The Parton Shower in ...

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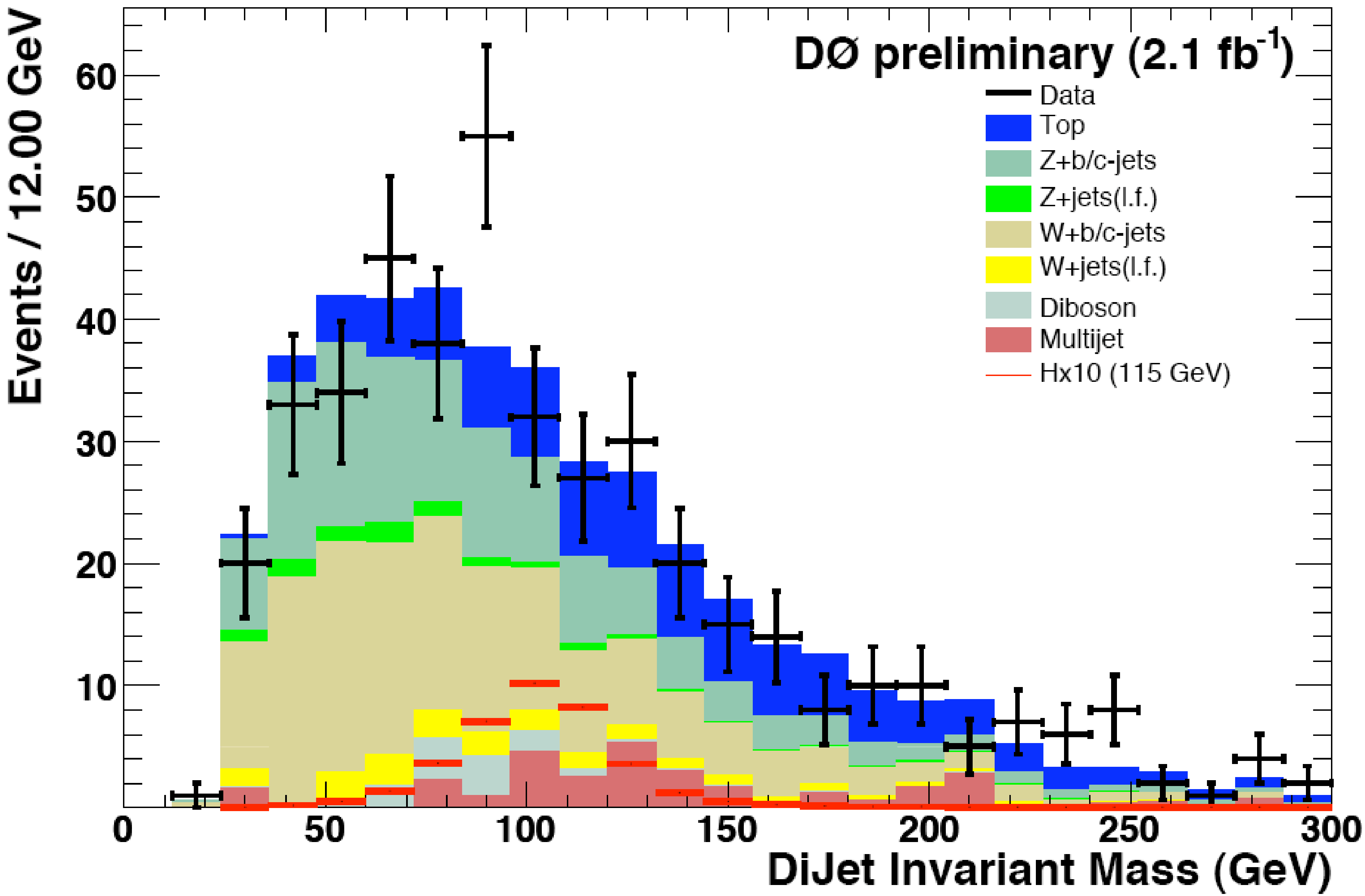
The problem of matching of parton showers to QCD matrix elements is a very important one for LHC physics.

It is not possible to obtain reliable predictions for BSM background processes such as W + jets and tt + jets from parton shower Monte Carlos alone.   On the other hand, exclusive n-parton calculations do not capture the complexity of LHC events.

In our discussion of top tagging, we have already encountered examples where hard parton emissions must be combined with parton showers to produce a simulation realistic enough the test the techniques under discussion.

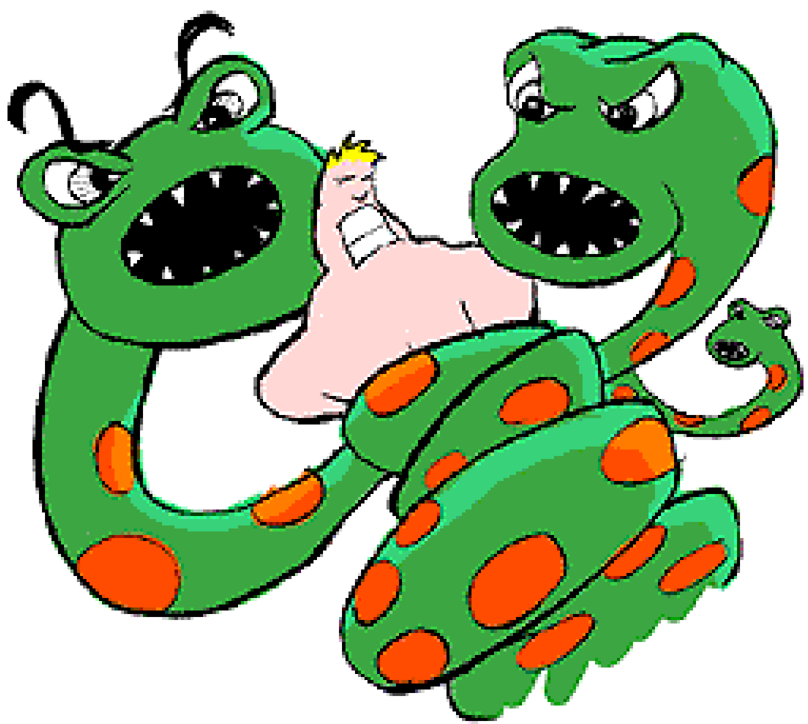
DO Higgs search in $p\bar{p} \rightarrow b\bar{b} + \text{invisible}$ (2008)



A large amount of effort has gone into the creation of codes with QCD matrix element - parton shower matching.

Still, I think there is room for more ideas.

In this talk, I will describe my attempts to contribute to this problem. The talk will mainly deal with the simplest parton shower, the shower in $h^0 \to ng$. At the end, I will discuss also some aspects of $e^+e^- \to q\bar{q}$ and $e^+e^- \to t\bar{t}$.



Hercules and the Hydra -- Mark Fiore

Initial-state radiation is harder to treat, and I am not quite there with these techniques.

There are two basic approaches to matching:

Additive:

Use matrix elements in a particular (hard scattering) region of phase space; use parton showers in the rest of phase space.

e.g. method of CKKW (Catani-Krauss-Kuhn-Webber):
use matrix elements when partons momentum transfers are greater than Q_0^2; use parton showers when the momentum transfers are less.

Multiplicative:

Use partons showers in all of phase space, but reweight events to take matrix elements into account.

All of the codes actually used now by experimenters to analyze data are of the additive type:

W, Z, t + jets, with matrix elements up to 4-jet emission

ALPGEN, MADEVENT, SHERPA, HELAC

correction of parton showers to incorporate exact 1-loop calculations

MC@NLO Frixione, Webber, Nason
POWHEG Frixione, Nason, Oleari

Any codes matched to PYTHIA or HERWIG are necessarily additive, because these showers do not cover all of phase space.

For concreteness, concentrate on the simplest case:

the shower in $h^0 \to ng$

Begin by writing -- using $h^0 \to ng$ QCD tree amplitudes only:

$$\mathrm{Prob}(n) = \int d\Pi_n |\mathcal{M}(h \to ng)|^2$$

After the 2 gluon term, all contributions to the sum are infinite.

This is corrected by inclusion of loop amplitudes. These combine with tree amplitudes to cancel infrared divergences. The finite terms left over give $\mathcal{O}(\alpha_s)$ corrections (e.g. 'K-factors').

However, QCD loop amplitudes are difficult to compute and, in a Monte Carlo, expensive to evaluate.

A parton shower deals with this in the following way:

Let t be an ordering variable among the parton emissions, e.g. $t = \log(m_h^2/s_{ij}),\ s_{ij} = (k_i + k_j)^2$ ('virtuality ordering'). Each emission is assigned a definite value of t.

Then let $S(t_n, t_{n+1}) = \int_{t_n}^{t_{n+1}} d\Pi\ |m(\rightarrow g)|^2$

for one emission. This is the Sudakov integral. The probability that there is no emission between t_n and t_{n+1} is

$$\exp[-S(t_n, t_{n+1})]$$

Including these probabilities or Sudakov factors, the total probability of a Higgs decay becomes

$$\mathrm{Prob}(n) = \int d\Pi_n |\mathcal{M}(h \rightarrow ng)|^2 \prod_i e^{-S(t_i, t_{i+1})}$$

With appropriate choice of $S(t_i, t_{i+1})$, $\sum_n \mathrm{Prob}(n) = 1$

This method incorporates the most important effects of loop diagrams, though it does not capture the K-factors and other finite radiative corrections.

In a parton shower, the full emission amplitude is taken to factorize by stages. At each stage, one takes the emission amplitude to be the Altarelli-Parisi splitting function. This is correct in the collinear limit (only).

My goal here is to apply a formula

$$\mathrm{Prob}(n) = \int d\Pi_n |\mathcal{M}(h \to ng)|^2 \prod_i e^{-S(t_i, t_{i+1})}$$

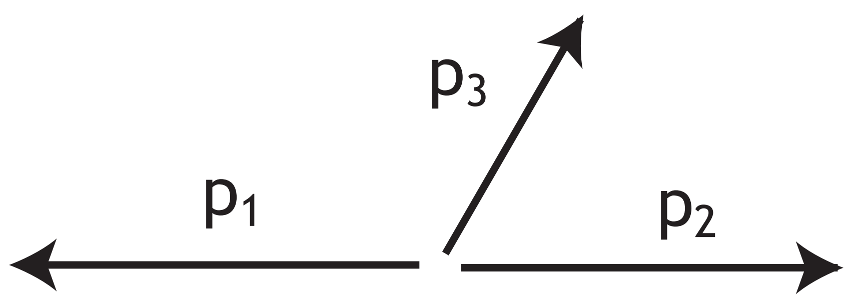
where $\mathcal{M}(h \to ng)$ is the exact QCD tree amplitude to as high a level as my computer has the strength to compute it.

[Caution: Here, 'exact' = leading order in N_c only.]

Bauer, Tackmann, and Thaler have emphasized that, to use the multiplicative method, it is necessary for the parton shower to exactly cover phase space. Here is my solution to this problem. To be most effective, I should preferentially generate points in phase space in the soft and collinear regions.

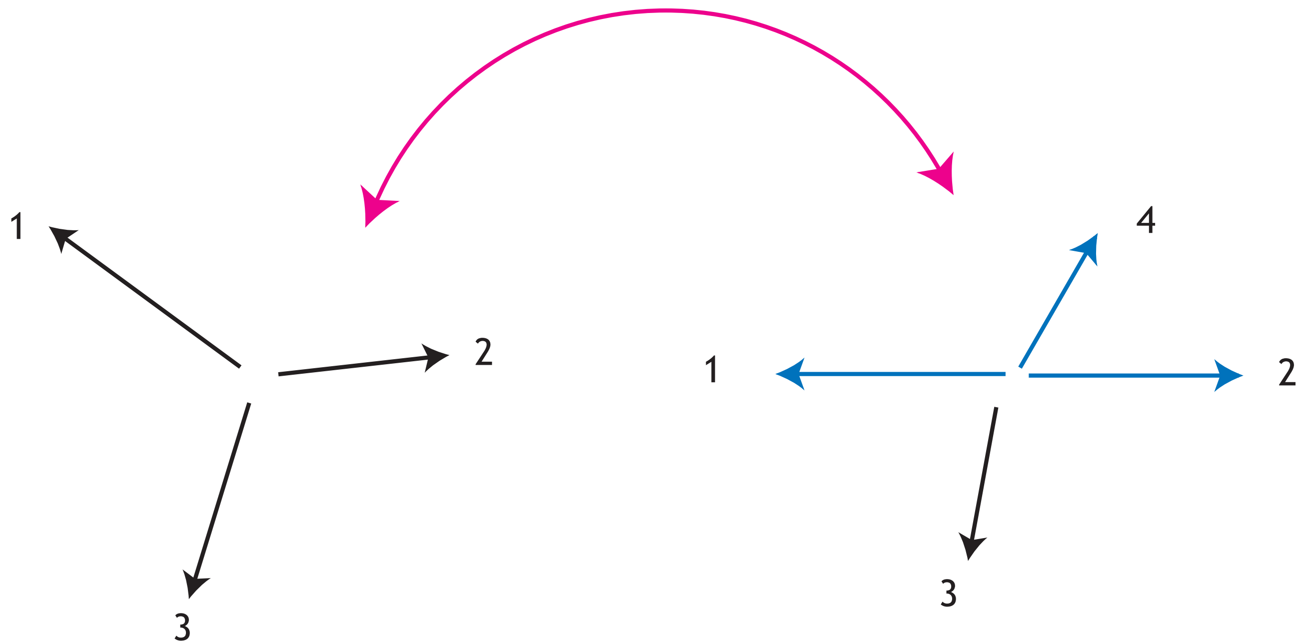
An effective trick has been introduced by Draggiotis, van Hameren, and Kleiss as the basis of their SARGE algorithm

Start with two back-to-back lightlike vectors. Add a third lightlike vector $p_3 = \xi_1 p_1 + \xi_2 p_2 + p_\perp$



Then boost and rescale to the original CM frame and energy.

To add the fourth vector, pick two neighbors, boost these back-to-back, add a vector as before, and then boost the entire system back to the CM frame.



Effectively, the entire event recoils when a new vector is added.

The logarthmic integral over the parameters reproduces massless phase space

$$\int \frac{d^3 p_3}{(2\pi)^2 2p_3} \frac{2p_1 \cdot p_2}{2p_1 \cdot p_3 \, 2p_3 \cdot p_2} = \frac{1}{(4\pi)^2} \int \frac{d\xi_1}{\xi_1} \int \frac{d\xi_2}{\xi_2} \int \frac{d\phi}{2\pi}$$

Applying this operation repeatedly, we build up phase space with all of the QCD denominators for emission of final-state radiation that are found in the exact, leading- N_c amplitudes.

$$\int d\Pi_n \frac{1}{2p_1 \cdot p_2 \, 2p_2 \cdot p_3 \cdots 2p_n \cdot p_1}$$
$$= \frac{1}{8\pi Q^4} \prod_i \left[\frac{1}{(4\pi)^2} \int \frac{d\xi_{1i}}{\xi_{1i}} \int \frac{d\xi_{2i}}{\xi_{2i}} \int \frac{d\phi}{2\pi} \right]$$

This is an exact formula for massless phase space with QCD denominators, but only if we integrate over every point in phase space exactly once.

Draggiotis, van Hameren, and Kleiss suggested adding the vectors 1, 2, 3 in fixed (color) order. This requires very large values for the ξ_i to reproduce some phase space configurations.

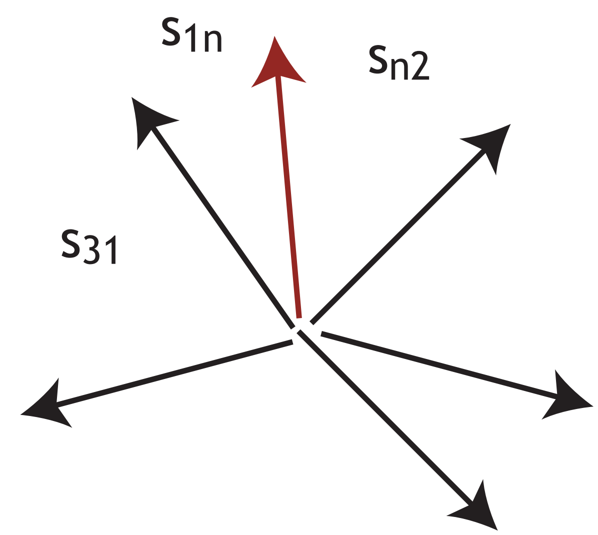
An alternative approach is to choose arbitrarily at each step one interval in which to insert a new vector. We call the set of such choices a chamber. It is then necessary to define the limits of each chamber so that the full set of chambers tiles phase space.

Here is a useful definition of a chamber:

Let the nth vector be inserted between 1 and 2. Then allow all values of ξ_1, ξ_2, ϕ such that

s_{1n} is the smallest invariant mass of two neighbors, and $s_{n2} < s_{13}$

Reversing the inequality defines a second chamber in which n is radiated on the left side of 1.

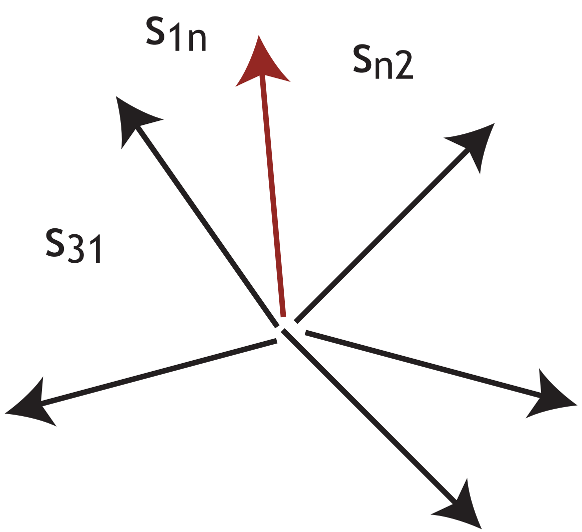


These prescriptions put reasonable upper limits on the ξ_{1j} integrals.

The ordering of virtualities s_{ij} is similar to the ordering in a parton shower. In fact, we can identify s_{ij} with the evolution variable of a parton shower.

Here is the proof that this method tiles phase space:

Just go backward. For an n-gluon configuration,

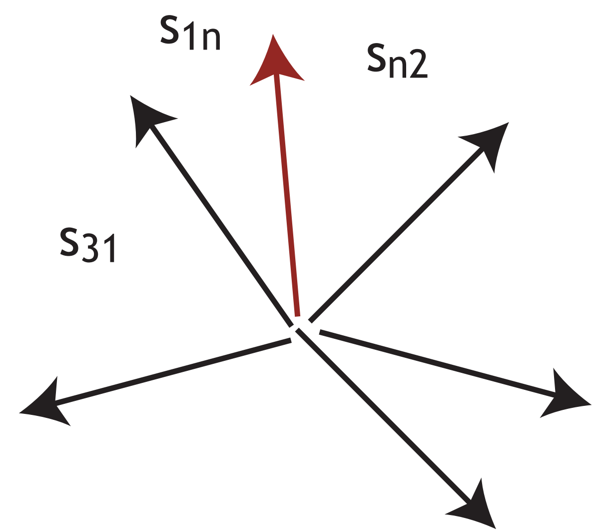


pick the smallest $s_{i,i+1}$ to be the emission chamber, and choose the smaller of the $s_{i,i+1}$ on the two sides to complete the dipole. Proceding in this way, each point in phase space gives a unique path back to the 2-gluon state.

We can look at the emission in the chamber

between 1 and 2, on the side of 1

as an emission from the gluon 1
in the antenna (in the sense of Skands, Weinzierl, et al.) of gluons 1 and 2.



At each stage in the shower, I choose an antenna and an emission side at random.

The correspondence to Altarelli-Parisi is

$$(1-z) = \frac{1}{(1+\xi_1+\xi_2)}$$

and

$$\int \frac{d\xi_2}{\xi_2} \int \frac{d\xi_1}{\xi_1} \approx \int \frac{dQ^2}{Q^2} \int \frac{dz}{z(1-z)}$$

So, using the SARGE measure and choosing all weights = 1 corresponds to the formula

$$\mathrm{Prob}(n) = \int d\Pi_n |\mathcal{M}(h \to ng)|^2 \prod_i e^{-S(t_i, t_{i+1})}$$

with
$$|\mathcal{M}(h \to ng)|^2 = \frac{m_h^8}{s_{12} s_{23} \cdots s_{n1}}$$

This is very convenient, because it is an exact result in QCD that

$$\mathcal{M}(h \to g_1^+ g_2^+ \cdots g_n^+) = \frac{m_h^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

Dixon, Glover, Khoze

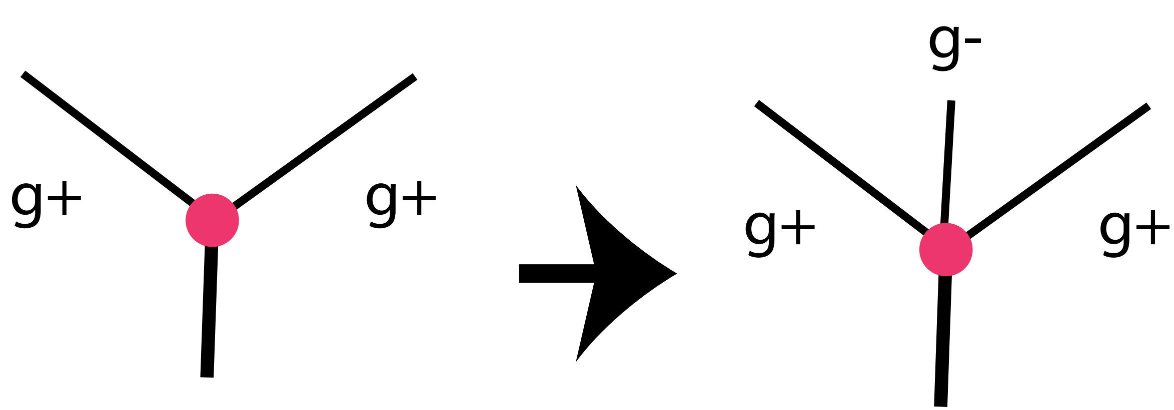
In addition, each antenna automatically has color-coherence it its emission

$$\int d\Pi \frac{1}{s_{12} s_{23}}$$

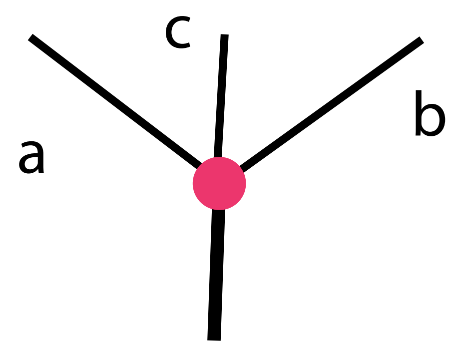
This is the more correct expression of the physics implemented in PYTHIA and HERWIG by angular ordering.

What is the splitting probability, more exactly ? It is natural in this approach to consider gluons as radiated from color dipoles. This type of parton shower, with the basic element being a 2➛3 splitting process, is called an 'antenna shower' by the authors of VINCIA. An antenna shower needs a 2➛3 splitting function that generalizes the Altarelli-Parisis splitting functions.

For matching to QCD, it is very convenient to assign each gluon a definite helicity. Then we need polarized 2➛3 splitting functions. These do not seem to have been given previously in the literature.



Actually, there is no unique or universal definition of a 2→3 splitting function. These functions must take a universal form when c and a become collinear, when c and b become collinear, and when c becomes soft, but this still leaves some freedom.



In 1990, Andersson, Gustafson, and Lonnblad, the authors of ARIADNE, proposed the following forms for unpolarized functions:

$$
\begin{aligned}
S(q\bar{q} \to qg\bar{q}) &= \frac{x_a^2 + x_b^2}{(1-x_a)(1-x_b)} \\
S(qg \to qgg) &= \frac{x_a^2 + x_b^3}{(1-x_a)(1-x_b)} \\
S(gg \to ggg) &= \frac{x_a^3 + x_b^3}{(1-x_a)(1-x_b)}
\end{aligned}
$$

Some more complicated forms for these functions, agreeing in the relevant limits, were proposed by Campbell and Glover, Catani and Grazzini, and Gehrmann-de Ritter, Gehrmann, and Glover from ingredients in NNLO QCD calculations.

Andrew Larkoski and I propose the following simple and systematic approach to the polarized functions: Write a local operator that creates the polarized 2-parton system, then evaluate its 3-parton matrix element. This gives very simple results with the correct limits:

$$S(AB \to acb) = \frac{\mathcal{N}(z_a, z_b, z_c)}{y_{ab} y_{ac} y_{bc}}$$

where $y_{ac} = s_{ac}/s_{AB} = (1 - z_b)$ etc.

For gg➤ggg, we can use the operators

$$\mathrm{tr}[(\sigma \cdot F)(\overline{\sigma} \cdot F)] \qquad \mathrm{tr}[\gamma^m (\overline{\sigma} \cdot F)\gamma^n (\overline{\sigma} \cdot F)]$$

for the initial states g+g+, g-g+, respectively.

The results are

$$\mathcal{N}(g_+ g_+) = (1, y_{ac}^4, y_{ab}^4, y_{bc}^4)$$

for (abc) = (+++, ++-, +-+, -++)

$$\mathcal{N}(g_- g_+) = ((y_{bc}/z_b)^4, z_a^4, z_b^4, (y_{ac}/z_a)^4))$$

for (abc) = (+-+, -++, --+, +--)

Using the first set of splitting functions to create the 3rd gluon in Higgs decay gives exact agreement with tree level QCD.

These results are very easy to include in the parton shower based on SARGE. The algorithm described above gives the $2 \to 3$ splitting with $\mathcal{N} = 1$.

So, generate the shower with partons of definite helicity, and, for each branching generated, reject it with probability $(1 - \mathcal{N})$. It is shown in the PYTHIA manual that this procedure gives not only the emission probabilities with given $\mathcal{N}$ but also the correct Sudakov factor corresponding to these emission probabilities.

In showers that include quarks, we need the quark-gluon splitting functions. These are generated with the operators

$$\overline{q}(\overline{\sigma} \cdot F) \qquad \overline{q}\gamma^m(\sigma \cdot F)$$

The complete set of numerators is given on the next slide.

$g_+g_+ \to ggg$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
1	y_{ac}^4	y_{ab}^4	y_{bc}^4	0	0	0	0
$g_-g_+ \to ggg$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
0	0	$(y_{ab}/z_b)^4$	z_a^4	z_b^4	$(y_{ab}/z_a)^4$	0	0
$g_+g_+ \to \overline{q}qg$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
-	-	$y_{ab}^3 y_{bc}$	$y_{ab} y_{bc}^3$	-	0	0	-
$g_-g_+ \to \overline{q}qg$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
-	-	$y_{ab} y_{bc}^3$	$z_a^2 z_b^2 y_{ab} y_{bc}$	-	0	0	-
$q_-\overline{q}_+ \to qg\overline{q}$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
-	-	-	$y_{ab} z_b^2$	$y_{ab} z_a^2$	-		-
$q_-\overline{q}_- \to qg\overline{q}$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
-	-	-	-	-	y_{ab}^3	-	y_{ab}
$q_-g_- \to qgg$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
-	-	-	0	y_{ac}^4/z_a	$y_{ab}^3 z_b$	-	z_a
$q_-g_+ \to qgg$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
-	-	-	z_a^3	$y_{ab} z_b^3$	y_{ac}^4/z_a^3	-	0
$q_-g_- \to q\overline{q}q$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
-	-	-	-	$y_{ab} y_{ac}^3/z_a$	$y_{ab}^2 y_{ac} z_b$	-	-
$q_-g_+ \to q\overline{q}q$							
$+++$	$++-$	$+-+$	$-++$	$--+$	$-+-$	$+--$	$---$
-	-	-	-	$z_a y_{ab} y_{ac} z_b^2$	$y_{ab} y_{ac}^3/z_a$	-	-

Now look at some results from the simulation:

All results refer to a Higgs of mass 1000 GeV, showered to an infrared cutoff scale of 2 GeV. Since we are doing shower physics, not Higgs physics, I use the effective interaction

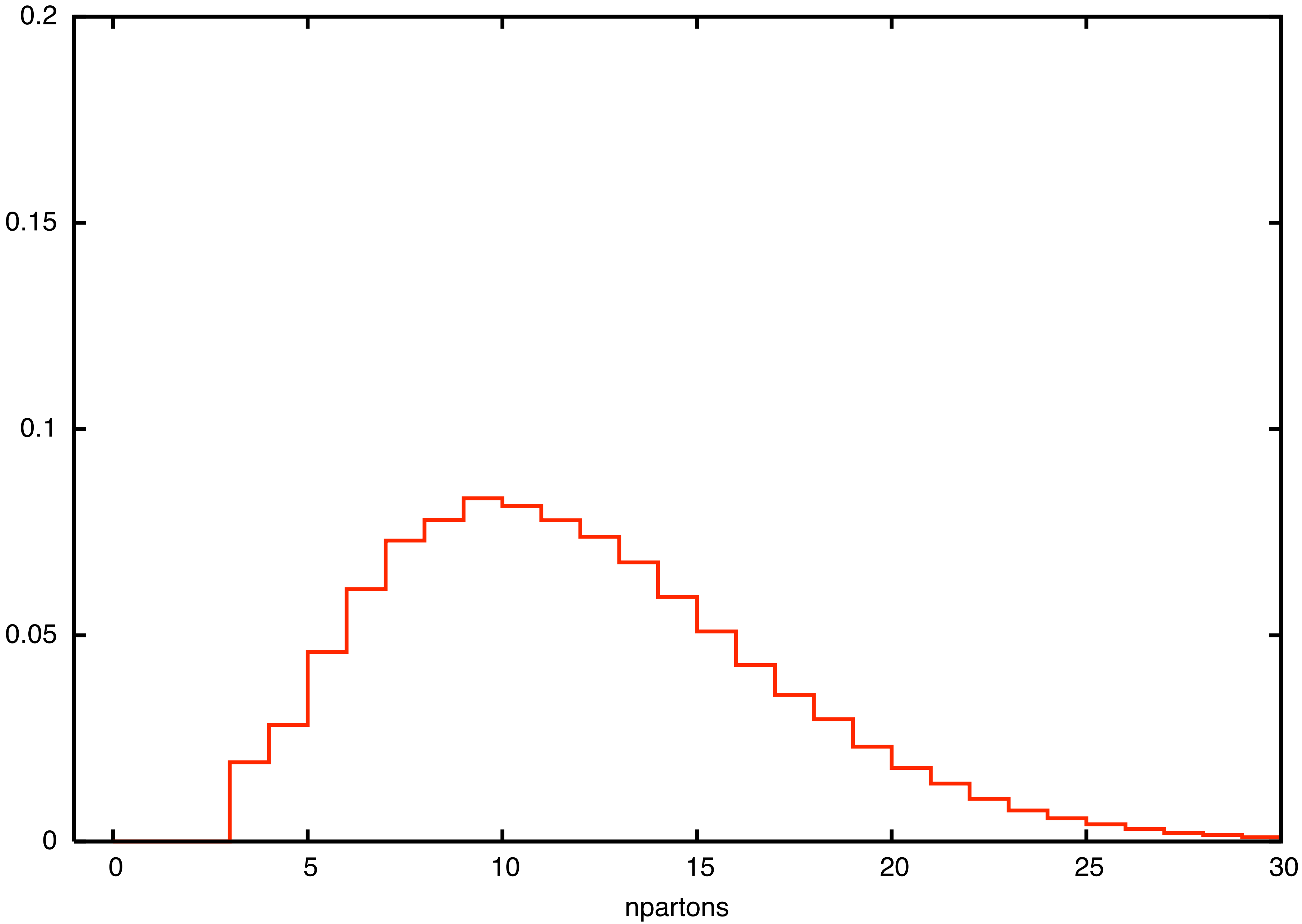
$$\delta\mathcal{L} = \frac{\alpha_s}{12\pi v} h\, F_{\mu\nu}F^{\mu\nu}$$

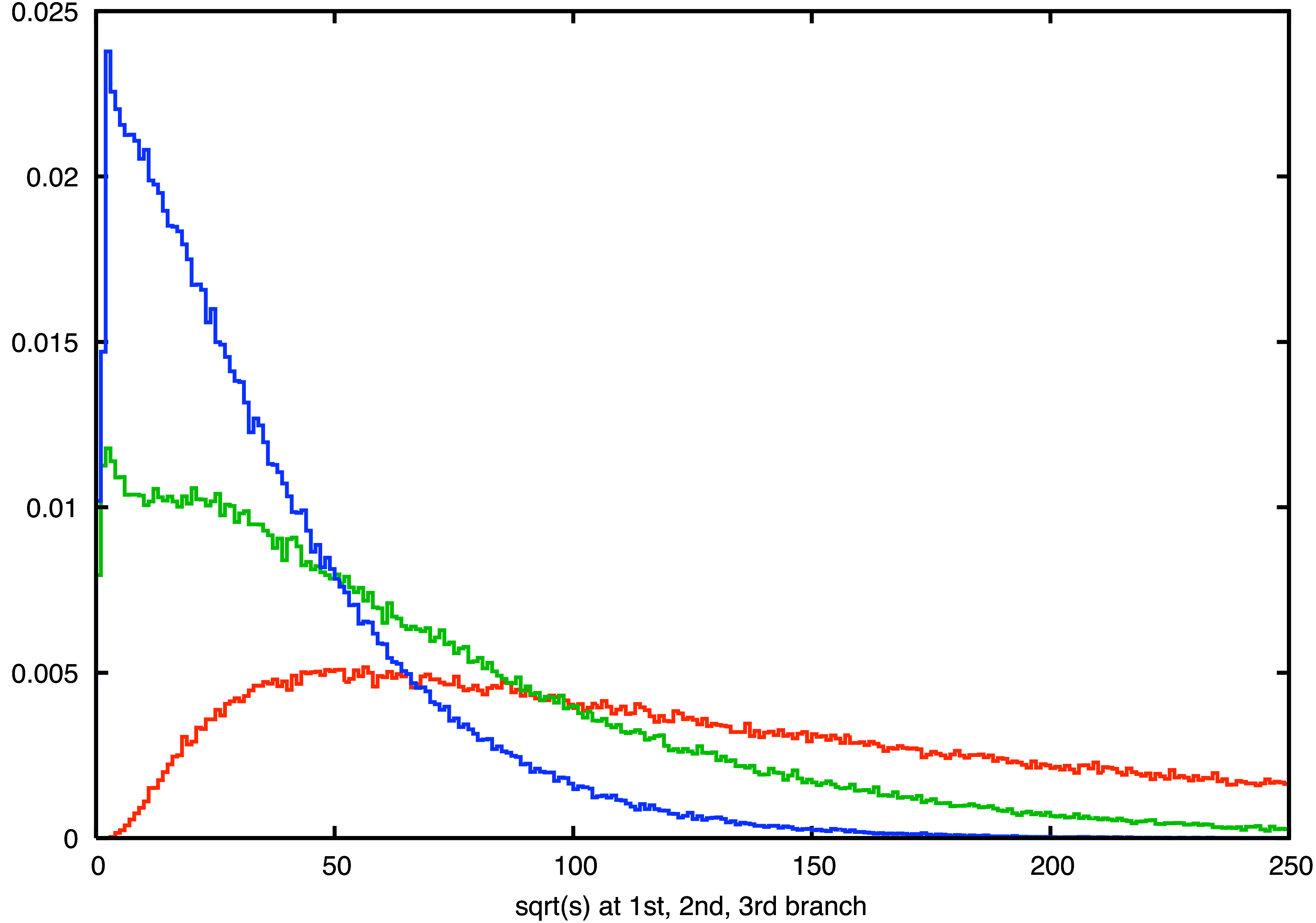
without apology.

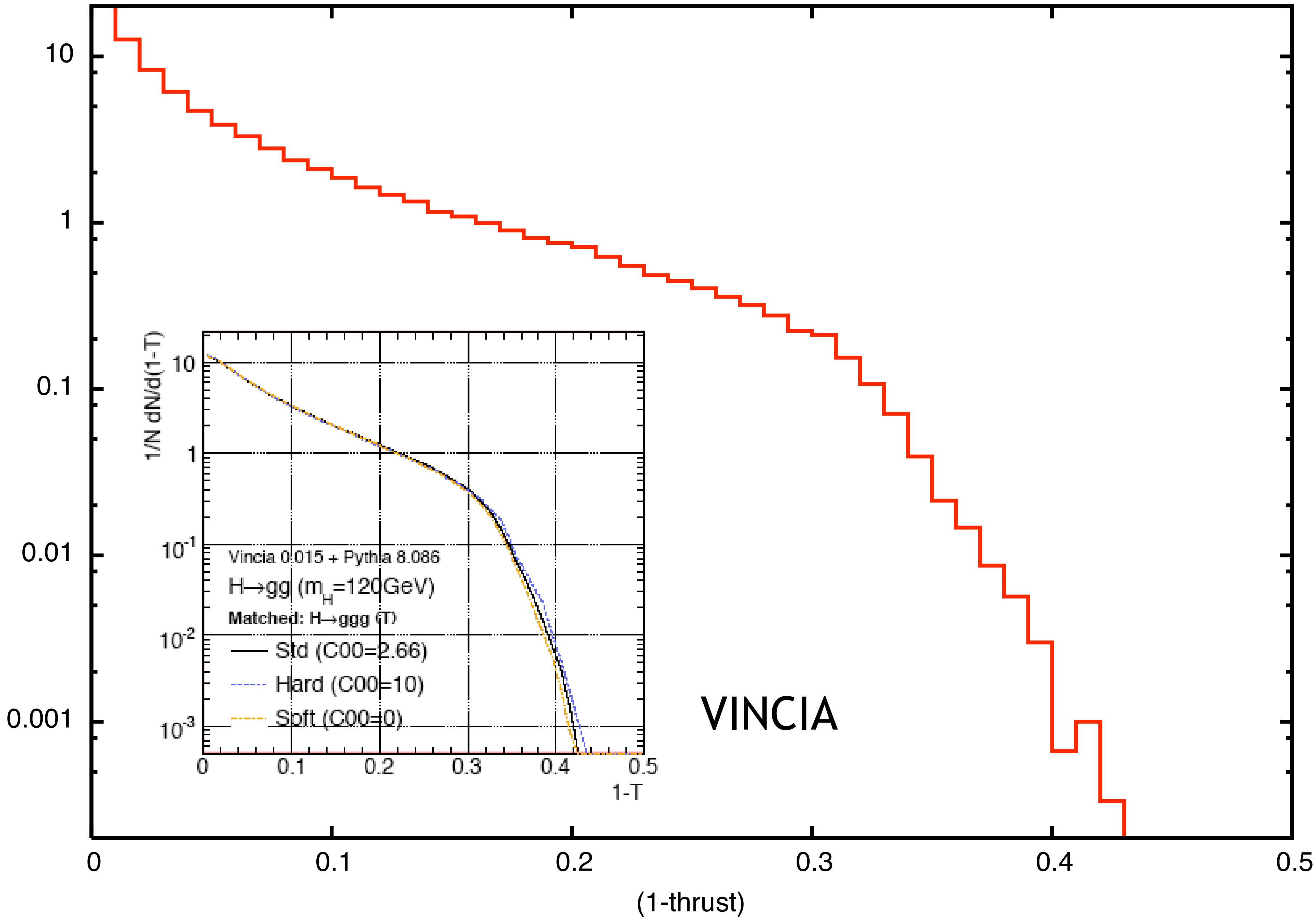
First, the simple shower without matching. This runs at

4 events / msec

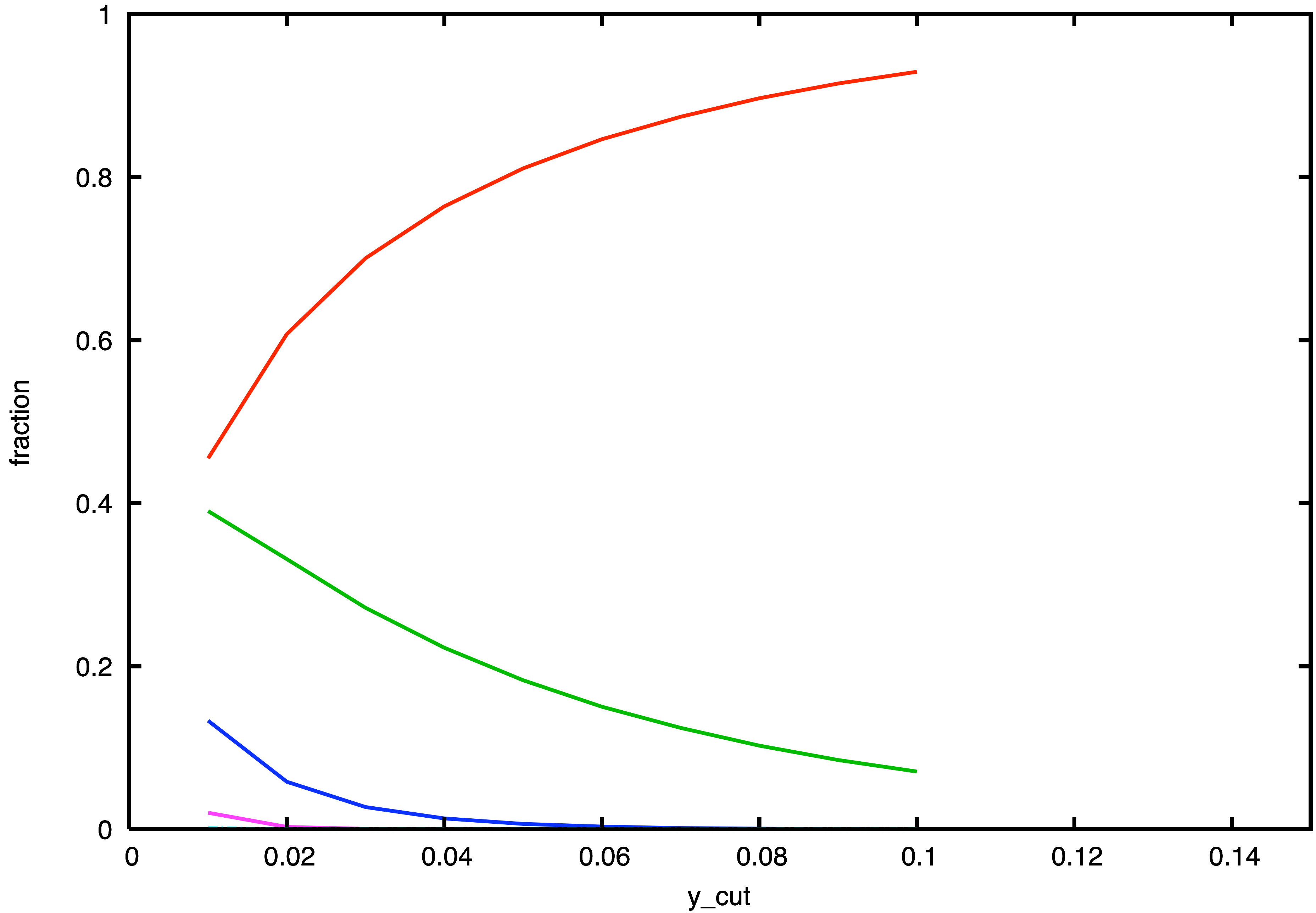
on my MacBook.


0.2
0.15
0.1
0.05
0
0
5
10
15
20
25
30
npartons

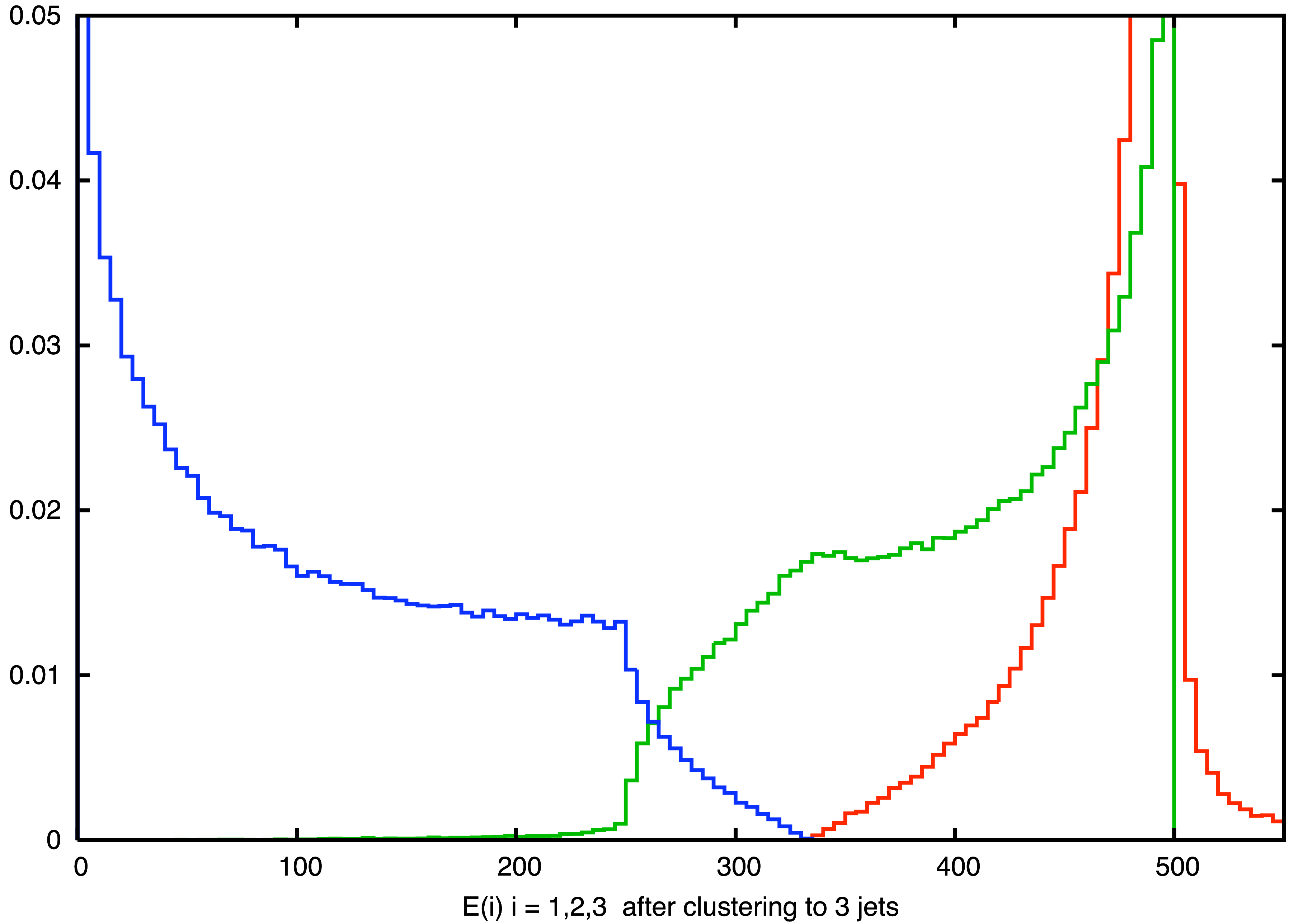

0.025
0.02
0.015
0.01
0.005
0
0
50
100
150
200
250
sqrt(s) at 1st, 2nd, 3rd branch



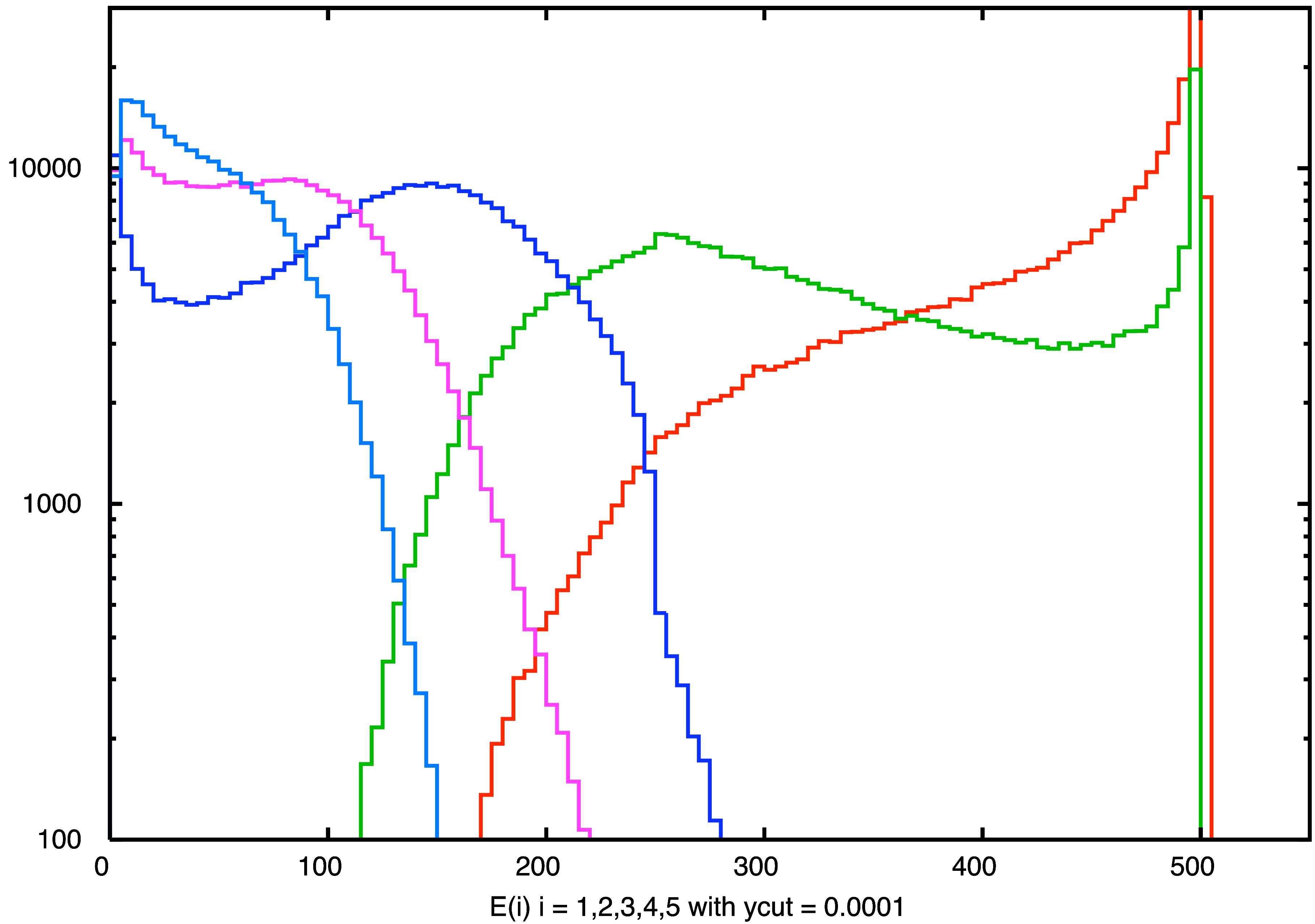
10
1
0.1
0.01
0.001
0
0.1
0.2
0.3
0.4
0.5
(1-thrust)
VINCIA
1/N dN/d(1-T)
Vincia 0.015 + Pythia 8.086
H→gg (m_H=120GeV)
Matched: H→ggg (T)
Std (C00=2.66)
Hard (C00=10)
Soft (C00=0)
1-T



1
0.8
0.6
0.4
0.2
0
fraction
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
y_cut



0.05
0.04
0.03
0.02
0.01
0
0
100
200
300
400
500
E(i) i = 1,2,3 after clustering to 3 jets


10000
1000
100
0
100
200
300
400
500
E(i) i = 1,2,3,4,5 with ycut = 0.0001

We are now set up to implement matrix element - parton shower matching. At each branching, instead of taking the weight for the emission to be $\mathcal{N}$, we can take it to be:

$$w = \left| \frac{\mathcal{M}(h \to ng)}{\mathcal{M}(h \to (n-1)g)} \right|^2 \cdot \frac{s_{1n} s_{n2}}{s_{12}}$$

where the matrix elements in the numerator are the full (color-ordered or leading Nc) tree-level QCD matrix elements.

Implementing the weights w by rejection, we obtain the QCD emission probabilities and the corresponding Sudakov factors.

To generate QCD tree amplitudes, I use the Britto-Cachazo-Feng recursion formula for on-shell, color-ordered amplitudes:

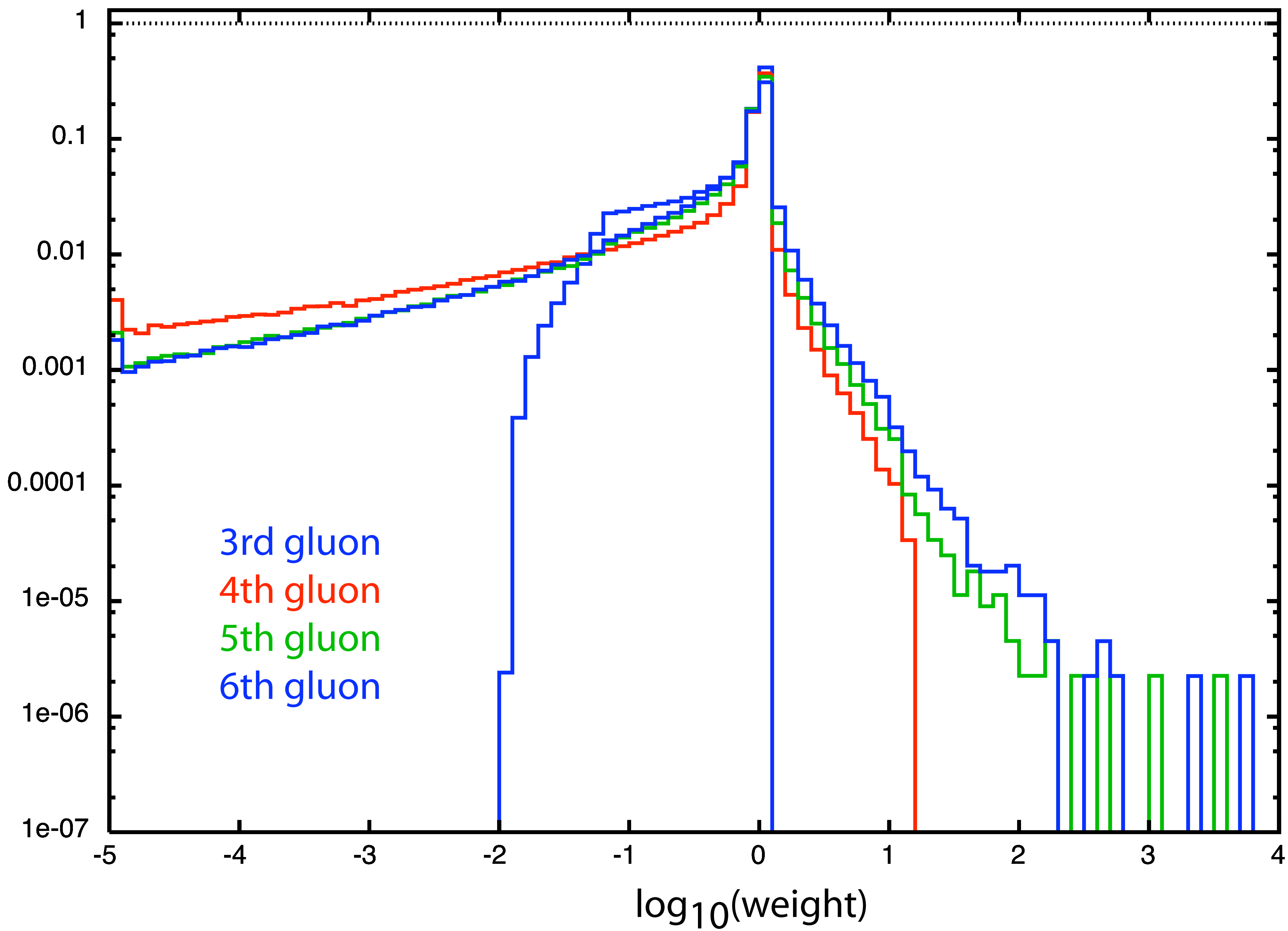
$$i\mathcal{M}(1\cdots n) = \sum_{splits} i\mathcal{M}(b+1\cdots \hat{i}\cdots a-1\ -\hat{Q}) \cdot \frac{1}{s_{a\cdots b}} \cdot i\mathcal{M}(a\cdots \hat{j}\cdots b\,\hat{Q})$$

The BCF formula recursively breaks amplitudes down (numerically, on the fly) to the simpler exact results for $h^0 \to 2g, 3g$, and $h^0 \to$ all + or all - gluons.

Now add matching to matrix elements.

There is a small problem here. For the PYTHIA rejection algorithm to work properly, we must run the shower, before rejection, with a weight constant weight A in the numerator that is an upper bound to all possible weights that can appear in the event generation. But, large weights can appear !

One large weight W means that typical branches are selected with probability $1/W$. This dramatically slows the process.

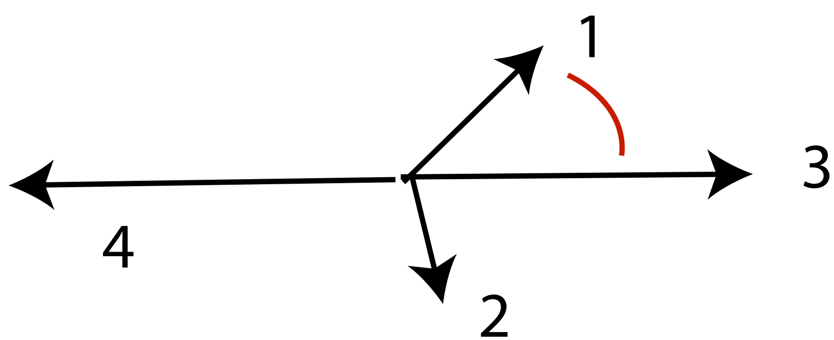

1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
-5
-4
-3
-2
-1
0
1
2
3
4
3rd gluon
4th gluon
5th gluon
6th gluon
log10(weight)

In the formula

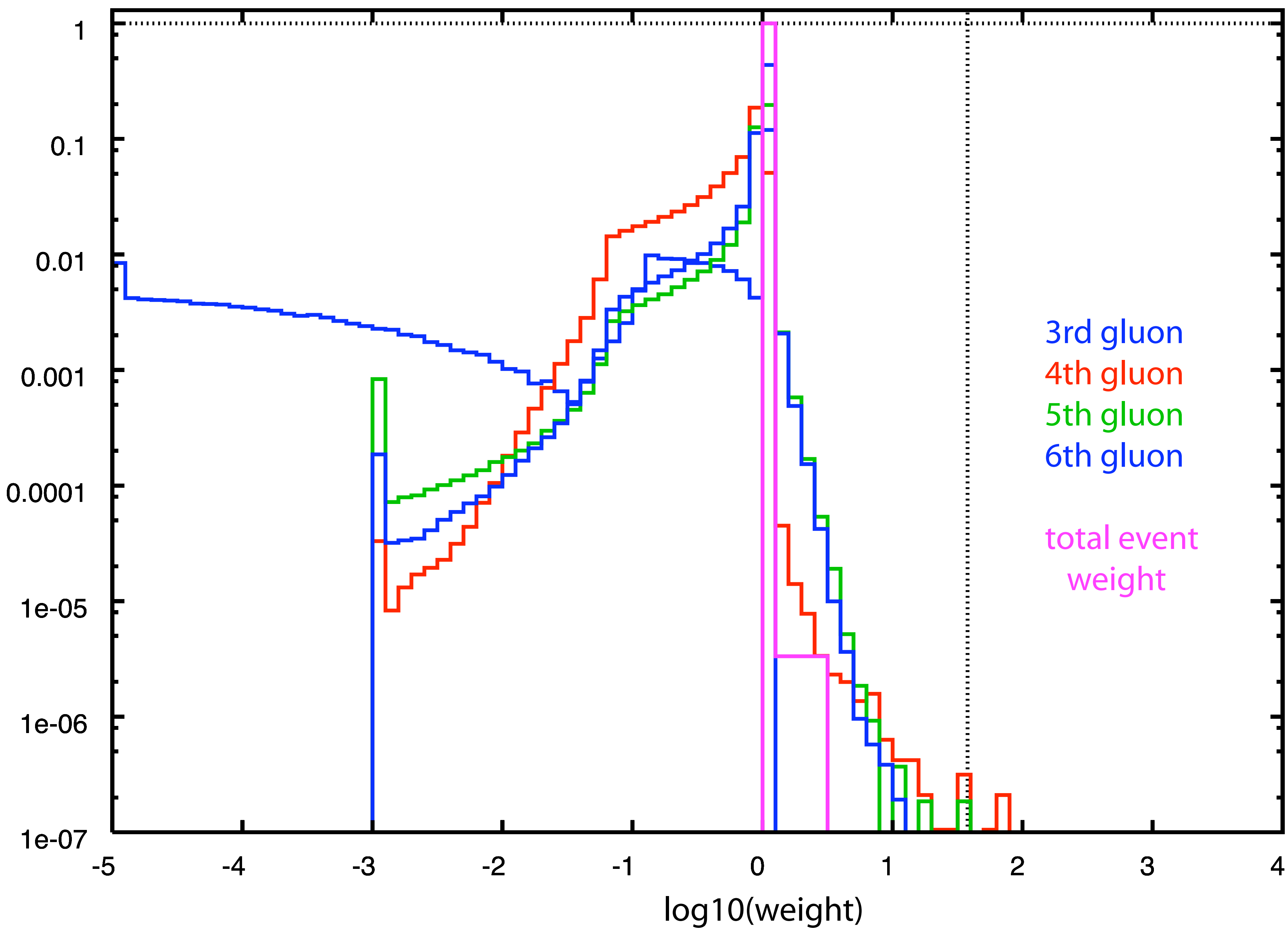
$$\mathcal{M} = \frac{A m_h^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}$$

$|A|^2$ never gets bigger than about 1.5. However, some A's are very small, and, in a series of emissions, a large A can follow a small A. This leads to large weights $|A_n/A_{n-1}|^2$.

The problem occurs because the chamber prescription above sometimes emits a high-energy gluon between two lower-energy gluons.



A better prescription is: $z_1 < z_3$ or $(s_{12} + s_{14}) < (s_{32} + s_{34})$
This also tiles phase space precisely.

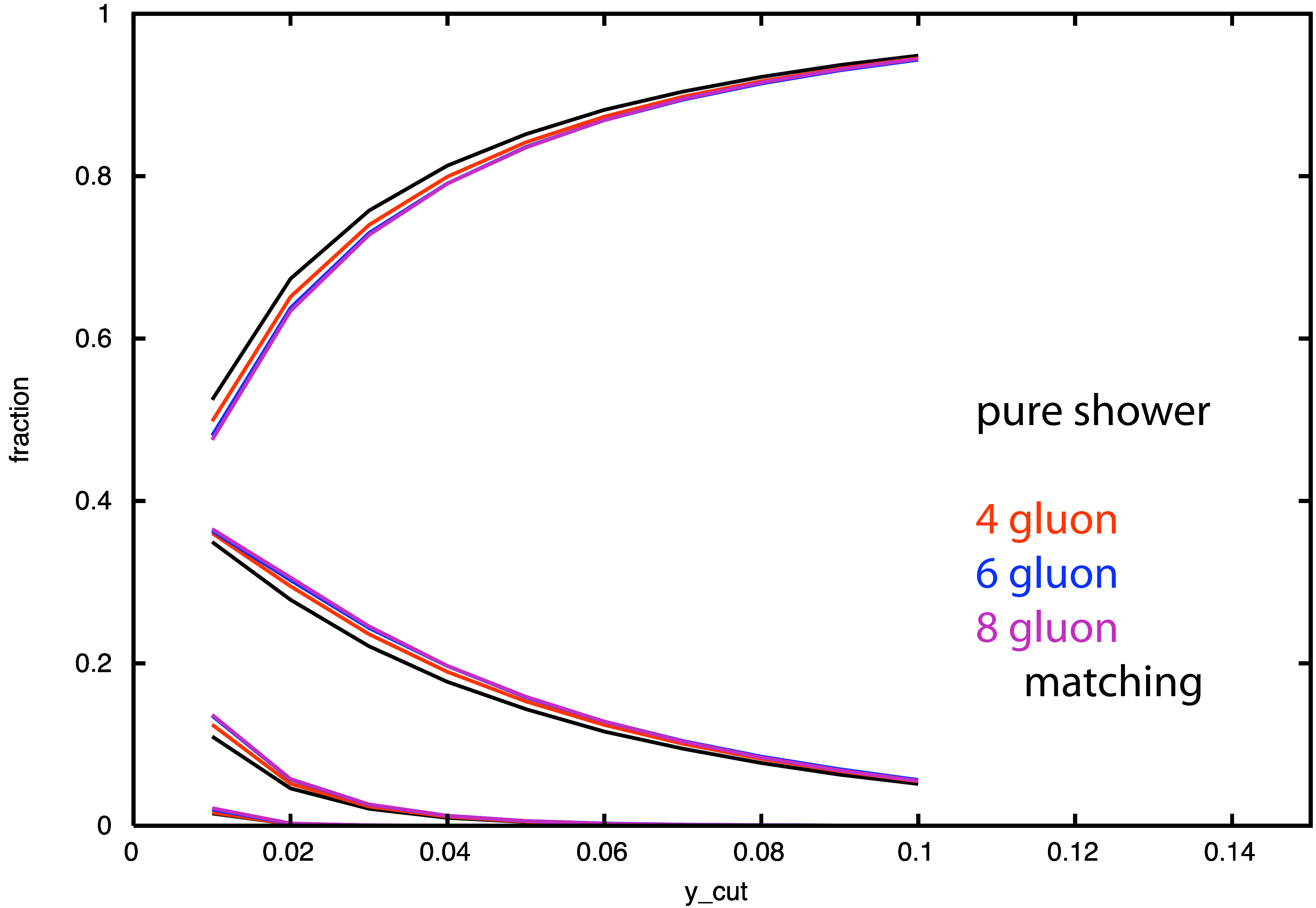

1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
-5
-4
-3
-2
-1
0
1
2
3
4
log10(weight)
3rd gluon
4th gluon
5th gluon
6th gluon
total event
weight

With this change, here is the speed of event generation (msec/event)

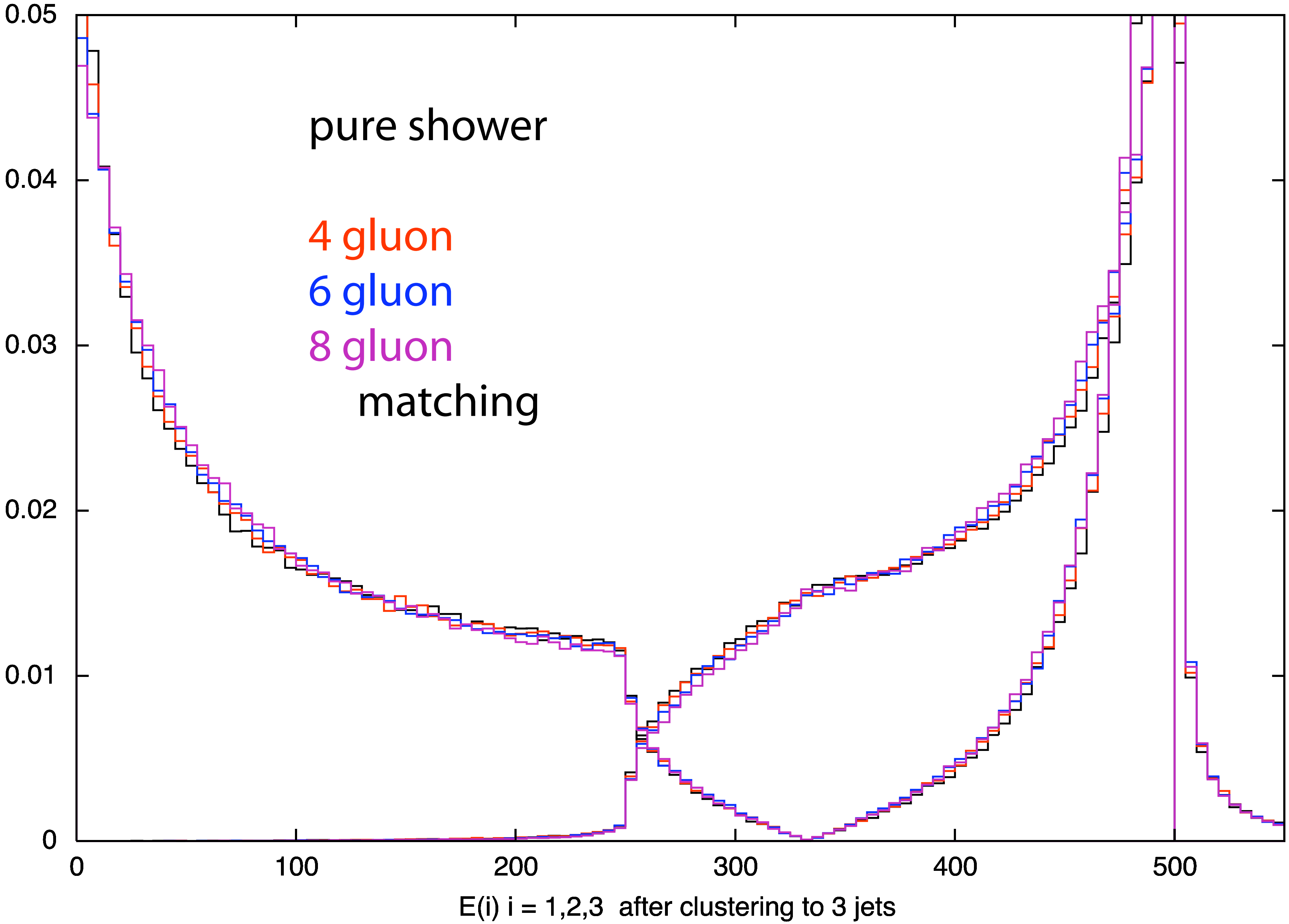
pure shower: 0.42

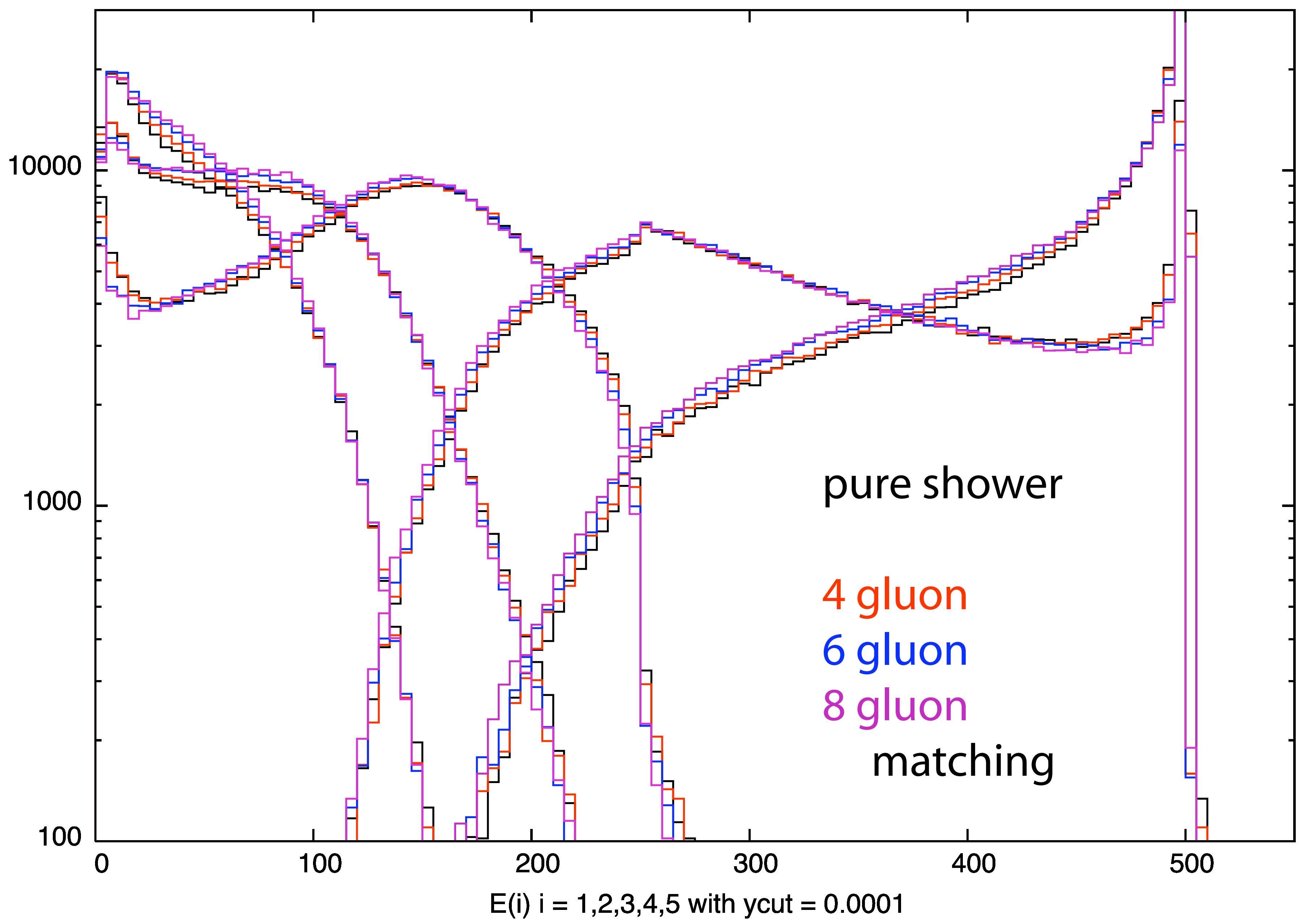
matching to	4 gluons	6 gluons	8 gluons
	2.1	6.4	78.

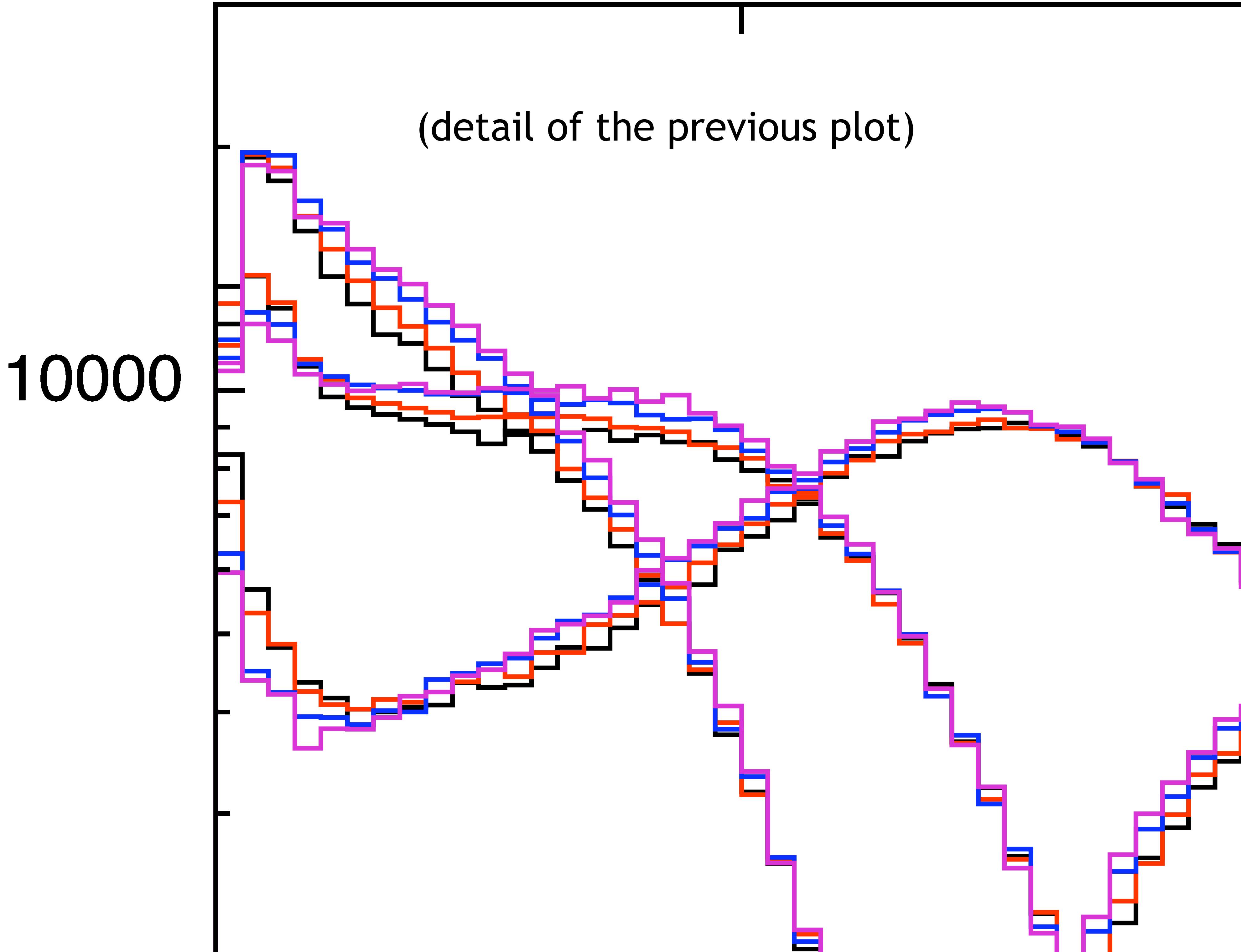
Here are some results of these simulations.



1
0.8
0.6
0.4
0.2
0
fraction
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
y_cut
pure shower
4 gluon
6 gluon
8 gluon
matching


pure shower
4 gluon
6 gluon
8 gluon
matching
0.05
0.04
0.03
0.02
0.01
0
0
100
200
300
400
500
E(i) i = 1,2,3 after clustering to 3 jets


pure shower
4 gluon
6 gluon
8 gluon
matching
10000
1000
100
0
100
200
300
400
500
E(i) i = 1,2,3,4,5 with ycut = 0.0001



(detail of the previous plot)
10000

This is the end of the results. More codes based on these ideas will be working soon, I hope.

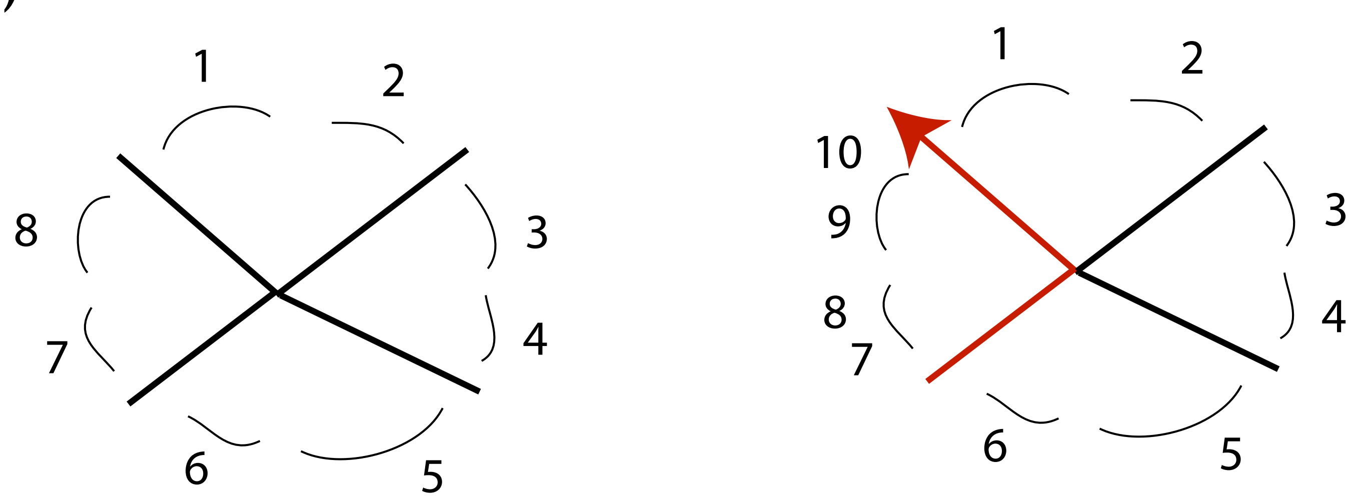
However, I would like to make some comments on generalizations to parton showers that are closer to practical interest / reality.

The most straightforward generalization is to the shower in

$$e^+e^- \to q + ng + \overline{q}$$

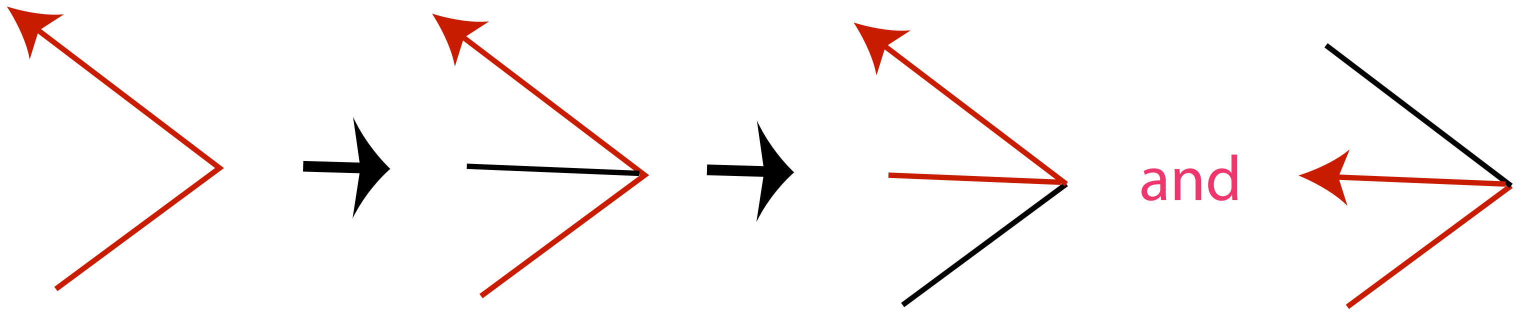
However, in this case, there is a counting problem. In the pure gluon shower, the transition from n to (n+1) gluons has 2n chambers. This is the correct counting for identical gluons. It leads to the factor (1/n) for identical particles in the total rate.

For the quark shower, the quark and antiquark positions are distinguished, so this factor does not appear. This requires that, at the transition from n to (n+1) particles, there should be 2(n+1) chambers.



The extra chambers arise when we use the quark and antiquark to generate the new vector with the SARGE algorithm.   In this case, the vectors must be relabeled to have a sensible color ordering.

There are two ways to do this. The correct answer is to do both. This gives a complete coverage of phase space.



These last four chambers are not populated by the parton shower. However, they are populated, with amplitudes that are not enhanced by collinear singularities, by exact QCD diagrams.

The generalization to massive quarks is a little more complicated.

It is possible to derive the splitting functions for massive quarks using the technique that I have described above, but I do not understand the answers very well. A sample result is

$$S(t_- g- \to t_- g_+ g_-) = z_b^2 \left| \frac{\langle a^\flat b \rangle [cab\rangle}{(s_{ac} - m^2)\langle bc \rangle} \right|^2$$

where $a^\flat$ is a massless vector along the top direction, a la Schwinn and Weinzierl. This has the correct denominator structure and the required 'dead cone' along the top direction.

Some of the other cases are not so simple. Maybe I can make some progress while I am here.

Conclusions:

This is a proof of principle for a new way to incorporate exact matrix elements into a parton shower. Still, there is promise that this method might develop into an interesting tool for modeling multijet QCD processes.

I’m a worthless check, a total wreck, a flop!
But if, baby, I’m the bottom, you’re the top.

-- Cole Porter