**Negative mass instability [2]** Particles with $v \approx c$ cannot travel much faster by acceleration but will increase their momentum and thus move with a larger radius of curvature. Above transition energy, particles which normally repel each other, seem to experience an attractive force. This is known as *negative mass effect* for its similarity with the formation of planetary rings with an otherwise attractive gravitational force outside the *Roche limit.* In uniform beams this may lead to self-bunching and particle loss when the current exceeds a threshold

$$I_{\rm thresh} = F' \frac{E_0}{e} \frac{|\eta|}{\gamma Z/n} \left( \frac{\Delta p_{\rm FWHM}}{m_0 c} \right)^2 \quad (3)$$

where $\Delta p_{\rm FWHM}$ is the momentum spread (full width at half maximum). The form factor $F'$, of order unity, depends on the particle distribution (see Tab. 1).

**Resistive wall instability** Currents are induced in metallic vacuum chamber walls by the beam. Due to the finite resistivity of the walls, they extend behind the position of ultra-relativistic charges. Their EM fields act on charged particles arriving later and may increase their oscillation amplitudes, and cause instabilities in bunched or unbunched beams, in the longitudinal [3] or a transverse direction [4].

The skin depth in a metal of resistivity $\rho_c$ and permeability $\mu$ is $\delta_s = \sqrt{2\rho_c/(\mu\omega)}$. For metal walls thicker than $\delta_s$, the *resistive wall impedances* of a circular pipe of radius $b$ are

$$\frac{Z_\parallel}{n}(\omega) = \frac{\mu Z_0}{2\mu_0 b}\delta_s, \quad Z_\perp(\omega) = \frac{\mu R Z_0}{\mu_0 b^3}\delta_s \quad (4)$$

where $2\pi R$ is accelerator circumference. Since $Z_\parallel/n \propto Z_\perp \propto \delta_s \propto \omega^{-1/2}$, the largest impedance occurs at the lowest frequency. For transverse instabilities, this lowest frequency occurs at

$$\omega_{\rm min} = \omega_0 \min\{\nu - [\nu], [\nu+1] - \nu\} \quad (5)$$

where $[\nu]$ = non-integer part of the betatron tune $\nu$. See Eq.(11), Sec.2.5.7 for growth rate. This single spectral line has narrow band width, and thus oscillates for a long time and may cause coupled bunch instabilities. Impedances at all other betatron frequencies may be grouped into a wide frequency band. These fields decay rapidly, hence drive only single bunch instabilities such as head-tail modes discussed below.

Longitudinal coupled-bunch instabilities can also be caused by a resistive wall, but are usually weaker. In general, impedances due to all elements should be added to determine beam stability, as can be done with computer programs [5, 6].

Table 1: Form factors $F'$ for various distributions.

| $F'$ | arg $Z = \tan^{-1}$ (Im $Z$ /Re $Z$) | | | |
|---|---|---|---|---|
| Distrib. | $-\pi/2$ | 0 | $\pi/2$ | Remarks |
| Parabolic | 1.047 | 0.798 | 0 | sharp-edge |
| 3/2 power | 1.061 | 1.061 | .0611 | Keil-Schnell |
| Quartic | 1.073 | 1.194 | 2.203 | rounded |
| Gaussian | 0.942 | 1.359 | $\infty$ | rounded |
| Triangular | 0 | 2.0 | 0 | sharp-edge |
| Rounded triangle | 0.555 | 1.98 | 1.265 | with 1% rounding |

**Microwave instability** In addition to potential well bunch lengthening (Sec.2.5.5), an increase of bunch current $I_b$ can lead to longitudinal instabilities of a single bunch, independent of the presence of other bunches in the machine. [See also Eqs.(12-15), Sec.2.5.7] Because it is often accompanied by high-frequency signals, it is also called microwave instability. Oscillation frequencies change with amplitude due to non-linearities, hence this instability is usually self-limiting and only rarely leads to particle loss, but may reduce luminosity in colliders since bunches do not keep an optimum distribution.

For long bunches, e.g. of proton beams, the impedance is mainly inductive and the threshold current is given by the *Boussard criterion* [7],

$$\frac{Z_\parallel}{n} \le F' \frac{E_0}{e} \frac{\eta\gamma\sigma}{I_b} \left( \frac{\Delta p_{\rm FWHM}}{p_\parallel} \right)^2 \quad (6)$$

This agrees with the (simplified) unbunched-beam or *Keil-Schnell criterion* [33],

$$\frac{Z_\parallel}{n} \le F' \frac{E_0}{e} \frac{|\eta|}{\gamma I_0} \left( \frac{\Delta p_{\rm FWHM}}{m_0 c} \right)^2 \quad (7)$$

applied to the *local values of current and energy spread* in a bunch. The form factor $F'$, shown in Tab.1, depends on particle distribution and ratio of real to imaginary part of the impedance [9, 10].

For short bunches, the impedance seen by the beam is dominated by resonances at higher frequencies, and may lead to *turbulent bunch lengthening*. A tentative explanation for this is *longitudinal mode coupling* [11] (Sec.2.4.7).

Operating with $\eta < 0$ can reduce bunch lengthening since the usually inductive impedance then shortens it [12]. However, theoretical and experimental investigations show that the turbulent threshold is lower with the shorter bunch, giving a larger energy spread.

**Head-tail instability** Short-range transverse wake fields, excited by particles at the head of a bunch, may excite oscillations at its tail. Synchrotron motion brings these particles again to the head and they continue to excite particles behind. These oscillations will grow (head-tail instability) if they add in phase due to a finite chromaticity $\xi = \Delta\nu/\delta$ [13, 14] and if the growth rate exceeds radiation and Landau damping.

In the lowest mode, $m = 0$, all particles in a bunch oscillate in phase, which corresponds to a rigid dipole oscillation at the betatron frequency. The $m = 0$ mode is unstable for $\xi < 0$. Since the natural chromaticity of circular machines is negative, one must correct it with sextupoles. *Bunch shape* modes with $m \ge 1$ oscillate at synchrotron side-band frequencies $\omega_\beta \pm m\omega_s$. In the $m = 1$ mode, particles at head and tail have opposite phases (when $\xi = 0$), the bunch "toggles" about its middle, while higher modes ($m \ge 2$) oscillate with $m$ *nodes* over the bunch length. Modes with $m \ge 1$ may become unstable for $\xi > 0$, but their growth rates are usually small and easily stabilized by damping except for large machines with very strong transverse wakes.

For short bunches, the growth rate of the $m$-th mode is (ignoring radial modes) [15]

$$\frac{1}{\tau_m} = \frac{N_B r_0 c}{2\pi^2\gamma} \frac{\xi\omega_\beta\hat{z}}{c\eta\nu_\beta^2} \times \int_0^\infty d\omega {\rm Re} Z_\perp(\omega) J_m(\frac{\omega\hat{z}}{c}) J'_m(\frac{\omega\hat{z}}{c}) \quad (8)$$

with $N_B$ number of particles per bunch, $\hat{z}$ the $z$-amplitude of synchrotron oscillation (airbag model), $r_0$ classical particle radius, $\omega_\beta$ the betatron frequency.

Example For resistive wall [14],

$$\frac{1}{\tau_m} = -\frac{\sqrt{2}J_m}{\pi^{5/2}} \frac{r_0 c}{\sqrt{\mu\sigma_c}} \frac{N_B \xi\sqrt{\hat{z}}}{\eta\gamma b^3 \nu_\beta} \quad (9)$$

with $\mu$ permeability, $\sigma_c$ conductivity, $b$ chamber half-height. The factor $J_m = \left(\int_0^{\pi/2} dx\sqrt{\sin x}\right) \left(\int_0^\pi d\psi \cos m\psi \sqrt{\sin\frac{\psi}{2}}\right)$ for $m$ = 0,1,2 is approximately 2.9, -0.57, -0.21. For negative chromaticity $\xi < 0$, the $m = 0$ mode is strongly unstable, while the higher modes are weakly unstable for $\xi > 0$.

Example For the case with constant wake-function $W_1(z) = -W_0$,

$$\frac{1}{\tau_m} = \frac{8}{\pi(4m^2-1)} \frac{N_B r_0 W_0 \xi\hat{z}}{c\eta\gamma T_0 Z_0 \nu_\beta} \quad (10)$$





**Longitudinal head-tail instability** This instability appears when the momentum compaction has an appreciable nonlinear dependence on energy. It has been seen at the CERN-SPS [16]. The growth rate of the instability can be written [17]

$$\frac{1}{\tau_m} = \frac{16m^2\eta_1 N_B r_0}{3\eta_0\gamma C Z_0} \int_{-\infty}^\infty d\omega \frac{{\rm Re} Z_\parallel(\omega)}{\sigma^2} \times [\sigma J_m(\sigma) J_{m+1}(\sigma) + (1-m) J_m^2(\sigma)] \quad (11)$$

where $\sigma = \omega\hat{z}/(2c)$, $\hat{z}$ is maximum $z$-amplitude of synchrotron oscillation of all particles (waterbag model), $\eta = \eta_0 + \eta_1\delta + \cdots$ is the phase slip factor (Sec.2.3.11). For the case $m = 1$ (dipole mode), Eq.(11) can be rewritten as

$$\frac{1}{\tau_1} = -\frac{c\eta_1\hat{z}}{3\eta_0 C E} \frac{d\Delta E}{d\hat{z}} \quad (12)$$

where $\Delta E$ is the energy loss of the beam bunch to the impedance. Its dependence on bunch length $\hat{z}$ characterizes the growth rate of the longitudinal head-tail instability.

**Transverse mode coupling instability (TMCI)** TMCI, also called *fast head-tail instability* or *strong head-tail instability*, occurs when the frequencies of two neighboring head-tail modes approach each other due to detuning with increasing current during accumulation. The original name *transverse turbulence* [18] is more appropriate for beams being injected into machine well above threshold, when a large number of modes may become coupled simultaneously.

Its threshold current is lowest in the $y$-plane since many elements are less high than wide,

$$I_{\rm thresh} = \frac{2\pi\nu_s E/e}{\sum_i \beta_i Z_{\perp i}} F(\sigma_z) \quad (13)$$

The form factor $F \approx 1$ for short bunches. For longer bunches, it increases proportionally to $\sigma_z$, and is essentially the ratio of the machine impedance to the effective impedance. For Gaussian bunches and broad-band resonator impedances, $I_{\rm thresh}$ can be expressed with the transverse loss or *kick factor* $\kappa_\perp(\sigma_z)$ which eliminates the need for a bunch length correction factor,

$$I_{\rm thresh} = \frac{C_1 f_s E/e}{\sum_i \beta_i \kappa_{\perp i}(\sigma_z)} \quad (14)$$

The constant $C_1 \approx 8$, but is often replaced by $2\pi$ [10] as in Eq.(13). For more exact calculations of the threshold one should use computer