# Coupled dark energy: a dynamical analysis with complex scalar field

Ricardo C. G. Landim

University of São Paulo

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# Interacting dark energy

$$\dot{\rho_\phi} + 3H(\rho_\phi + p_\phi) = -\mathcal{Q}, \quad (1)$$

$$\dot{\rho_m} + 3H\rho_m = \mathcal{Q}, \quad (2)$$

$$\dot{\rho_r} + 4H\rho_r = 0, \quad (3)$$

The case of $\mathcal{Q} > 0$ corresponds to dark energy transformation into dark matter, while $\mathcal{Q} < 0$ is the transformation in the opposite direction.

- Quintessence and phantom: $\mathcal{Q} = Q\rho_m\dot{\phi}$ [Wetterich 1995, Amendola 2000]
- Tachyon: $\mathcal{Q} = Q\rho_m\rho_\phi\dot{\phi}/H$

# Dynamical system theory: a quick review

System of differential equations with dimensionless variables

$$X' = f[X], \tag{4}$$

The critical points $X_c$ satisfies $X' = 0$.
Linear perturbation $U$ thus $X = X_c + U$.

$$U' = \mathcal{J} U, \tag{5}$$

where $\mathcal{J}$ is the Jacobian matrix. The critical points are

- stable (all eigenvalues $< 0$),
- unstable (all eigenvalues $> 0$) or
- saddle points (if at least one eigenvalue is positive and the others are negative, or vice-versa).

# Quintessence and Phantom

$$\mathcal{L} = -\sqrt{-g}\left(\frac{\epsilon}{2}\partial^{\mu}\Phi\partial_{\mu}\Phi + V(|\Phi|)\right), \tag{6}$$

where $\epsilon = +1$ for quintessence and $\epsilon = -1$ for phantom. $\Phi = \phi e^{i\theta}$

$$\epsilon\ddot{\phi} + 3\epsilon H\dot{\phi} + V'(\phi) - \epsilon\phi\dot{\theta}^2 = 0, \tag{7}$$

$$\epsilon\ddot{\theta} + \left(3H + \frac{2\dot{\phi}}{\phi}\right)\dot{\theta} = 0. \tag{8}$$

# Friedmann equations

$$H^2 = \frac{1}{3}\left(\frac{\epsilon}{2}\dot{\phi}^2 + \frac{\epsilon}{2}\phi^2\dot{\theta}^2 + V(\phi) + \rho_m + \rho_r\right), \tag{9}$$

$$\dot{H} = -\frac{1}{2}\left(\epsilon\dot{\phi}^2 + \epsilon\phi^2\dot{\theta}^2 + \rho_m + \frac{4}{3}\rho_r\right), \tag{10}$$

The equation of state becomes

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 + \phi^2\dot{\theta}^2 - 2\epsilon V(\phi)}{\dot{\phi}^2 + \phi^2\dot{\theta}^2 + 2\epsilon V(\phi)}. \tag{11}$$

# Dynamical analysis

$$x_1 \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad x_2 \equiv \frac{\phi\dot{\theta}}{\sqrt{6}H}, \quad x_3 \equiv \frac{\sqrt{6}}{\phi}, \quad y \equiv \frac{\sqrt{V(\phi)}}{\sqrt{3}H},$$
$$z \equiv \frac{\sqrt{\rho_r}}{\sqrt{3}H}, \quad \lambda \equiv -\frac{V'}{V}, \quad \Gamma \equiv \frac{VV''}{V'^2}. \tag{12}$$

$$\Omega_\phi \equiv \frac{\rho_\phi}{3H^2} = \epsilon x_1^2 + \epsilon x_2^2 + y^2, \quad \Omega_\phi + \Omega_m + \Omega_r = 1, \tag{13}$$

where $\Omega_i = \rho_i/(3H^2)$. $0 \leq \Omega_\phi \leq 1$ implies

$$0 \leq \epsilon x_1^2 + \epsilon x_2^2 + y^2 \leq 1 \tag{14}$$

$$w_\phi = \frac{\epsilon x_1^2 + \epsilon x_2^2 - y^2}{\epsilon x_1^2 + \epsilon x_2^2 + y^2}, \tag{15}$$

$$w_{eff} = \frac{p_\phi + p_r}{\rho_\phi + \rho_m + \rho_r} = \epsilon x_1^2 + \epsilon x_2^2 - y^2 + \frac{z^2}{3}, \tag{16}$$

$$\frac{dx_1}{dN} = -3x_1 + x_2^2 x_3 + \frac{\sqrt{6}}{2}\epsilon y^2\lambda - \frac{\sqrt{6}}{2}\epsilon Q(1 - x_1^2 - x_2^2 - y^2 - z^2) - x_1 H^{-1}\frac{dH}{dN}, \tag{17}$$

$$\frac{dx_2}{dN} = -3x_2 - x_1 x_2 x_3 - x_2 H^{-1}\frac{dH}{dN}, \tag{18}$$

$$\frac{dx_3}{dN} = -x_1 x_3^2, \tag{19}$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}x_1 y\lambda - yH^{-1}\frac{dH}{dN}, \tag{20}$$

$$\frac{dz}{dN} = -2z - zH^{-1}\frac{dH}{dN}, \tag{21}$$

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2 x_1 \left(\Gamma - 1\right), \tag{22}$$

where

$$H^{-1}\frac{dH}{dN} = -\frac{3}{2}(1 + \epsilon x_1^2 + \epsilon x_2^2 - y^2) - \frac{z^2}{2}. \tag{23}$$

# Critical points

| Point | Existence | $x_1$ | $x_2$ | $x_3$ | $y$ | $z$ | $w_\phi$ | $\Omega_\phi$ | $w_{eff}$ |
|---|---|---|---|---|---|---|---|---|---|
| (a) | $Q=0$ | 0 | 0 | any | 0 | 0 | – | 0 | 0 |
| (b) | $\epsilon=+1$ | $\frac{-\sqrt{6}Q}{3}$ | 0 | 0 | 0 | 0 | 1 | $\frac{2Q^2}{3}$ | $\frac{2Q^2}{3}$ |
| (c) | any | 0 | 0 | any | 0 | 1 | – | 0 | $\frac{1}{3}$ |
| (d) | $\epsilon=+1$ | $\frac{-1}{\sqrt{6}Q}$ | 0 | 0 | 0 | $\sqrt{1-\frac{1}{2Q^2}}$ | 1 | $\frac{1}{6Q^2}$ | $\frac{1}{3}$ |
| (e) | $\epsilon=+1$ | $\frac{2\sqrt{6}}{3\lambda}$ | 0 | 0 | $\frac{2\sqrt{3}}{3\lambda}$ | $\sqrt{1-\frac{4}{\lambda^2}}$ | $\frac{1}{3}$ | $\frac{4}{\lambda^2}$ | $\frac{1}{3}$ |
| (f) | $\epsilon=+1$ | any | $\sqrt{1-x_1^2}$ | 0 | 0 | 0 | 1 | 1 | 1 |
| (g) | any | $\frac{\sqrt{6}}{2(\lambda+Q)}$ | 0 | 0 | $\sqrt{\frac{2Q(Q+\lambda)+3\epsilon}{2(\lambda+Q)^2}}$ | 0 | $\frac{-Q(Q+\lambda)}{Q(Q+\lambda)+3\epsilon}$ | $\frac{Q(Q+\lambda)+3\epsilon}{(\lambda+Q)^2}$ | $\frac{-Q}{\lambda+Q}$ |
| (h) | any | $\frac{\epsilon\lambda}{\sqrt{6}}$ | 0 | 0 | $\sqrt{1-\frac{\epsilon\lambda^2}{6}}$ | 0 | $-1+\frac{\epsilon\lambda^2}{3}$ | 1 | $-1+\frac{\epsilon\lambda^2}{3}$ |

$$V(\phi) = V_0 e^{-\lambda\phi} \tag{24}$$

[Copeland et al 1998, Amendola 2000, Gumjudpai et al 2005]

# Summary

- For almost all fixed points, but (a) and (c), $x_3 = 0$, which means $\phi \rightarrow \infty$. However, this limit implies that $x_2 \propto \phi\dot{\theta}/H \rightarrow \infty$ as well.
- Only (a) and (c) are physically viable and they describe the sequence: radiation $\rightarrow$ matter. Both of them are unstable, however there does not exist a point that describe the dark-energy-dominated universe.
- The extra degree of freedom due to the phase $\theta$ spoils the physically acceptable fixed points that exist for the case of real scalar field, indicating that the dynamical system theory is not a good tool when one tries to analyze the complex quintessence (phantom).

# Tachyon dynamics

$$\mathcal{L}_{BI} = -\sqrt{-g}V(|\Phi|)\sqrt{1-\partial^\mu\Phi\partial_\mu\Phi}, \tag{25}$$

$$\frac{\ddot{\phi}}{1-\phi^2\dot{\theta}^2-\dot{\phi}^2}+\frac{\ddot{\theta}+\dot{\phi}\dot{\theta}\phi^{-1}}{(1-\phi^2\dot{\theta}^2-\dot{\phi}^2)}\frac{\phi^2\dot{\phi}\dot{\theta}}{(1-\phi^2\dot{\theta}^2)}+\frac{3H\dot{\phi}-\phi\dot{\theta}^2}{1-\phi^2\dot{\theta}^2}+\frac{V'(\phi)}{V(\phi)}=0, \tag{26}$$

$$\frac{\ddot{\theta}}{1-\phi^2\dot{\theta}^2-\dot{\phi}^2}+3H\dot{\theta}+\frac{2\dot{\phi}\dot{\theta}}{\phi(1-\phi^2\dot{\theta}^2-\dot{\phi}^2)}=0, \tag{27}$$

# Friedmann equations

$$H^2 = \frac{1}{3}\left(\frac{V(\phi)}{\sqrt{1-\phi^2\dot{\theta}^2-\dot{\phi}^2}} + \rho_m + \rho_r\right), \tag{28}$$

$$\dot{H} = -\frac{1}{2}\left(\frac{V(\phi)(\dot{\phi}^2+\phi^2\dot{\theta}^2)}{\sqrt{1-\phi^2\dot{\theta}^2-\dot{\phi}^2}} + \rho_m + \frac{4}{3}\rho_r\right), \tag{29}$$

The equation of state for the tachyon field is

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \dot{\phi}^2 + \phi^2\dot{\theta}^2 - 1, \tag{30}$$

thus, the tachyon behavior is between the cosmological constant one ($w_\phi = -1$) and matter one ($w_\phi = 0$).

$$\mathcal{Q} = Q\rho_m\rho_\phi\dot{\phi}/H \tag{31}$$

# Dynamical analysis

$$x_1 \equiv \dot{\phi}, \quad x_2 \equiv \phi\dot{\theta}, \quad x_3 \equiv \frac{1}{H\phi}, \quad y \equiv \frac{\sqrt{V(\phi)}}{\sqrt{3}H},$$
$$z \equiv \frac{\sqrt{\rho_r}}{\sqrt{3}H}, \quad \lambda \equiv -\frac{V'}{V^{3/2}}, \quad \Gamma \equiv \frac{VV''}{V'^2}. \tag{32}$$

Since $\dot{\phi}$ and $\theta$ are dimensionless variables, $\phi$ has dimension of time.

$$\Omega_\phi \equiv \frac{\rho_\phi}{3H^2} = \frac{y^2}{\sqrt{1 - x_1^2 - x_2^2}}, \tag{33}$$

Due to $0 \leq \Omega_\phi \leq 1$

$$0 \leq x_1^2 + x_2^2 + y^4 \leq 1. \tag{34}$$

$$w_\phi = x_1^2 + x_2^2 - 1, \tag{35}$$

$$w_{eff} = \frac{p_\phi + p_r}{\rho_\phi + \rho_m + \rho_r} = -y^2\sqrt{1 - x_1^2 - x_2^2} + \frac{z^2}{3}, \tag{36}$$

$$\frac{dx_1}{dN} = -(1 - x_1^2 - x_2^2) \times \left[3x_1 - \sqrt{3}y\lambda + 3Q\left(1 - z^2 - \frac{y^2}{\sqrt{1 - x_1^2 - x_2^2}}\right)\right] + x_2^2 x_3, \tag{37}$$

$$\frac{dx_2}{dN} = -x_1 x_2 x_3 - 3x_2(1 - x_1^2 - x_2^2), \tag{38}$$

$$\frac{dx_3}{dN} = -x_1 x_3^2 + \frac{x_3}{2}\left[3 + z^2 - \frac{3y^2(1 - x_1^2 - x_2^2)}{\sqrt{1 - x_1^2 - x_2^2}}\right], \tag{39}$$

$$\frac{dy}{dN} = \frac{y}{2}\left[-\sqrt{3}x_1 y\lambda + 3 + z^2 - \frac{3y^2(1 - x_1^2 - x_2^2)}{\sqrt{1 - x_1^2 - x_2^2}}\right], \tag{40}$$

$$\frac{dz}{dN} = -2z + \frac{z}{2}\left[3 + z^2 - \frac{3y^2(1 - x_1^2 - x_2^2)}{\sqrt{1 - x_1^2 - x_2^2}}\right], \tag{41}$$

$$\frac{d\lambda}{dN} = -\sqrt{3}\lambda x_1 y\left(\Gamma - \frac{3}{2}\right). \tag{42}$$

# Critical points

| Point | $x_1$ | $x_2$ | $x_3$ | $y$ | $z$ | $w_\phi$ | $\Omega_\phi$ | $w_{eff}$ |
|---|---|---|---|---|---|---|---|---|
| (a) | 1 | 0 | $\frac{3}{2}$ | 0 | 0 | 0 | 0 | 0 |
| (c) | 1 | 0 | 2 | 0 | $\pm 1$ | 0 | 0 | $\frac{1}{3}$ |
| (d) | 0 | 0 | any | 0 | $\pm 1$ | $-1$ | 0 | $\frac{1}{3}$ |
| (e) | 0 | 0 | any | 1 | 0 | $-1$ | 1 | $-1$ |
| (f) | $\frac{\lambda y_c}{\sqrt{3}}$ | 0 | $\frac{\sqrt{3}\lambda y_c}{2}$ | $y_c$ | 0 | $\frac{\lambda^2 y_c^2}{3} - 1$ | 1 | $w_\phi$ |
| (h) | $x_f$ | 0 | $\frac{\sqrt{3\lambda y_f}}{2}$ | $y_f$ | 0 | $x_f^2 - 1$ | $\frac{w_{eff}}{w_\phi}$ | $\frac{x_f y_f \lambda}{\sqrt{3}} - 1$ |

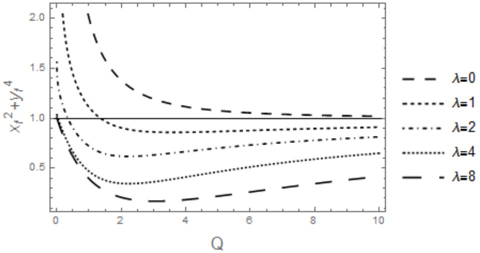
$$y_c = \sqrt{\frac{\sqrt{\lambda^4 + 36} - \lambda^2}{6}}, \tag{43}$$

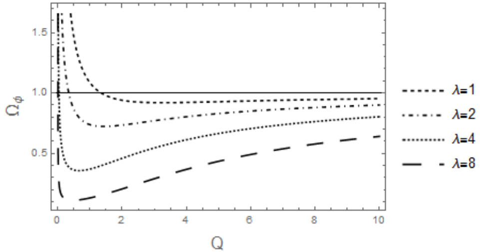
$$x_f = -\frac{Q}{2} \pm \frac{\sqrt{Q^2 + 4}}{2}, \quad y_f = \frac{-\lambda x_f + \sqrt{\lambda^2 x_f^2 + 12\sqrt{1 - x_f^2}}}{\sqrt{12(1 - x_f^2)}}. \tag{44}$$

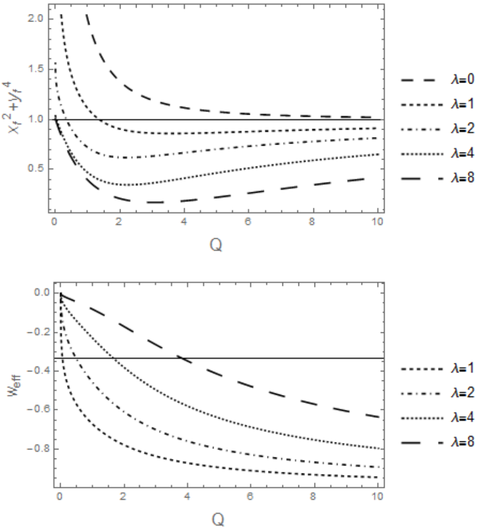
$$V(\phi) = V_0 \phi^{-2} \tag{45}$$

[Aguirregabiria 2004, Gumjudpai et al 2005, Copeland et al 2005]

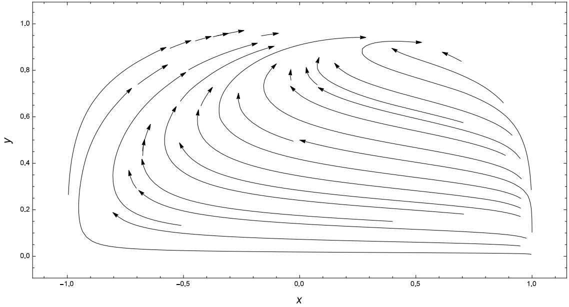
| Point | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\mu_5$ | Stability |
|---|---|---|---|---|---|---|
| (a) | $6(1+Q)$ | $-\frac{3}{2}$ | $-\frac{3}{2}$ | $\frac{3}{2}$ | $-\frac{1}{2}$ | saddle |
| (c) | $6$ | $-2$ | $-2$ | $2$ | $1$ | saddle |
| (d) | $-3$ | $-3$ | $2$ | $4$ | $1$ | saddle |
| (e) | $-3$ | $-3$ | $0$ | $-3$ | $-2$ | stable |
| (f) | $\sqrt{3}Q\lambda y_c - 3\left(1-\frac{\lambda^2 y_c^2}{3}\right)$ | $\frac{\lambda^2 y_c^2}{2} - 3$ | $-\frac{\lambda^2 y_c^2}{2}$ | $\frac{\lambda^2 y_c^2}{2} - 3$ | $\frac{\lambda^2 y_c^2}{2} - 2$ | stable for $\lambda < 0$ or $Q = 0$ |
| (h) | $3\left(x_f^2 - \frac{x_f y_f \lambda}{\sqrt{3}}\right)$ | $-\frac{\sqrt{3}}{2}\lambda x_f y_f - 3(1-x_f)$ | $-\frac{\sqrt{3}}{2}\lambda x_f y_f$ | $\frac{3}{2}\left(\frac{x_f y_f \lambda}{\sqrt{3}} - 2\right)$ | $\frac{3}{2}\left(\frac{x_f y_f \lambda}{\sqrt{3}} - \frac{4}{3}\right)$ | stable |


2.0
1.5
1.0
0.5
$x_f^2+y_f^4$
0
2
4
6
8
10
Q
λ=0
λ=1
λ=2
λ=4
λ=8


1.5
1.0
0.5
$\Omega_\phi$
0
2
4
6
8
10
Q
λ=1
λ=2
λ=4
λ=8




We have $\Omega_\phi = 1$ for $\lambda = 1$ and $Q \approx 1.35$, and for $\lambda = 2$ and $Q \approx 0.35$, for instance. The effective equation of state for these two values is $w_{eff} = -0.72$ and $-0.29$, respectively.




Phase plane $x_f - y_f$ for $\lambda = 1$ and $Q = 1.35$. The tachyonic-dominated solution (h) is a stable point at $x \approx 0.53$ and $y \approx 0.92$.

# Summary

- The cosmological transition radiation $\rightarrow$ matter $\rightarrow$ dark energy is achieved considering the following sequence of fixed points: (c) or (d) $\rightarrow$ (a) $\rightarrow$ (f) [$\lambda$ dependent] or (h) [$Q$ and $\lambda$ dependent]