programs solving for the coherent modes [19], including bunch lengthening [7] and potential-well deformations with current.

For the case of constant wake with $W_1(z) = -W_0$ and a two-particle model beam bunch, the TMCI threshold occurs at

$$I_{\rm thresh} = \frac{8}{\pi}\frac{\omega_\beta \omega_s (E/e)}{cW_0} \qquad (15)$$

where $\omega_{\beta,s}$ are the betatron and synchrotron frequencies.

TMCI is a severe limitation of single bunch current for large high-energy electron storage rings which have a high transverse impedance $Z_\perp \approx (2R/b^2)Z/n$ because of their large radius R and the large number of rf cavities.

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2.5.7 Sacherer Formulae

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In longitudinal phase space $x = r\cos\theta$, $\dot{x} = \omega_{s0} r \sin\theta$, a bunch shape mode $\sim \exp(-im\theta)$ is specified by the azimuthal mode number m ($m = 1$ dipole, $m = 2$ quadrupole etc.), when only the lowest radial mode is considered. Its projection onto real space $\lambda_m(x)$ is called line density mode. The coherent frequency ω_m can be obtained from the (complex) coherent frequency shift $\Delta\omega = \omega_{\rm coh} - \omega_{\rm inc}$, if the incoherent oscillation frequency $\omega_{\rm inc}$ is known [1]. For $n_B > 1$ bunches, the coupled bunch mode number $0 \le n < n_B$ gives the phase-shift $2\pi n/n_B$ between oscillations of adjacent bunches.

Longitudinal bunch oscillations For n_B equally spaced and populated bunches, with total beam current $I_{\rm tot} = n_B I_B$, the coherent frequency shift of for $m \ge 1$ is [2]

$$\Delta\omega_m = -i\frac{m\,\omega_s}{m+1}\frac{I_{\rm tot}\,Z_{\rm eff}^{(m)}}{3B^3 h V_T \cos\varphi_s} \qquad (1)$$

The incoherent synchrotron frequency $\omega_s = \nu_s\omega_0$ is a function of beam current, ω_{s0} is its value for vanishing current, $V_T = V_{\rm rf}\omega_s^2/\omega_{s0}^2$ = the total voltage seen by the beam, $V_{\rm rf}$ = applied rf voltage with harmonic number $h = \omega_{\rm rf}/\omega_0$, and φ_s = stable phase angle. Above transition, $90° < \varphi_s < 180°$ is needed for stability, hence $\cos\varphi_s < 0$. An inductance, for which $Z_{\rm eff} = -i\omega_0 L$, then gives a positive coherent frequency shift, compensating almost exactly its negative incoherent frequency shift. The bunching factor $B = n_B f_0 \tau_L$ is the ratio of bunch length $L = \beta c \tau_L$ to bunch spacing $2\pi R/n_B$.

Assuming the modes do not depend on impedance or beam current, the effective impedance (Sec.3.2.6) can be expressed

$$Z_{\rm eff}^{(m)} = \omega_0 \sum_{p=-\infty}^{\infty} \frac{Z_\parallel(\omega_{mp})}{\omega_{mp}} H_m(\omega_{mp}) \qquad (2)$$

where the spectral power density $H_m(\omega) \propto |\tilde{\lambda}_m(\omega)|^2$ is normalized by $\sum_p H_m(\omega_{mp}) = 1$. The oscillation spectrum is $\omega_{mp} = \omega_0(n_B p + n + m\nu_s)$. The summation over all frequencies can be replaced by one over only positive harmonics with upper and lower sidebands, as illustrated in Fig.1 for the case of 4 Gaussian bunches in a resonator impedance.

Proton bunches have mostly a parabolic distribution, and *Legendre modes* $\lambda_m \sim P_m(\tau)$ are eigenmodes in an inductive impedance. However, *sinusoidal modes* $\lambda(\tau) \sim \sin[\pi m\omega_0\tau]$ or $\cos[\pi(m+1)\omega_0\tau/2]$ (for m odd or even) give better agreement with observation. Their spectral power density is

$$H_m(\omega) = C_m(m+1)^2 \frac{1-(-)^{m+1}\cos(\omega\tau_L)}{\left[(\omega\tau_L/\pi)^2 - (m+1)^2\right]^2} \qquad (3)$$

with $C_m = 2n_B\omega_0\tau_L/\pi^3$ for normalization. Proton bunches are usually long compared to the vacuum chamber diameter; the impedance is mainly inductive, $Z_\parallel(\omega)/n = -i\omega_0 L$ is constant and is

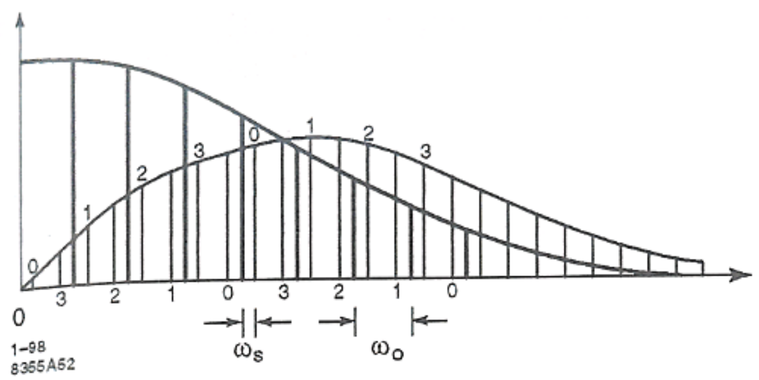

Figure 1: Effective impedance as summation over positive harmonics with upper and lower sidebands, weighted by spectral power density of a Gaussian. Indices 0,1,2,3 are multibunch mode number for $n_B = 4$.

equal to $Z_{\rm eff}$, hence the frequency shift is mostly real. When the bunches are short compared to the rf wavelength, the coherent shift of the dominant dipole mode ($m = 1$) is equal and opposite to the incoherent one, and thus $\omega_{1,\rm coh}$ remains constant with current.

Electron bunches usually have a Gaussian distribution, and oscillate in *Hermitian modes*,

$$H_m(\omega_{mp}) = D_m(\omega_{mp}\sigma)^{2m}\exp(-\omega_{mp}^2\sigma^2) \qquad (4)$$

$D_m = \Gamma(m+1/2)/n_B\omega_0$ for normalization.

Neglecting potential well deformation and turbulence, the frequency spread due to a sinusoidal rf voltage is

$$S = \omega_s \frac{\pi^2}{16}\frac{1+\frac{2}{3}\sin^2\varphi_s}{\cos^2\varphi_s}B^2 \qquad (5)$$

The *effective spread* for a parabolic bunch is approximately $mS/2(m+1)$, while it is $\sqrt{m}S/4$ for Gaussian bunches.

When modes with different mode numbers do not couple, an instability may occur if the total frequency shift $|\Delta\omega|$ is larger than the effective frequency spread; Landau damping is lost, and if the growth rate $1/\tau = -{\rm Im}\Delta\omega$ exceeds radiation damping.

Resonator impedance With resonant frequency ω_R, shunt impedance R_s and quality factor Q, the coherent frequency shift is

$$\frac{\Delta\omega_s}{\omega_0} = 0.159\frac{R_s\,I_0}{BhV\cos\varphi_s}F_m(\omega_R\tau_L)D(\alpha_d) \qquad (6)$$

The form factor $F_m(x)$ is proportional to x^m for short bunches, while it is $64m/\pi x^2$ for long ones. The factor D depends on the attenuation $\alpha_d = \pi\omega_R/n_B Q\omega_0$ between adjacent bunches,

$$D(x) = -i\alpha_d\left[\frac{1}{1-e^{x_+}} - \frac{1}{1-e^{x_-}}\right] \qquad (7)$$

where $x_\pm = -2\pi i(n \pm \omega_R/\omega_0)/n_B - \alpha_d$. The maximum value of $|D|$ is about unity for narrow band, and approaches zero for wide band resonators.

Transverse bunch oscillations The transverse mode frequencies are $\omega_{mp} = \omega_0(n_B p + n - m_t\nu_\beta + m\nu_s)$. The transverse mode number m_t is usually taken to be unity (dipole oscillations), while the longitudinal one ($m \ge 0$) should include the rigid bunch oscillation $m = 0$, which usually has the largest frequency shift and is most likely to become unstable [6, 7].

The coherent frequency shift of a bunch with total current $I_{\rm tot} = n_B I_B$ and energy E is [3, 4]

$$\Delta\omega_m = -\frac{iec}{m+1}\frac{I_{\rm tot}Z_{\perp\rm eff}^{(m)}}{4\pi\nu_\beta EB} \qquad (8)$$

with the effective transverse impedance

$$Z_{\perp\rm eff}^{(m)} = \sum_{p=-\infty}^{\infty} Z_\perp(\omega_{mp})H_m(\omega_{mp} - \omega_\xi) \qquad (9)$$

H_m is evaluated at the mode frequency ω_{mp} shifted by the *chromatic frequency* $\omega_\xi = \xi\omega_0/\eta\nu_\beta$ where $\xi = d\nu_\beta/\delta$.

(i) For a constant impedance $Z_{\perp\rm eff} \approx Z_\perp H_m(\omega_\xi)$, one obtains the unbunched beam result divided by the bunching factor, and by $(m+1)$ since higher modes are more difficult to drive.

(ii) For a narrow-band impedance, when only a single resonance at frequency ω_R contributes to the sum for the effective impedance, the coherent frequency shift becomes

$$\Delta\omega_m = -\frac{iec}{m+1}\frac{I_{\rm tot}Z_\perp(\omega_R)}{4\pi\nu_\beta E_0}F'(\chi - \omega_R\tau_L) \qquad (10)$$

where $\chi = \omega_\xi\tau_L$ is the chromatic phase-shift across the bunch, and $F'(\omega\tau_L) = H_m(\omega)/B$.

(iii) The resistive wall impedance has a single large peak at $\omega_{\rm min}$ [Eq.(5), Sec.2.5.6]. The narrow-band (or long range) contribution is approximately

$$\Delta\omega_m = -\frac{iec}{m+1}\frac{Z_\perp(\omega_0)I_{\rm tot}}{4\pi\nu_\beta E_0\tau_L}\sqrt{\frac{\pi}{n_B B}} \qquad (11)$$

The short range contribution can be approximated by replacing the infinite sum for the effective impedance by an integral over all frequencies, and is usually negligible.

Longitudinal mode coupling Different oscillation modes can strongly couple when detuning makes two mode frequencies approach each