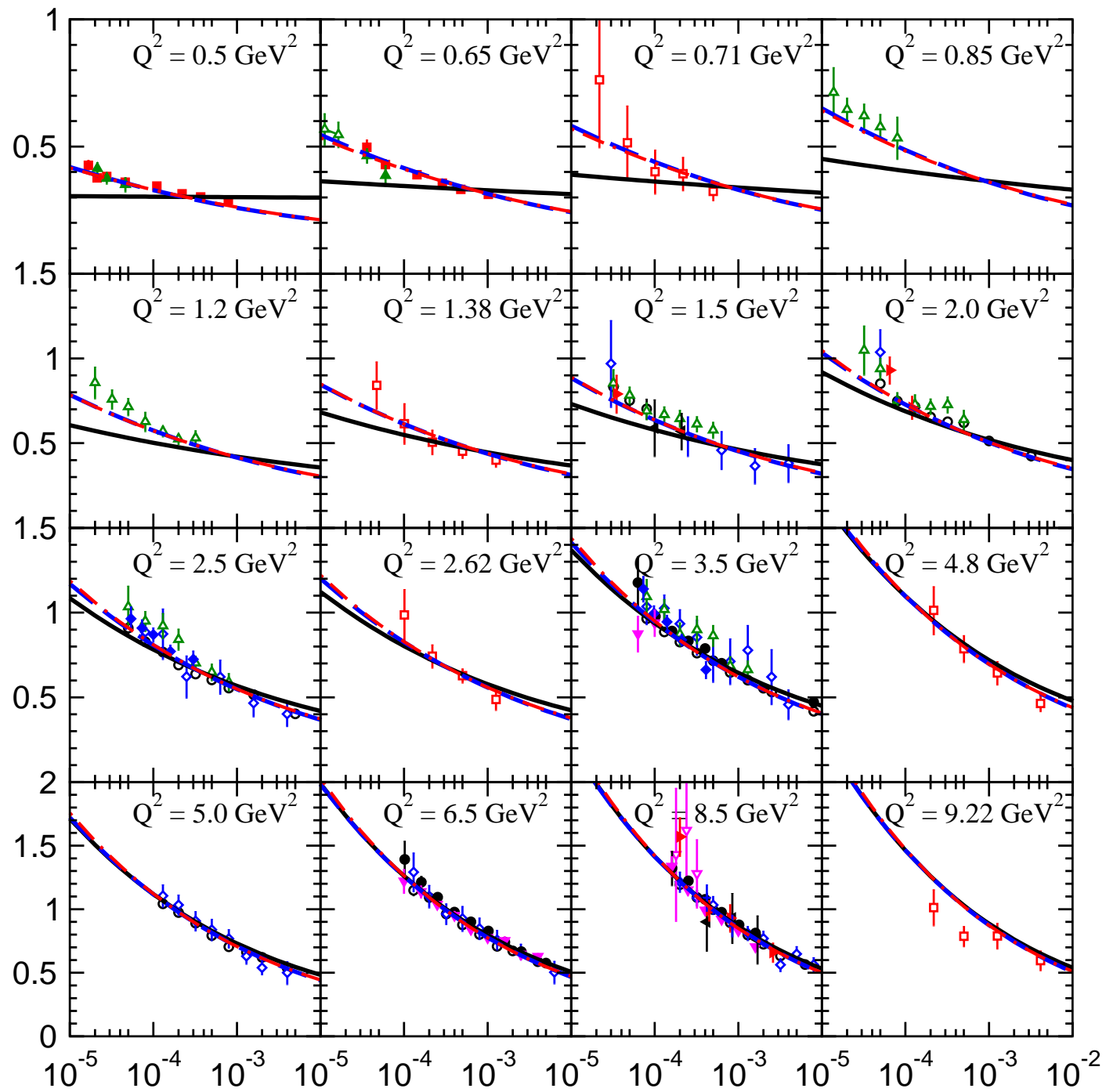
There is a difference between our formulas and the experimental data at low Q^2 values.

There are two possibility to improve our analysis:

1. higher twist (HT) corrections;
2. Infrared modifications of the strong coupling constant.

Infrared renormalon model (E. Stein, M. Maul, L. Mankiewicz, A. Schafer, 1998) with arbitrary magnitudes has been used to estimate for twist-4 and twist-6 corrections.

Now there is a good agreement with experimental data at $Q^2 \leq 2.5 \text{ GeV}^2$. But there are four additional parameters: the magnitudes $A_q^{R\tau 4}$, $A_g^{R\tau 4}$, $A_q^{R\tau 6}$ and $A_g^{R\tau 6}$ ($A_b^{R\tau m} = a_b^{\tau m} A_b$, ($b = q, g$) and ($m = 4, 6$)).



5. Analytical and “frozen” coupling constants

Two modifications of the coupling constant (G.Cvetic, A.Yu.Illarinov, B.A. Kniehl, and A.V.K., 2009); (A.V.K. and B.G. Shaikhatdenov, 2012,2013,2015)

A. More phenomenological.

(G.Curci, M.Greco and Y.Sristava, 1979), (M.Greco, G. Penso and Y.Sristava, 1980), (N.N.Nikolaev and B.M.Zakharov, 1991,1992), (B.Badelek,J.Kwiecinski and A.Stasto, 1997), (A.M.Badalian and Yu.A.Simonov, 1997)

We introduce freezing of the coupling constant by changing its argument $Q^2 \rightarrow Q^2 + M_\rho^2$, where M_ρ is usually the ρ -meson mass. Thus, in the formulae of the previous Sections we should do the following replacement

$$a_s(Q^2) \rightarrow a_{fr}(Q^2) \equiv a_s(Q^2 + M_\rho^2) \quad (7)$$

B. Theoretical approach.

Incorporates the Shirkov-Solovtsov idea ([D.V.Shirkov and L.I.Solovtsov, 1997](#)), about analyticity of the coupling constant that leads to the additional its power dependence.

([K.A.Milton, A.V. Nesterenko, O.Solovtsova, G. Cvetič, C. Valenzuela, I. Schmidt, O. Teryaev, N. Stefanis, A. Bakulev, S. Mikhailov, ...](#))

$$a(Q^2) = \frac{1}{\pi} \int_{\sigma=-\Lambda_{LO}^2}^{\infty} d\sigma \frac{\rho^{pt}(\sigma)}{\sigma + Q^2}$$
$$a_{an}(Q^2) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} d\sigma \frac{\rho^{pt}(\sigma)}{\sigma + Q^2}$$

Then, in the formulae of the previous Section the coupling constant $a_s(Q^2)$ should be replaced as follows

$$a_{an}^{LO}(Q^2) = a_s(Q^2) - \frac{1}{\beta_0} \frac{\Lambda_{LO}^2}{Q^2 - \Lambda_{LO}^2} \quad (8)$$

at the LO approximation and

$$a_{an}(Q^2) = a_s(Q^2) - \frac{1}{2\beta_0} \frac{\Lambda^2}{Q^2 - \Lambda^2} + \dots \quad (9)$$

at the NLO approximation, where the symbol ... marks numerically small terms.

The replacement (8) and (9) is applicable only for rather large values of $Q^2!!!$

For lower Q^2 values it is better to use the fraction analytic perturbation theory

([A. Bakulev, S. Mikhailov, N. Stefanis, 2005](#)) , but its direct application is rather difficult. [Now it is in progress.](#)

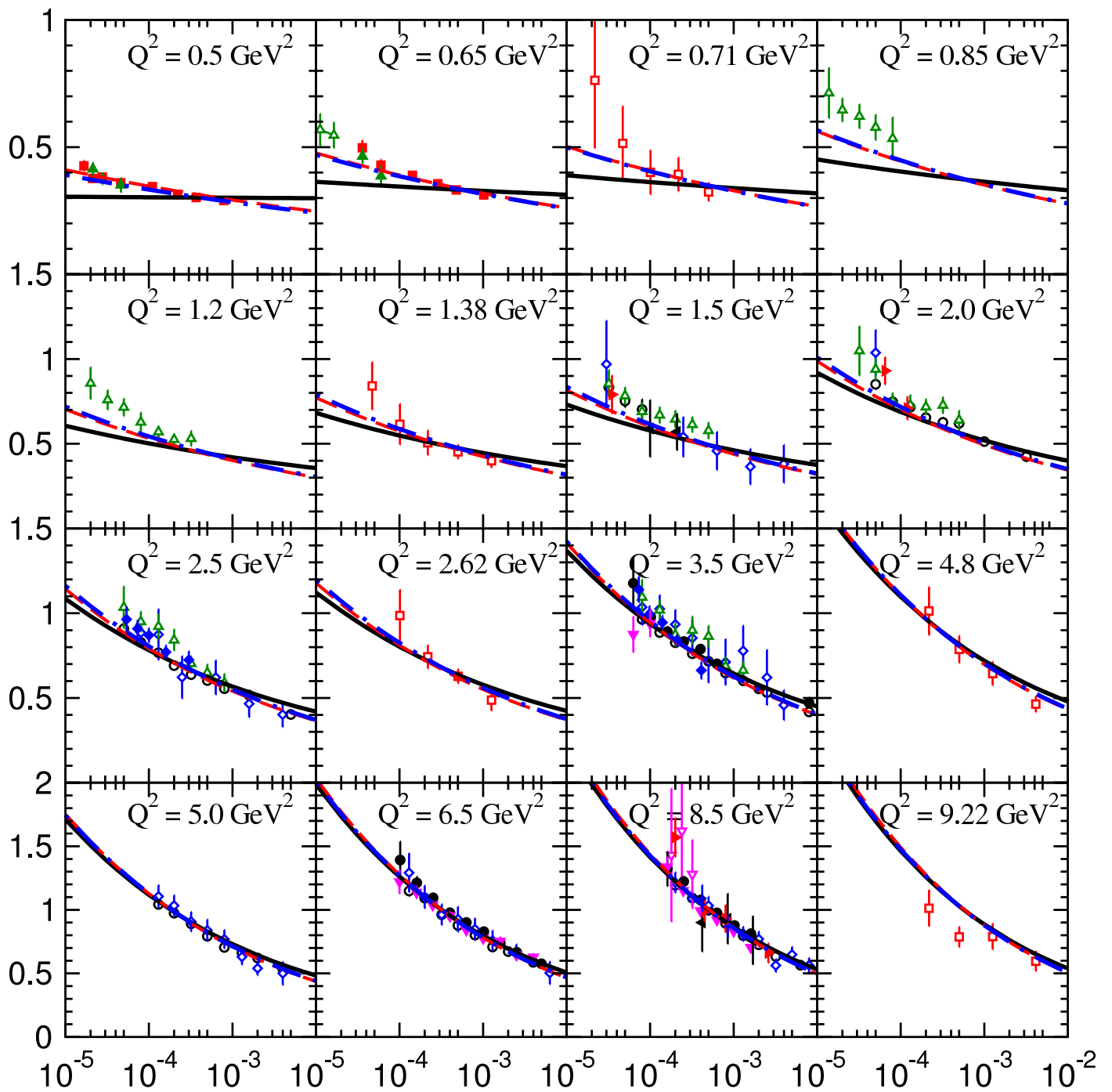


Table 1: The result of the LO and NLO fits to H1 and ZEUS data for different low Q^2 cuts. In the fits f is fixed to 4 flavors.

	A_g	A_q	Q_0^2 [GeV 2]	$\chi^2/n.o.p.$
$Q^2 \geq 1.5\text{GeV}^2$				
LO	$0.784 \pm .016$	$0.801 \pm .019$	$0.304 \pm .003$	754/609
LO&an.	$0.932 \pm .017$	$0.707 \pm .020$	$0.339 \pm .003$	632/609
LO&fr.	$1.022 \pm .018$	$0.650 \pm .020$	$0.356 \pm .003$	547/609
NLO	$-0.200 \pm .011$	$0.903 \pm .021$	$0.495 \pm .006$	798/609
NLO&an.	$0.310 \pm .013$	$0.640 \pm .022$	$0.702 \pm .008$	655/609
NLO&fr.	$0.180 \pm .012$	$0.780 \pm .022$	$0.661 \pm .007$	669/609
$Q^2 \geq 0.5\text{GeV}^2$				
LO	$0.641 \pm .010$	$0.937 \pm .012$	$0.295 \pm .003$	1090/662
LO&an.	$0.846 \pm .010$	$0.771 \pm .013$	$0.328 \pm .003$	803/662
LO&fr.	$1.127 \pm .011$	$0.534 \pm .015$	$0.358 \pm .003$	679/662
NLO	$-0.192 \pm .006$	$1.087 \pm .012$	$0.478 \pm .006$	1229/662
NLO&an.	$0.281 \pm .008$	$0.634 \pm .016$	$0.680 \pm .007$	633/662
NLO&fr.	$0.205 \pm .007$	$0.650 \pm .016$	$0.589 \pm .006$	670/662

- Usage of the analytical and “frozen” coupling constants leads to improvement with data: χ^2 decreased twicely
- Really, no difference between results based on the analytical and “frozen” coupling constants.

!!! One example of application the analytical and “frozen” coupling constants: (A.V.Kotikov, A.V.Lipatov and N.P.Zotov, 2004)

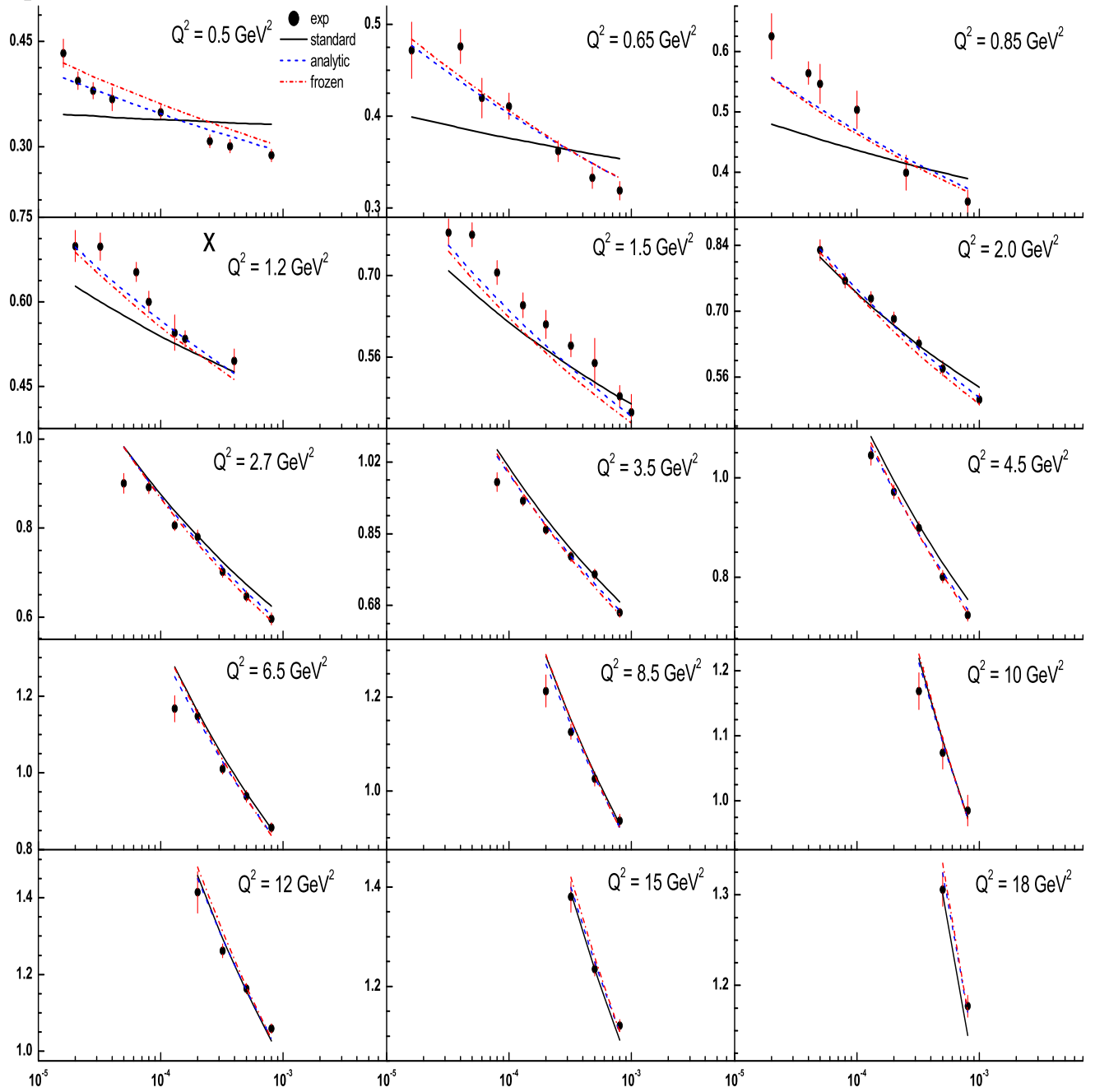
New *H1&ZEUS* (2010) experimental data for F_2 :
(F.D. Aaron *et al.*, 2010)

there is a good agreement for $Q^2 \geq 0.5 \text{ GeV}^2$.

Table 2: The results of LO and NLO fits to H1 & ZEUS data with various lower cuts on Q^2 ; in the fits the n is fixed to 4.

	A_g	A_q	Q_0^2 [GeV 2]	$\chi^2/n.d.f.$
$Q^2 \geq 5\text{GeV}^2$				
LO	0.623±0.055	1.204±0.093	0.437±0.022	1.00
LO&an.	0.796±0.059	1.103±0.095	0.494±0.024	0.85
LO&fr.	0.782±0.058	1.110±0.094	0.485±0.024	0.82
NLO	-0.252±0.041	1.335±0.100	0.700±0.044	1.05
NLO&an.	0.102±0.046	1.029±0.106	1.017±0.060	0.74
NLO&fr.	-0.132±0.043	1.219±0.102	0.793±0.049	0.86
$Q^2 \geq 3.5\text{GeV}^2$				
LO	0.542±0.028	1.089±0.055	0.369±0.011	1.73
LO&an.	0.758±0.031	0.962±0.056	0.433±0.013	1.32
LO&fr.	0.775±0.031	0.950±0.056	0.432±0.013	1.23
NLO	-0.310±0.021	1.246±0.058	0.556±0.023	1.82
NLO&an.	0.116±0.024	0.867±0.064	0.909±0.330	1.04
NLO&fr.	-0.135±0.022	1.067±0.061	0.678±0.026	1.27
$Q^2 \geq 2.5\text{GeV}^2$				
LO	0.526±0.023	1.049±0.045	0.352±0.009	1.87
LO&an.	0.761±0.025	0.919±0.046	0.422±0.010	1.38
LO&fr.	0.794±0.025	0.900±0.047	0.425±0.010	1.30
NLO	-0.322±0.017	1.212±0.048	0.517±0.018	2.00
NLO&an.	0.132±0.020	0.825±0.053	0.898±0.026	1.09
NLO&fr.	-0.123±0.018	1.016±0.051	0.658±0.021	1.31
$Q^2 \geq 0.5\text{GeV}^2$				
LO	0.366±0.011	1.052±0.016	0.295±0.005	5.74
LO&an.	0.665±0.012	0.804±0.019	0.356±0.006	3.13
LO&fr.	0.874±0.012	0.575±0.021	0.368±0.006	2.96
NLO	-0.443±0.008	1.260±0.012	0.387±0.010	6.62
NLO&an.	0.121±0.008	0.656±0.024	0.764±0.015	1.84
NLO&fr.	-0.071±0.007	0.712±0.023	0.529±0.011	2.79

$F_2(x, Q^2)$



NEW SCALE

In the generalized DAS approach parton densities tend to some constant value at $x \rightarrow 0$ and at some initial value Q_0^2 .

The main ingredients of the results are:

1. Both, the gluon and quark singlet densities are presented in terms of two components (" + " and " - ") which are obtained from the analytic Q^2 -dependent expressions of the corresponding (" + " and " - ") PDF moments.

2. The twist-two part of the " − " component is constant at small x at any values of Q^2 , whereas the one of the " + " component grows at $Q^2 \geq Q_0^2$ as

$$\sim I_m(\sigma) \sim e^\sigma, \quad \sigma = 2 \sqrt{\left[|\hat{d}_+| s - \left(\hat{d}_{++} + |\hat{d}_+| \frac{\beta_1}{\beta_0} \right) p \right] \ln \left(\frac{1}{x} \right)},$$

$$\rho = \frac{\sigma}{2 \ln(1/x)},$$

where σ and ρ are the generalized Ball–Forte variables,

$$s = \ln \left(\frac{a_s(Q_0^2)}{a_s(Q^2)} \right), \quad p = a_s(Q_0^2) - a_s(Q^2),$$

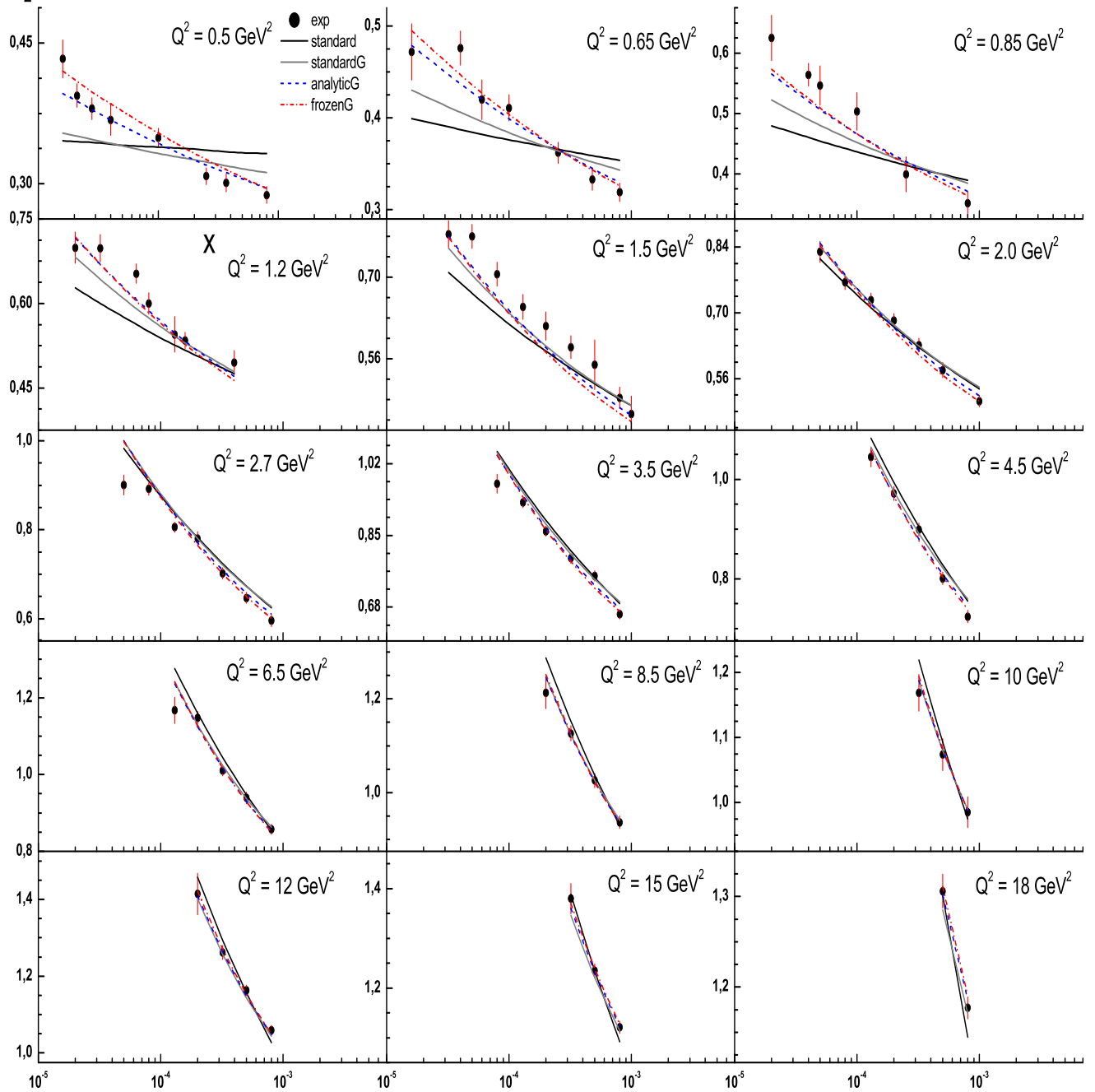
$$\hat{d}_+ = -\frac{12}{\beta_0}, \quad \hat{d}_{++} = \frac{412}{27\beta_0}.$$

At low Q^2 , the value of the strong coupling constant is large. The most important terms, which are singular at $n \rightarrow 1$, where n is the Mellin moment, are collected in the result for σ^2 . So, it is convenient to choose the scale μ^2 of the strong coupling at which the NLO contribution $\sim p$ vanishes. This choice is

$$\mu^2 = Q^2 \exp \left[\left(\hat{d}_{++} + |\hat{d}_+| \beta_1 / \beta_0 \right) / (|\hat{d}_+| \beta_0) \right] \approx 3.89 Q^2$$

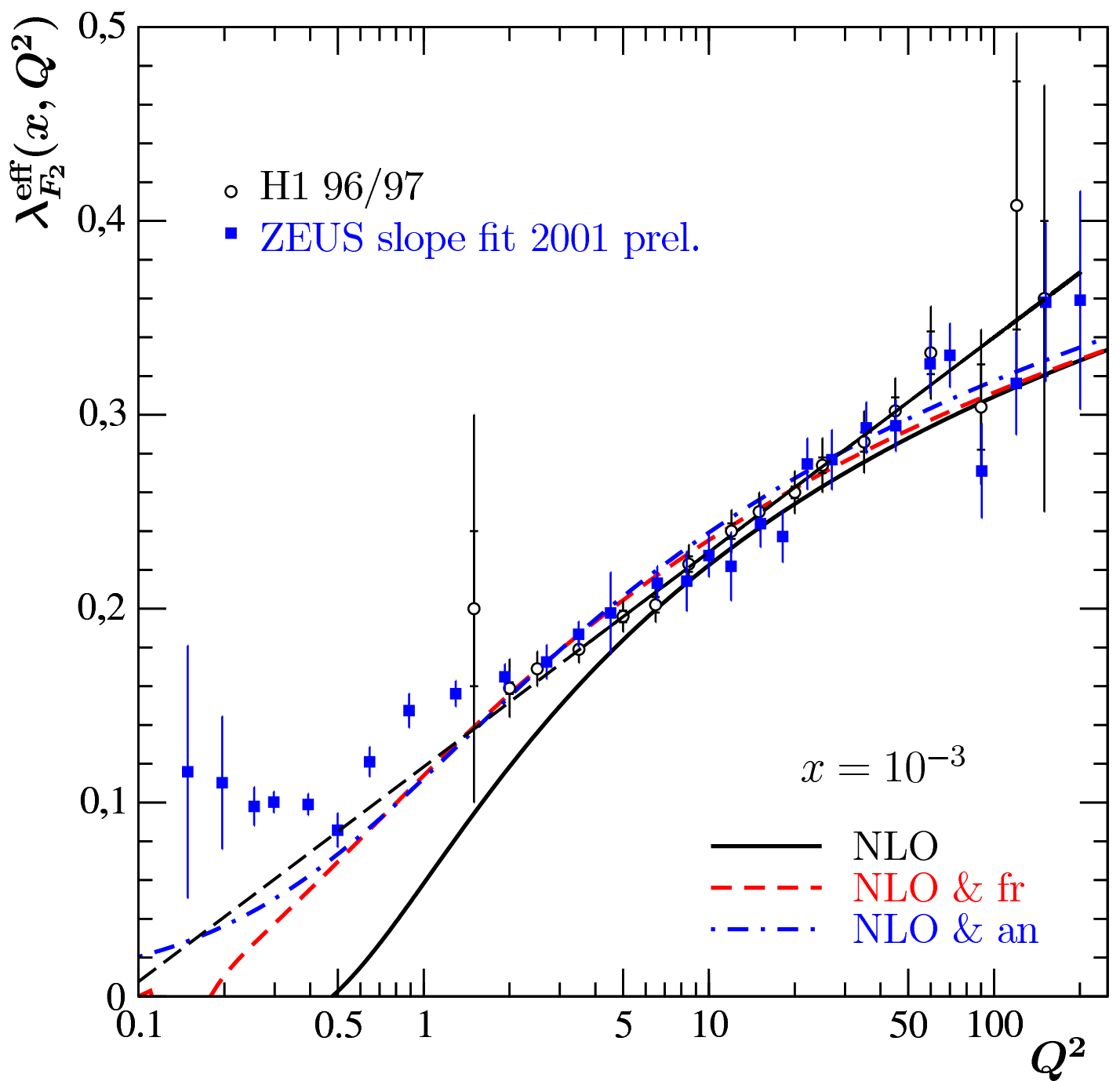
where the symbol \approx marks the case $f = 3$, which is relevant at low Q^2 values. We see that this choice of scale increases effectively the argument of coupling constant at low x values.

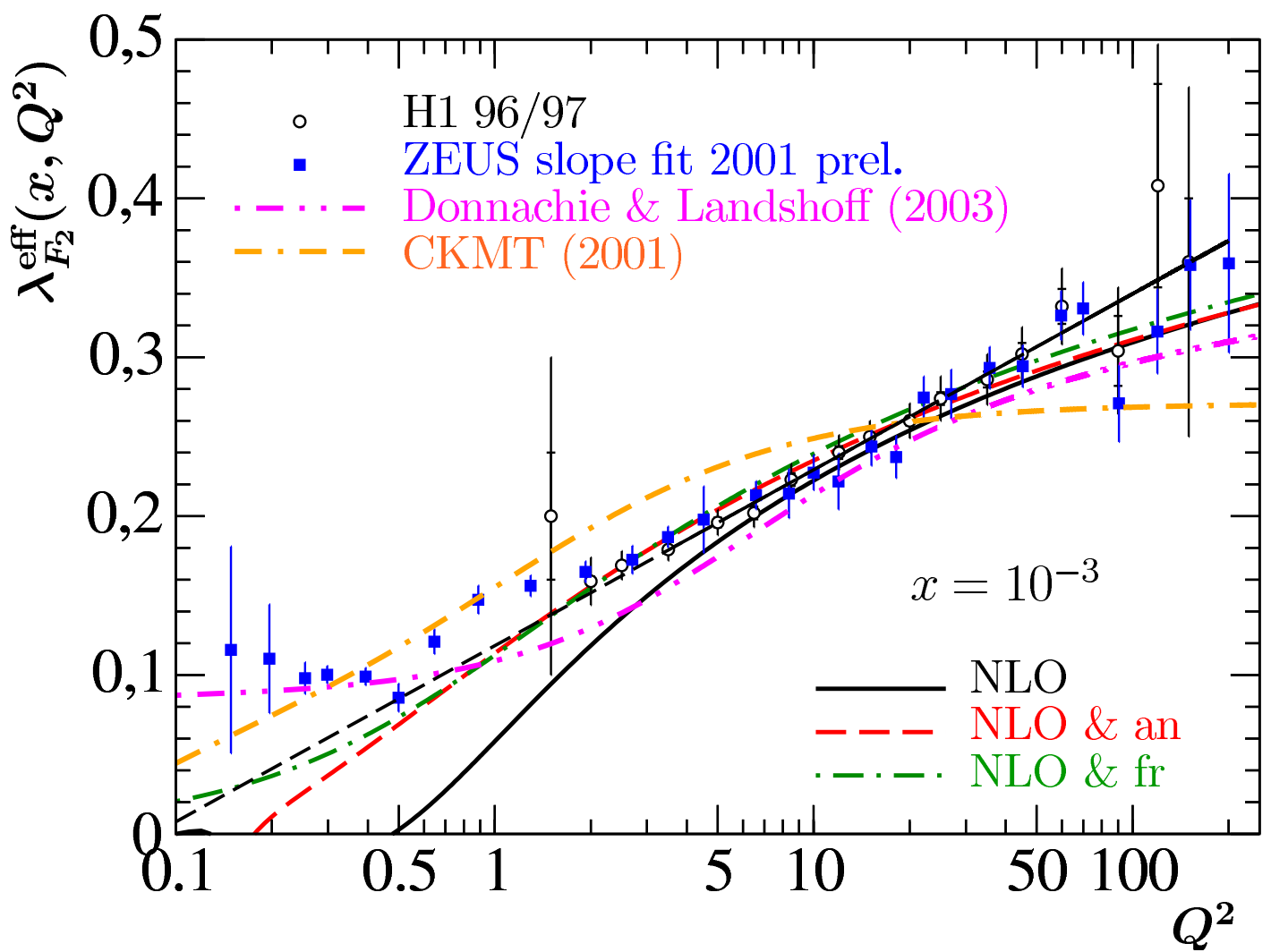
$F_2(x, Q^2)$

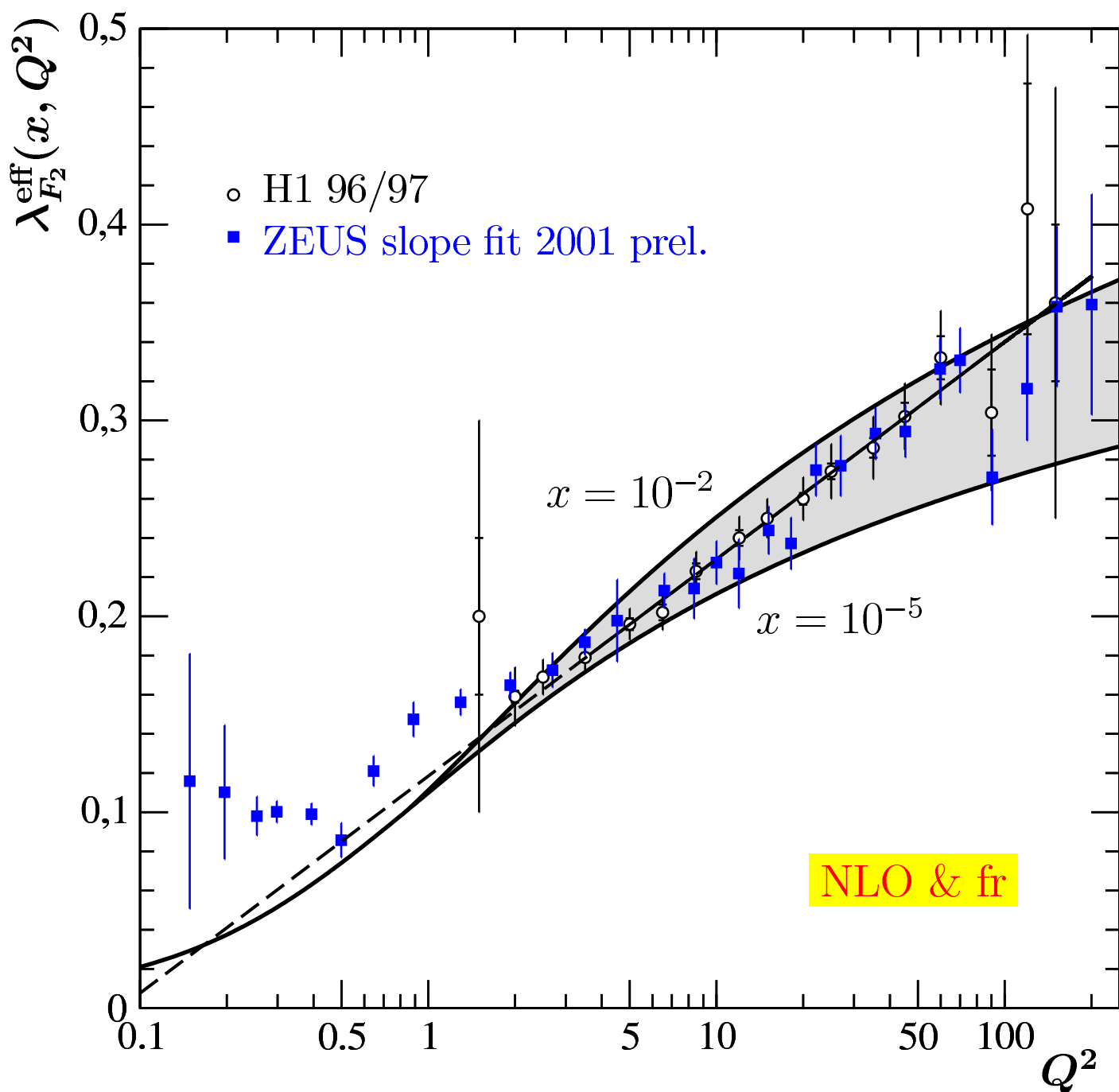


The results for F_2 and for the slope of the SF F_2
The double-logarithmic behaviour can mimic a power law shape
over a limited region of x, Q^2 .

$$f_a(x, Q^2) \sim x^{-\lambda_a^{eff}(x, Q^2)} \quad \text{and} \quad F_2(x, Q^2) \sim x^{-\lambda_{F_2}^{eff}(x, Q^2)}$$







6. Charm contribution

Consider the contribution of heavy quarks separately (really only charm one). So, at NLO ($f = 3$) in the framework of photon-gluon fusion (PGF)

$$\begin{aligned} F_2(x, Q^2) &= F_2^q(x, Q^2) + F_2^Q(x, Q^2), \\ F_2^q(x, Q^2) &= e_f \left[f_q(x, Q^2) + \frac{2f}{3} a_s(Q^2) f_g(x, Q^2) \right], \\ F_2^Q(x, Q^2) &\equiv F_2^c(x, Q^2) \approx M_{2,g}(1, Q^2, \mu^2) f_g(x, \mu^2), \end{aligned}$$

where $M_{2,g}(1, Q^2, \mu^2, m_c^2)$ is the first moment of the corresponding Wilson coefficient function $C_{2,g}(x, Q^2, \mu^2, m_c^2)$.

Through NLO, $M_{2,g}(1, Q^2, \mu^2, m_c^2)$ exhibits the structure

$$\begin{aligned} M_{2,g}(1, Q^2, \mu^2, m_c^2) &= e_f^2 a_s(\mu) \{ M_{2,g}^{(0)}(1, c) \\ &+ a_s(\mu) \left[M_{2,g}^{(1)}(1, c) + M_{2,g}^{(2)}(1, c) \ln \frac{\mu^2}{m_c^2} \right] \} + \mathcal{O}(a_s^3). \end{aligned}$$

2.3 LO results for F_2^c

The LO coefficient function of PGF can be obtained from the QED case (V.N. Baier, V.S. Fadin and V.A. Khoze, 1966), (V.G. Zima, 1972), (V.M. Budnev, I.F. Ginzburg, G.V. Meledin and V.G. Serbo, 1974) by adjusting coupling constants and color factors, and they read (E. Witten, 1975), (J.P. Leveille and T.J. Weile, 1977), (V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, 1978)

$$C_{2,g}^{(0)}(x, c) = -2x\{[1 - 4x(2 - c)(1 - x)]\beta - [1 - 2x(1 - 2c) + 2x^2(1 - 6c - 4c^2)]L(\beta)\},$$

where

$$c = \frac{m_c^2}{Q^2}, \quad \beta(x) = \sqrt{1 - \frac{4cx}{1-x}}, \quad L(\beta) = \ln \frac{1+\beta}{1-\beta}.$$

Performing the Mellin transformation

$$M_{2,g}^{(0)}(1, c) = \int_0^b dx x^{n-1} C_{2,g}^{(0)}(x, c),$$

we find

$$M_{2,g}^{(0)}(1, c) = \frac{2}{3}[1 + 2(1 - c)J(c)]$$

with

$$J(c) = -\sqrt{b} \ln t, \quad t = \frac{1 - \sqrt{b}}{1 + \sqrt{b}}, \quad b = \frac{1}{1 + 4c}.$$

2.4 NLO results

The NLO coefficient functions of PGF are rather lengthy and not published in print; they are only available as computer codes.

(E. Laenen, S. Riemersma, J. Smith and W.L. van Neerke, 1992)

For $x \ll 1$, it is possible to use the compact form (S. Catani, M. Ciafaloni and F. Hautmann, 1994)

$$C_{2,g}^{(j)}(x, c) = \beta R_{2,g}^{(j)}(1, c),$$

with

$$R_{2,g}^{(1)}(1, c) = \frac{8}{9}C_A[5 + (13 - 10c)J(c) + 6(1 - c)I(c)],$$

$$R_{2,g}^{(2)}(1, c) = -4C_A M_{2,g}^{(0)}(1, c),$$

where $C_A = N$ for the color gauge group SU(N), and

$$I(c) = -\sqrt{b} \left[\zeta(2) + \frac{1}{2} \ln^2 t - \ln(bc) \ln t + 2 \text{Li}_2(-t) \right],$$

with $\text{Li}_2(x) = -\int_0^1 (dy/y) \ln(1 - xy)$ is the dilogarithmic function.

The Mellin transforms of $C_{k,g}^{(j)}(x, c)$ exhibit singularities in the limit $n \rightarrow 1$. So, now we have the terms involving $1/\delta_{\pm}$, which correspond to singularities of the Mellin moments $M_{2,g}^{\pm}(n)$ at $n \rightarrow 1$ and depend on the exact form of the subasymptotic low- x behavior encoded in $f_g^{\pm}(x, \mu^2)$. The modification is simple:

$$\frac{1}{n-1} \rightarrow \frac{1}{\tilde{\delta}_{\pm}}, \quad \frac{1}{\tilde{\delta}_{\pm}} = \frac{1}{f_g^{\pm}(\hat{x}, \mu^2)} \int_{\hat{x}}^1 \frac{dy}{y} f_g^{\pm}(y, \mu^2),$$

where $\hat{x} = x/b$. In the considered case

$$\frac{1}{\tilde{\delta}_+} \approx \frac{1}{\rho(\hat{x})} \frac{I_1(\sigma(\hat{x}))}{I_0(\sigma(\hat{x}))}, \quad \frac{1}{\tilde{\delta}_-} \approx \ln \frac{1}{\hat{x}},$$

where σ and ρ were introduced earlier.

Because the ratio $f_g^-(x, Q^2)/f_g^+(x, Q^2)$ is rather small at the Q^2 values considered, then

$$F_2^c(x, Q^2) \approx \tilde{M}_{2,g}(1, \mu^2, c) x f_g(x, \mu^2),$$

where $\tilde{M}_{2,g}(1, \mu^2)$ is obtained from $M_{2,g}(n, \mu^2)$ by taking the limit $n \rightarrow 1$ and replacing $1/(n-1) \rightarrow 1/\tilde{\delta}_+$:

$$M_{2,g}^{(j)}(1, c) \rightarrow \tilde{M}_{2,g}^{(j)}(1, c) \approx \left[\frac{1}{\delta_+} - \ln(bc) - \frac{J(c)}{b} \right] R_{2,g}^{(j)}(1, c) \quad (j = 1, 2)$$

with $R_{2,g}^{(j)}(1, a)$ ($j = 1, 2$) are given above.

3 Results for F_2

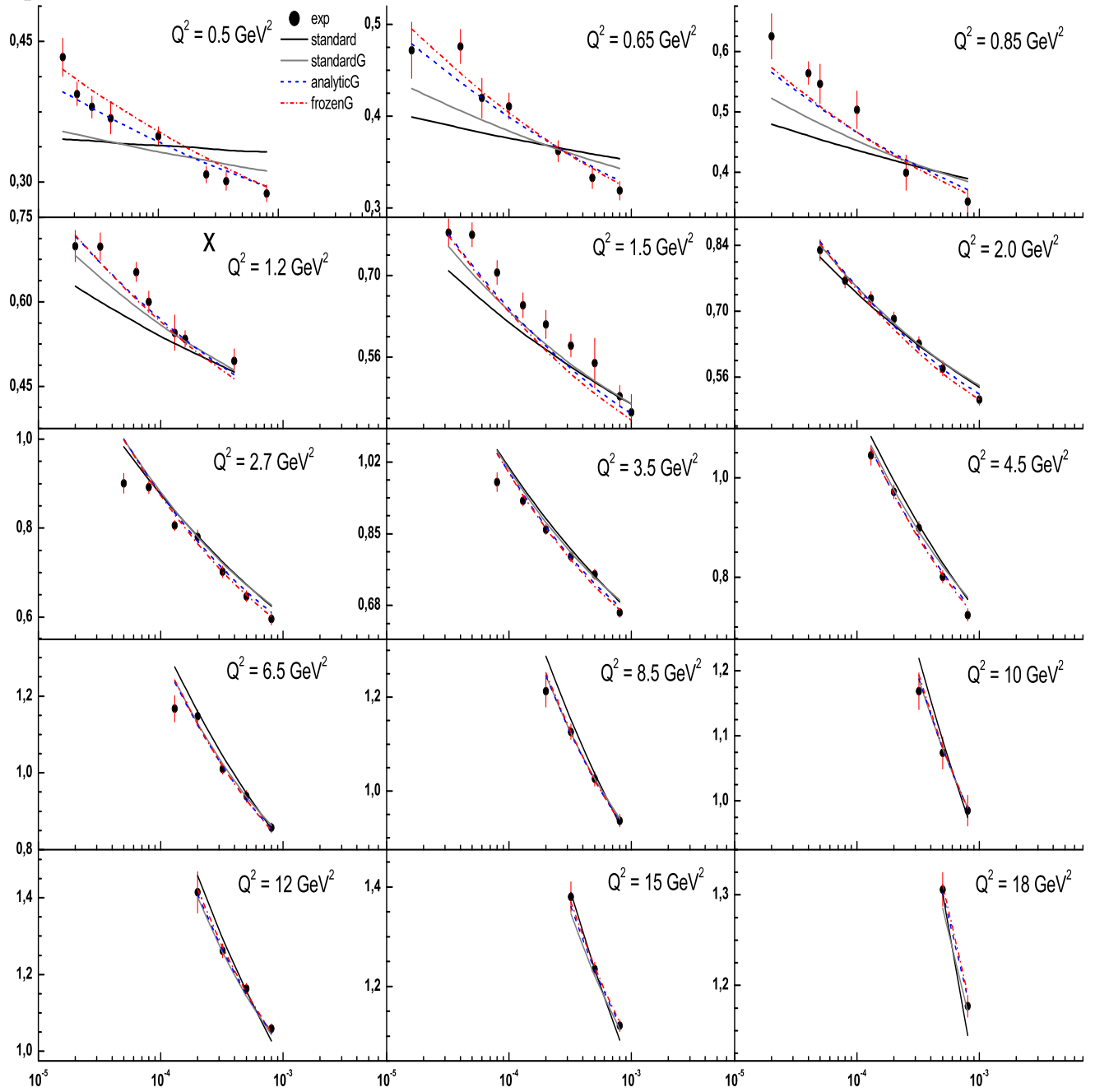
Using above equations, we fit the new *H1&ZEUS* data for F_2 .
(F.D. Aaron *et al.*, 2010) using our formulas

$$\begin{aligned} F_2(x, Q^2) &= F_2^q(x, Q^2) + F_2^c(x, Q^2), \\ F_2^q(x, Q^2) &= \frac{2}{9} \left[f_q(x, Q^2) + 2a_s(Q^2) f_g(x, Q^2) \right], \\ F_2^c(x, Q^2) &= \frac{4}{9} a_s(Q^2) M_{2,g}^{(0)}(1, c) f_g(x, \mu^2), \\ M_{2,g}^{(0)}(1, c) &= \frac{2}{3} [1 + 2(1 - c)J(c)] \end{aligned} \tag{10}$$

As for our input parameters, we choose $m_c = 1.25$ GeV in agreement with Particle Data Group.

The results are in rather good agreement in the case of the analytic and “frozen” coupling constants.

$F_2(x, Q^2)$



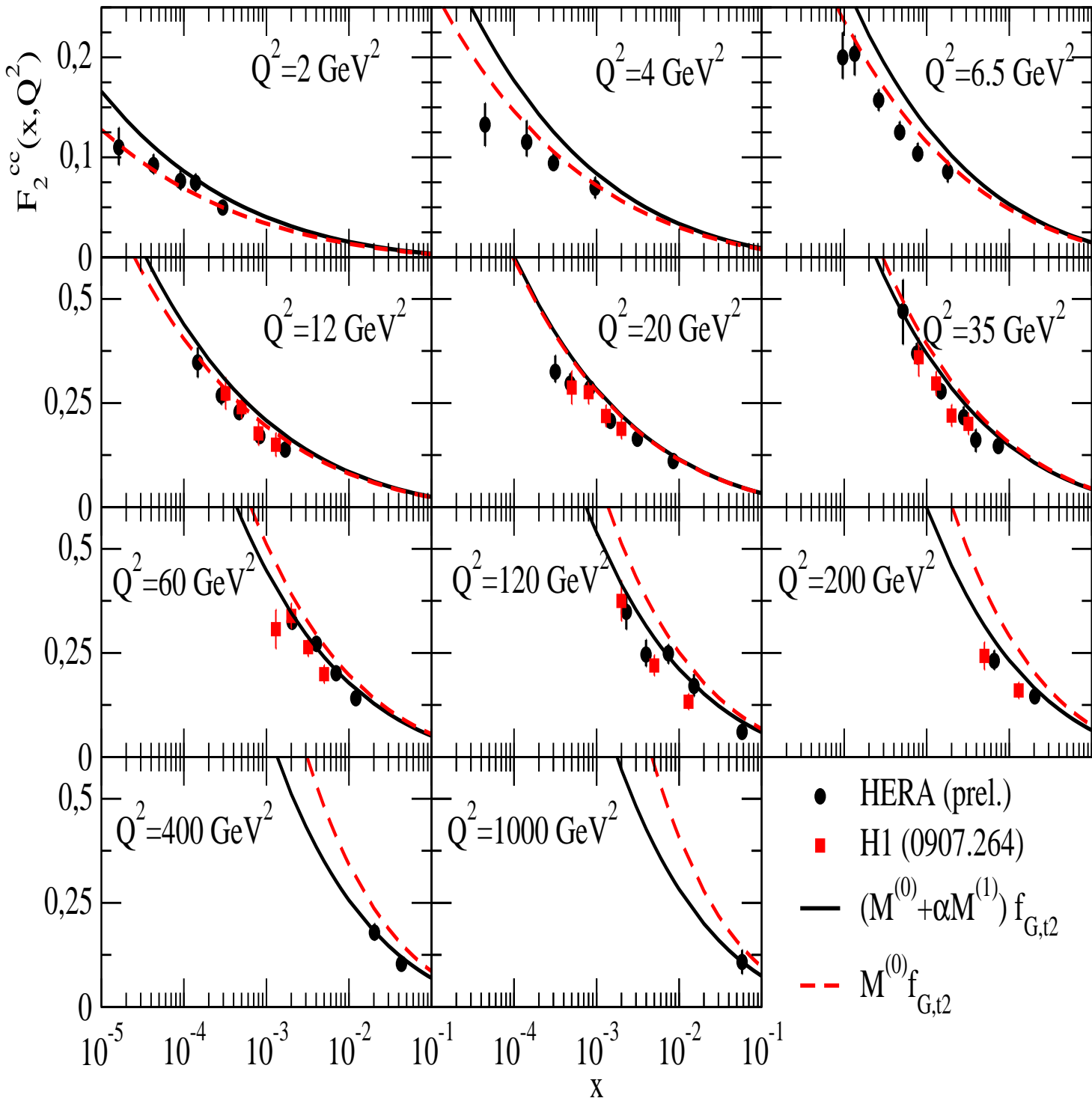
4 Results for F_2^c

We use the LO and NLO formulas

$$F_{2,LO}^c(x, Q^2) = \frac{4}{9}a_s(Q^2)M_{2,g}^{(0)}(1, c)f_g(x, \mu^2),$$
$$F_{2,NLO}^c(x, Q^2) = \frac{4}{9}a_s(Q^2)\{M_{2,g}^{(0)}(1, c) + a_s(\mu) \left[M_{2,g}^{(1)}(1, c) + M_{2,g}^{(2)}(1, c) \ln \frac{\mu^2}{m_c^2} \right]\} f_g(x, \mu^2).$$

We put $\mu^2 = Q^2 + 4m_c^2$, which is the standard scale in heavy quark production.

The PDF parameters μ_0^2 , A_q and A_g have been fixed in the fits of F_2 experimental data (of *H1* Collaboration). So, the analysis of F_2^c of *H1* Collaboration has no free parameters. We have got a rather good agreement with *H1* experimental data.



As a next step. we will study an agreement with the new *H1&ZEUS* data for F_2^C (H. Abramowitz *et al.*, 2012) using the corresponding analysis of *H1&ZEUS* data for F_2 (F.D. Aaron *et al.*, 2010)

Is it possible to fit the mass $m_c(m_c)$.

6. BFKL corrections

(V.S. Fadin, E.A. Kuraev and L.N. Lipatov, 1975, 1976, 1977),

(I.I. Balitsky and L.N. Lipatov, 1978)

(V.S. Fadin and L.N. Lipatov, 1998)

Idea:

(in this section it is convenient to change $\gamma^{\text{DIS}}/2 \rightarrow -\gamma$)

$$\gamma_+(n) = \frac{\hat{\gamma}_+}{\omega} + \bar{\gamma}_+(n), \quad (\omega = n - 1)$$

$$\frac{\hat{\gamma}_+}{\omega} \rightarrow \gamma_{\text{BFKL}}(\omega) = \gamma(\omega)$$

$$\omega = 4A_s \chi(\gamma), \quad \chi(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1 - \gamma),$$

$$A_s = C_A a_s, \quad \Psi(\gamma) = \frac{d(\ln(\Gamma(\gamma)))}{d\gamma},$$

where $\chi(\gamma)$ is so-called Lipatov characteristic function.

I. *Expansions*

(A.V.Kotikov, 2012)

$$\chi(\gamma) = \frac{1}{\gamma} + 2 \sum_{k=1}^{\infty} \zeta(2k+3) \gamma^{2k+2},$$

that is equal

$$\gamma = \gamma_0 \left[1 + 2 \sum_{k=1}^{\infty} \zeta(2k+3) \gamma_0^{2k+3} \right], \quad \gamma_0 = \frac{4A_s}{\omega} \quad (11)$$

or equal to

$$\gamma = \gamma_0 + \sum_{k=3}^{\infty} C_k \gamma_0^{k+1}$$

with the following nonzero C_k :

$$C_3 = 2\zeta(3), C_5 = 2\zeta(5), C_6 = 12\zeta^2(3), C_7 = 2\zeta(7), \dots \quad (12)$$

For the renormalization exponent we have the following replacement

$$\begin{aligned} & \exp\left\{-\frac{1}{\beta_0} \int_{a_s(Q_0^2)}^{a_s(Q^2)} \frac{da}{a^2} \frac{\hat{\gamma}_+(a)}{\omega}\right\} \\ \rightarrow & \exp\left\{-\frac{1}{\beta_0} \int_{a_s(Q_0^2)}^{a_s(Q^2)} \frac{da}{a^2} \gamma(a)\right\} \end{aligned}$$

and the following additional contribution

$$-\frac{1}{\beta_0} \int \frac{da}{a^2} \sum_{k=3}^{\infty} C_k \gamma_0^{k+1}(a) = -\frac{1}{\beta_0} \frac{1}{a} \sum_{k=3}^{\infty} \frac{C_k}{k} \gamma_0^{k+1}(a)$$

So, there is the additional factor k in denominator. We hope that in this form the BFKL corrections will be not so large!!!

NLO BFKL corrections can be evaluated in the same way.

II. *Exact form (LO BFKL)*

(H. Kowalski, L.N. Lipatov and D.A. Ross, 2012),

(A.V.Kotikov, 2012)

So, we should consider the contribution

$$\exp\left\{-\frac{1}{\beta_0} \int_{a_s(Q_0^2)}^{a_s(Q^2)} \frac{da}{a^2} \gamma(a)\right\}$$

Using integration by parts procedure:

$$\int \frac{da}{a^2} \gamma(a) = -\frac{1}{a} \gamma(a) + \int \frac{da}{a} \frac{d\gamma(a)}{da} = -\frac{1}{a} \gamma(a) + \int \frac{d\gamma}{a}$$

BFKL:

$$\frac{1}{a} = \frac{4C_A}{\omega} \chi(\gamma),$$

and

$$\int d\gamma \chi(\gamma) = 2\Psi(1)\gamma + \ln \frac{\Gamma(1-\gamma)}{\Gamma(\gamma)}$$

So, we have

$$\exp \left\{ -\frac{1}{\beta_0} \int_{a_s(Q_0^2)}^{a_s(Q^2)} \frac{da}{a^2} \gamma(a) \right\} = \frac{R(Q^2)}{R(Q_0^2)},$$

where

$$R(Q^2) = \left[\frac{\Gamma(\gamma)}{\Gamma(1-\gamma)} \right]^{d_0} \exp \{ -d_0 \gamma (\Psi(\gamma) + \Psi(1-\gamma)) \} \quad (13)$$

with

$$d_0 = \frac{4C_A}{\beta_0 \omega} = -\frac{\hat{d}_+}{\omega} = -\frac{\hat{d}_{gg}}{\omega}$$

At $a_s \rightarrow 0$ (i.e. $\gamma \rightarrow 0$) we recover the singular part of the LO contributions

$$R(Q^2) \rightarrow \gamma^{-d_0} \rightarrow a_s^{-d_0}$$

The following steps:

- To perform the inverse Mellin transformation (in some approximate form, because the structure of (13) is rather complicated).
- To perform fits with the terms (13) taken into account.

Conclusion

- I have demonstrated the low x asymptotics of parton densities and SF F_2 and F_2^C .
- Low x asymptotics of F_2 are in good agreement with data from HERA at $Q^2 \geq 2.5 \text{ GeV}^2$.
- Usage of the analytical and “frozen” coupling constants leads to improvement with data from HERA at $Q^2 \leq 2.5 \text{ GeV}^2$, including the new H1+ZEUS data for F_2 .
(F.D. Aaron *et al.*, 2010).
- BFKL corrections taken into account (only first steps).

Next steps:

- To add the NNLO corrections (which has $\sim 1/(n-1)^2$ poles at $n \rightarrow 1$). So, the NNLO small- x asymptotics $\sim \exp[\sim (\ln(1/x))^{2/3}]$ is more singular than the corresponding LO and NLO ones $\sim I_k(\sim \sqrt{\ln(1/x)}) \sim \exp[\sim \sqrt{\ln(1/x)}]$.
- To consider the new H1+ZEUS data for F_2^C and to fit $m_c(m_c)$.
(H. Abramowitz *et al.*, 2012).
- To finish with BFKL corrections (at LO and NLO levels of approximation).
- To investigate a usage of less restricted initial conditions