

# Cosmology of the de Sitter Horndeski Models

**Nelson Nunes**

*Instituto de Astrofísica e Ciências do Espaço*

in collaboration with: P. Martin-Moruno, and F. Lobo,  
arXiv:1505.06585, 1502.05878, 1502.03236, 1506.02497

# 100 years of General Relativity



# A brief cosmologist's view

$$\mathcal{L} = \frac{R}{2} + \mathcal{L}_m$$

# A brief cosmologist's view

$$\mathcal{L} = \frac{R}{2} + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \Lambda + \mathcal{L}_m$$

# A brief cosmologist's view

$$\mathcal{L} = \frac{R}{2} + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \Lambda + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \mathcal{L}_m$$

# A brief cosmologist's view

$$\mathcal{L} = \frac{R}{2} + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \Lambda + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - X - V(\phi) + \mathcal{L}_m$$

# A brief cosmologist's view

$$\mathcal{L} = \frac{R}{2} + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \Lambda + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - X - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

# A brief cosmologist's view

$$\mathcal{L} = \frac{R}{2} + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \Lambda + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - X - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = f(\phi)\frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

# A brief cosmologist's view

$$\mathcal{L} = \frac{R}{2} + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \Lambda + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - X - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = f(\phi)\frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = f(X, \phi)\frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

# A brief cosmologist's view

$$\mathcal{L} = \frac{R}{2} + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \Lambda + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - X - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = f(\phi)\frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = f(X, \phi)\frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = f(X, \phi)\frac{R}{2} + K(X, \phi) - V(\phi) - G(X, \phi)\square\phi + \mathcal{L}_m$$

# A brief cosmologist's view

$$\mathcal{L} = \frac{R}{2} + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \Lambda + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\nabla\phi)^2 - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} - X - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = \frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = f(\phi)\frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = f(X, \phi)\frac{R}{2} + K(X, \phi) - V(\phi) + \mathcal{L}_m$$

$$\mathcal{L} = f(X, \phi)\frac{R}{2} + K(X, \phi) - V(\phi) - G(X, \phi)\square\phi + \mathcal{L}_m$$



# The Horndeski Lagrangian

$$\mathcal{L} = \frac{R}{2} + \sum_{i=2}^4 \mathcal{L}_i + \mathcal{L}_m$$

It is the most general scalar field theory in 4D with second order equations of motion

- Found by Horndeski in 1975
- Rediscovered by Deffayet et al. in 2011
- Includes all other models (quintessence, k-essence, scalar-tensor, Galileon, etc.)

# The Horndeski Lagrangian

$$\mathcal{L} = \frac{R}{2} + \sum_{i=2}^4 \mathcal{L}_i + \mathcal{L}_m$$

It is the most general scalar field theory in 4D with second order equations of motion

$$\mathcal{L}_2 = K(\phi, X)$$

$$\mathcal{L}_3 = -G_3(\phi, X)\square\phi$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4,X} [(\square\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)]$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) -$$

$$\frac{1}{6}G_{5,X} [(\square\phi)^3 - 3(\square\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) +$$

$$2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)]$$

# Recipe for self-tuning Lagrangians, in concept

$$L(\phi, \dot{\phi}, a, \dot{a}) = a^3 \sum_{i=0}^3 Z_i(\phi, \dot{\phi}, a) H^i$$

where  $H$  is Hubble rate

$$Z_i(\phi, \dot{\phi}, a) = X_i(\phi, \dot{\phi}) - \frac{k}{a^2} Y_i(\phi, \dot{\phi})$$

and  $X_i$  and  $Y_i$  are functions of the Horndeski or Deffayet free functions.

- 1 The theory must admit "the vacuum" for any value of the cosmological constant;
- 2 This should remain true before and after the phase transition where the cosmological constant jumps instantaneously by a finite amount;
- 3 The theory allows for a non-trivial cosmology.

# Recipe for self-tuning Lagrangians, physically

We require that an abrupt change in the matter sector is absorbed by the scalar field leaving the vacuum unchanged.

- 1 The field equation must be trivially satisfied at the critical point to allow the field to self-adjust ( $L_{\text{cp}}(a, \phi, \dot{\phi}) = L_{\text{cp}}(a)$ );
- 2 At the critical point, the Hamiltonian must depend on  $\dot{\phi}$  so that the continuous field can absorb discontinuities of the vacuum energy ( $\mathcal{H} \propto \rho_{\text{vac}} \Rightarrow \mathcal{H}_{\text{cp}} \propto f(\dot{\phi})$ );
- 3 The scalar field equation of motion must depend on  $\dot{H}$ , such that the cosmological evolution is non-trivial before screening takes place ( $\dot{\phi} \propto \dot{H}$ ).

Charmousis et al. 2011

# The Fab 4

The Fab Four potentials (Charmousis *et al.*) are indeed able to self-tune for  $k = -1$

$$\mathcal{L}_{\text{John}} = V_J(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi,$$

$$\mathcal{L}_{\text{Paul}} = V_P(\phi)P^{\mu\nu\alpha\beta}\nabla_\mu\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi,$$

$$\mathcal{L}_{\text{George}} = V_G(\phi)R,$$

$$\mathcal{L}_{\text{Ringo}} = V_R(\phi)G,$$

- The cosmological models approach a patch of Minkowski with  $k = -1$  when it is an attractor, and describe matter domination before that.
- $V_J, V_P \sim$  “stiff fluid”;  $V_G \sim$  “radiation”; and  $V_R \sim$  “curvature”.
- Unclear how to obtain a late time accelerated universe.

# What are the de Sitter Horndeski models?

The most general scalar-tensor cosmological models with second order equations of motion that, regardless of the content of the Universe, have a de Sitter critical point.

# Self tuning to a spatially flat de Sitter vacuum

The Lagrangian for  $k = 0$ ,

$$L_H = a^3 \sum_{i=0}^3 X_i(\phi, \dot{\phi}) H^i, \quad L_m = -a^3 \rho_m$$

The Hamiltonian density

$$\mathcal{H}_H = \sum_{i=0}^3 \left[ (i-1) X_i + X_{i,\dot{\phi}} \right] H^i$$

The field equation

$$-\frac{d}{dt} \left[ a^3 \sum_{i=0}^3 X_{i,\dot{\phi}} H^i \right] + a^3 \sum_{i=0}^3 X_{i,\phi} H^i = 0$$

# Self tuning to a spatially flat de Sitter vacuum

At the critical point,  $H_{\text{cp}} = \sqrt{\Lambda}$ .

The Lagrangian that at the critical point that satisfies all the constraints, i.e.,  $L_{\text{cp}}(a, \phi, \dot{\phi}) = L_{\text{cp}}(a)$  and  $\mathcal{H}_{\text{cp}} \propto f(\dot{\phi})$ , is

$$\mathcal{L}_{\text{H}}^{\text{cp}} = \sum_{i=0}^3 X_i(\phi, \dot{\phi}) \Lambda^{i/2} = 3\sqrt{\Lambda} h(\phi) + \dot{\phi} h_{,\phi}(\phi)$$

Martin-Moruno, NJN, Lobo (2015)

# Self tuning to a spatially flat de Sitter vacuum

$$\mathcal{L}_H^{\text{CP}} = \sum_{i=0}^3 X_i(\phi, \dot{\phi}) \Lambda^{i/2} = 3\sqrt{\Lambda} h(\phi) + \dot{\phi} h_{,\phi}(\phi)$$

What are the  $X_i(\phi, \dot{\phi})$ ?

- ①  $X_i$  are terms linear in  $\dot{\phi}$

$$X_i = 3\sqrt{\Lambda} U_i(\phi) + \dot{\phi} W_i(\phi)$$

- ②  $X_i$  are terms with a non-linear dependence on  $\dot{\phi}$  which contribution has to vanish at the critical point, i.e.,  $\mathcal{L}_H^{\text{CP}} = 0$

# I.

## Linear models

# The linear Lagrangian

Considering also matter, the linear Lagrangian and Hamiltonian are

$$L = L_{\text{EH}} + L_{\text{linear}} + L_{\text{m}} \quad \mathcal{H} = \mathcal{H}_{\text{EH}} + \mathcal{H}_{\text{linear}} + \mathcal{H}_{\text{m}} = 0$$

where

$$L_{\text{linear}} = a^3 \sum_i \left( 3\sqrt{\Lambda} U_i(\phi) + \dot{\phi} W_i(\phi) \right) H^i$$

$i = 0, \dots, 3$ , subject to the constraint at the critical point,

$$\sum_i W_i(\phi) \Lambda^{i/2} = \sum_j U_{j,\phi}(\phi) \Lambda^{j/2},$$

**8 functions - 1 constraint = 7 free functions  $\Rightarrow$  Mag 7!**

$W_i$  and  $U_i$  are related to the  $\kappa_j$  functions of the Horndeski Lagrangian and  $G_j$  functions of the Deffayet et al. functions.

# Equations of motion

Together they give respectively the field equation for  $H'$  and the Friedmann equation

$$H' = 3 \frac{\sum_i H^i \left( \sqrt{\Lambda} U_{i,\phi}(\phi) - H W_i(\phi) \right)}{\sum_i i H^i W_i(\phi)}$$
$$\phi' = \sqrt{\Lambda} \frac{(1 - \Omega) H^2 - 3 \sum_i (i - 1) H^i U_i(\phi)}{\sum_i i H^{i+1} W_i(\phi)}$$

# General considerations

- 1 Only  $W_0 \neq 0$
- 2  $U_i, W_j$  pair
- 3  $W_i, W_j$  pair
- 4 Term-by-Term model (4 potentials)
- 5 Tripod model (3 potentials)

# 1. Only $W_0 \neq 0$

$$H' = 3 \frac{\sum_i H^i (\sqrt{\Lambda} U_{i,\phi}(\phi) - H W_i(\phi))}{\sum_i i H^i W_i(\phi)}$$

For  $W_0 \neq 0$  but  $W_1 = W_2 = W_3 = 0$  then  $H'$  is ill defined. This can be understood by inspecting

$$\begin{aligned} \mathcal{H}_{\text{linear}} &= \sum_i \left[ 3(i-1)\sqrt{\Lambda} U_i(\phi) + i \dot{\phi} W_i(\phi) \right] H^i \\ &= \sum_i \left[ 3(i-1)\sqrt{\Lambda} U_i(\phi) \right] H^i \end{aligned}$$

independent of  $\dot{\phi} \Rightarrow$  **The model does not screen dynamically.**  
Only de Sitter attractor exists.

## 2. $W_i, U_j$ pair

From the constraint equation  $W_i = U_{j,\phi} \Lambda^{(j-i)/2}$  and then

$$\frac{H'}{H} = -\frac{3}{i} \left[ 1 - \left( \frac{H}{\sqrt{\Lambda}} \right)^{j-i-1} \right]$$

which again does not depend on  $\phi$ .

When  $j - i - 1 < 0$  and  $H \gg \sqrt{\Lambda}$

$$\frac{H'}{H} = -\frac{3}{i}$$

means that we recover dust for  $i = 2$ .

We reach de Sitter when  $H \rightarrow \sqrt{\Lambda}$ .

### 3. $W_i, W_j$ pair

From the constraint equation  $W_i = -W_j \Lambda^{(j-i)/2}$  and then

$$\frac{H'}{H} = -3 \frac{1 - (H/\sqrt{\Lambda})^{i-j}}{j - i(H/\sqrt{\Lambda})^{i-j}}$$

again independent of  $\phi$ .

For  $j > i$  and  $H \gg \sqrt{\Lambda}$

$$\frac{H'}{H} = -\frac{3}{j}$$

means that we recover dust for  $j = 2$ .

We reach de Sitter when  $H \rightarrow \sqrt{\Lambda}$ .

## 4. Term-by-Term model

The constraint equation is satisfied for equal powers of  $\Lambda$ , i.e.  $W_i = U_{i,\phi}$ . We have then

**8 functions – 4 constraints = 4 free potentials**

Defining  $U_{i,\phi} = \Lambda^{-i/2} V_{i,\phi}$

$$\frac{H'}{H} = -3 \left( 1 - \frac{\sqrt{\Lambda}}{H} \right) \frac{\sum_i (H/\sqrt{\Lambda})^i V_{i,\phi}}{\sum_i i (H/\sqrt{\Lambda})^i V_{i,\phi}}$$

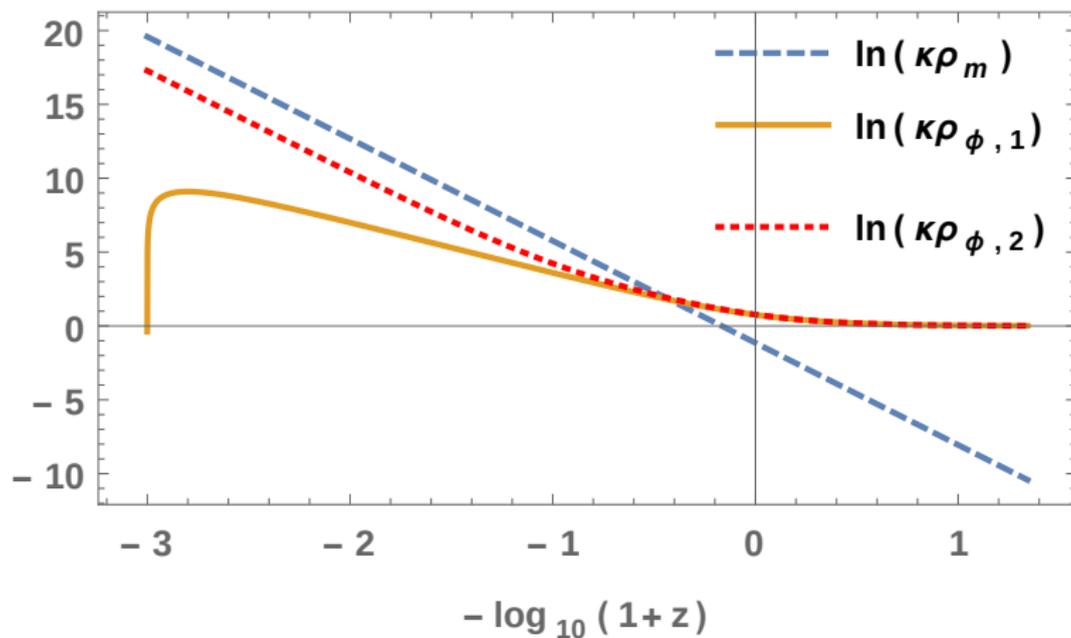
Here the field (and the background matter) contributes to the dynamics of the Universe!

For  $H \gg \sqrt{\Lambda}$  and only one  $i$  component dominates  $\frac{H'}{H} = -\frac{3}{i}$  means that we recover dust for  $i = 2$ .

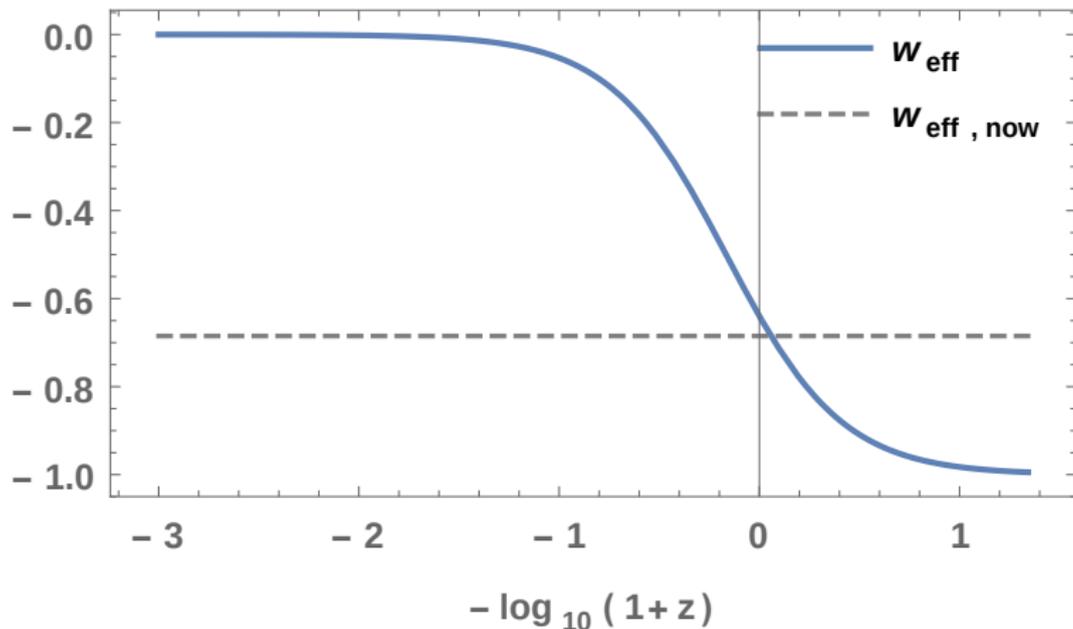
We reach de Sitter when  $H \rightarrow \sqrt{\Lambda}$ .

## 4. Term-by-Term model: Energy densities

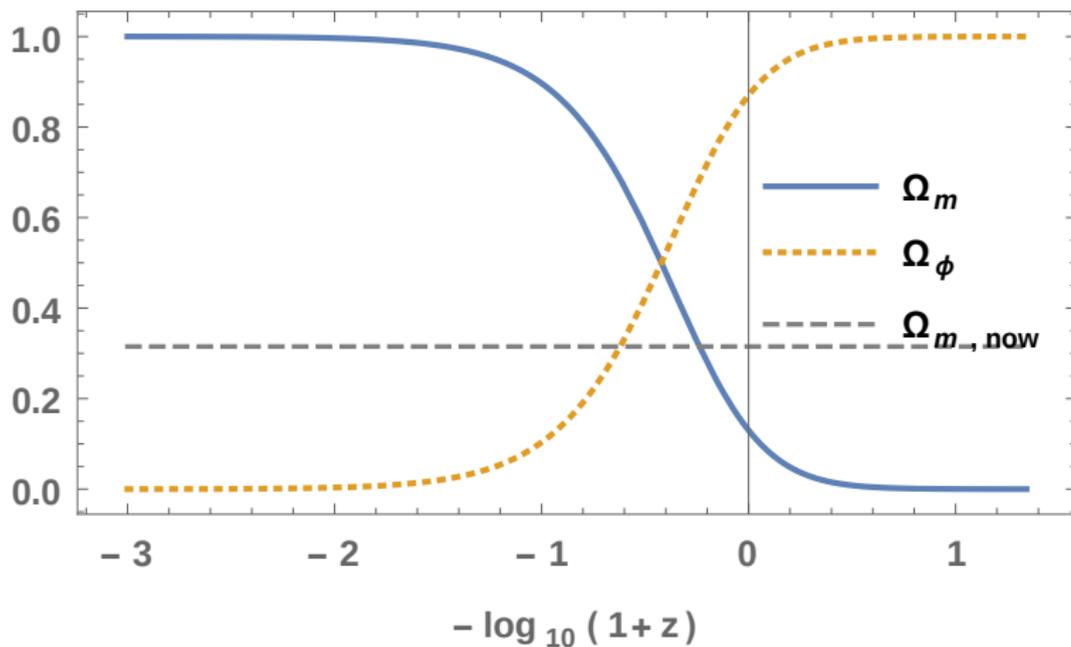
Example for  $V_0 = -10^{-6}\phi$  and  $V_2 = 10^{-6}e^\phi$



## 4. Term-by-Term model: Effective equation of state



## 4. Term-by-Term model: Abundances



## 5. Tripod model

Let us consider the 3 potentials  $U_2$ ,  $U_3$  and  $W_2$ . The constraint equation imposes  $U_{2,\phi}\Lambda + U_{3,\phi}\Lambda^{3/2} = W_2\Lambda$ , then

$$\frac{H'}{H} = -\frac{3}{2} \frac{U_{2,\phi}}{W_2} \left( 1 - \frac{\sqrt{\Lambda}}{H} \right)$$

For  $H \gg \sqrt{\Lambda}$ ,  $\frac{H'}{H} = -\frac{3}{2} \frac{U_{2,\phi}}{W_2}$ . Therefore we need:

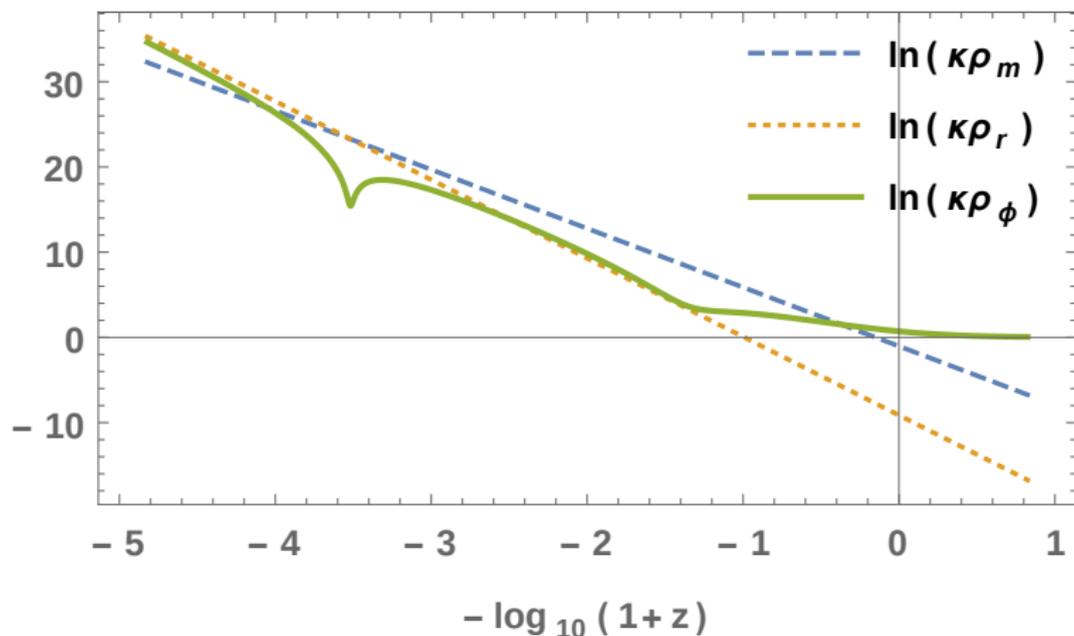
$$\frac{U_{2,\phi}}{W_2} = 1, \quad \text{for M.D.}$$

$$\frac{U_{2,\phi}}{W_2} = \frac{4}{3}, \quad \text{for R.D.}$$

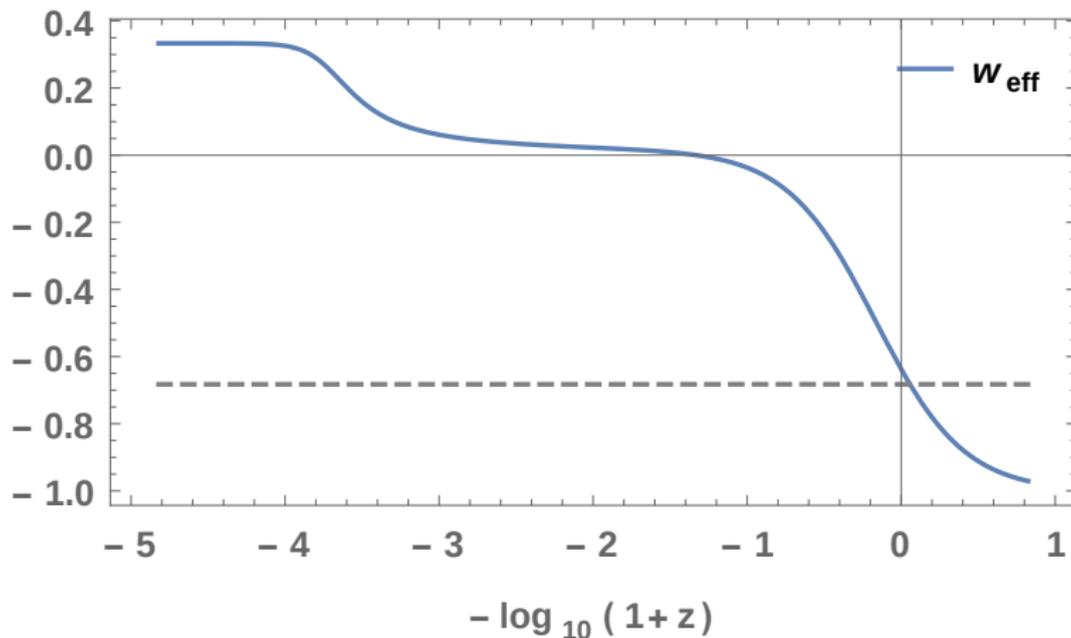
de Sitter is attained when  $H \rightarrow \sqrt{\Lambda}$ .

## 5. Tripod model: Energy densities

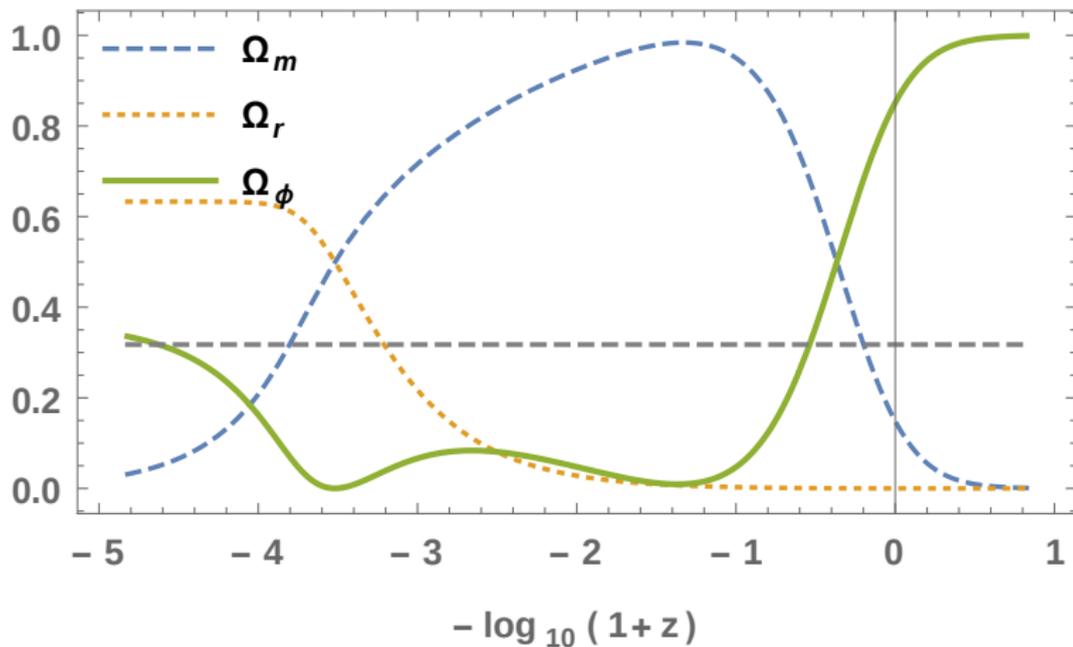
Example for:  $U_2 = e^{\lambda\phi} + \frac{4}{3}e^{\beta\phi}$  and  $W_2 = \lambda e^{\lambda\phi} + \beta e^{\beta\phi}$ .



## 5. Tripod model: Effective equation of state



## 5. Tripod model: Abundances



# Summary of Linear models

- ① **Only  $W_0 \neq 0$**   
No dynamics. Only de Sitter exists.
- ②  **$U_i, W_j$  pair and  $W_i, W_j$  pair**  
The evolution of the Universe does not depend on the material content or on the form of the field potentials.
- ③ **Term-by-Term model (4 potentials)**  
Do depend on the field evolution but does not provide a radiation dominated epoch.
- ④ **The tripod model (3 potentials)**  
Is the most promising but the field contribution seems to be too large at early times in the studied examples.

Look for non-linear models?

# II.

## Non-linear models

# Non-linear Lagrangian

$$L_{\text{nl}} = a^3 \sum_{i=0}^3 X_i(\phi, \dot{\phi}) H^i$$

To ensure that any non-linear dependence of the Lagrangian on  $\dot{\phi}$  to vanish at the critical point,

$$\sum_{i=0}^3 X_i(\phi, \dot{\phi}) \Lambda^{i/2} = 0$$

Again,  $X_i$  are related to the the  $\kappa_j$  functions of the Horndeski Lagrangian and  $G_j$  functions of the Deffayet et al. functions.

# Equations of motion

Proceed to shift-symmetric case and the redefinition  $\psi = \dot{\phi}$

$$H' = \frac{3(1+w)Q_0P_1 - Q_1P_0}{Q_1P_2 - Q_2P_1}$$

$$\psi' = \frac{3(1+w)Q_0P_2 - Q_2P_0}{Q_2P_1 - Q_1P_2}$$

where  $Q_0, Q_1, Q_2, P_0, P_1, P_2$ , are non-trivial functions of  $X_i$  and  $H$ , and the average equation of state parameter of matter fluids is

$$1 + w = \frac{\sum_s \Omega_s (1 + w_s)}{\sum_s \Omega_s}$$

# General considerations

- 1  $X_3 = \psi^n$  is the dominant contribution
- 2  $X_2 = \psi^n$  is the dominant contribution
- 3  $X_0$  and  $X_1$  are the sole contributions
- 4 Extension with  $X_0$ ,  $X_1$  and  $X_2$

# 1. $X_3 = \psi^n$ is the dominant contribution

When  $H \gg \sqrt{\Lambda}$ ,

$$1 + w_{\text{eff}} \simeq \frac{2}{3}(1 + w), \quad \text{for} \quad \frac{|(2X_3 + \psi X_{3,\psi}) X_{3,\psi\psi}|}{|(3X_{3,\psi} + \psi X_{3,\psi\psi}) X_{3,\psi}|} \gg 1$$

$$1 + w_{\text{eff}} \simeq \frac{2}{3} \quad \text{otherwise}$$

Neither allow for  $w_{\text{eff}}$  corresponding to radiation and matter domination.

## 2. $X_2 = \psi^n$ is the dominant contribution

When  $H \gg \sqrt{\Lambda}$ ,

$$w_{\text{eff}} \simeq w, \quad \text{for} \quad \frac{|(1 - X_2 - \psi X_{2,\psi}) X_{2,\psi\psi}|}{|(2X_{2,\psi} + \psi X_{2,\psi\psi}) X_{2,\psi}|} \gg 1,$$

$$w_{\text{eff}} \simeq 0, \quad \text{otherwise}$$

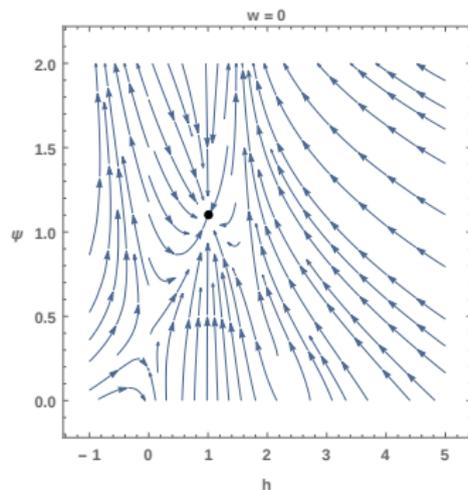
Either  $w_{\text{eff}}$  is too small today when compared with observational limits or,  $\Omega_\psi$  is too large in the early Universe.

### 3. $X_0$ and $X_1$ are the sole contributions

When  $H \gg \sqrt{\Lambda}$ ,

$$w_{\text{eff}} \simeq w,$$

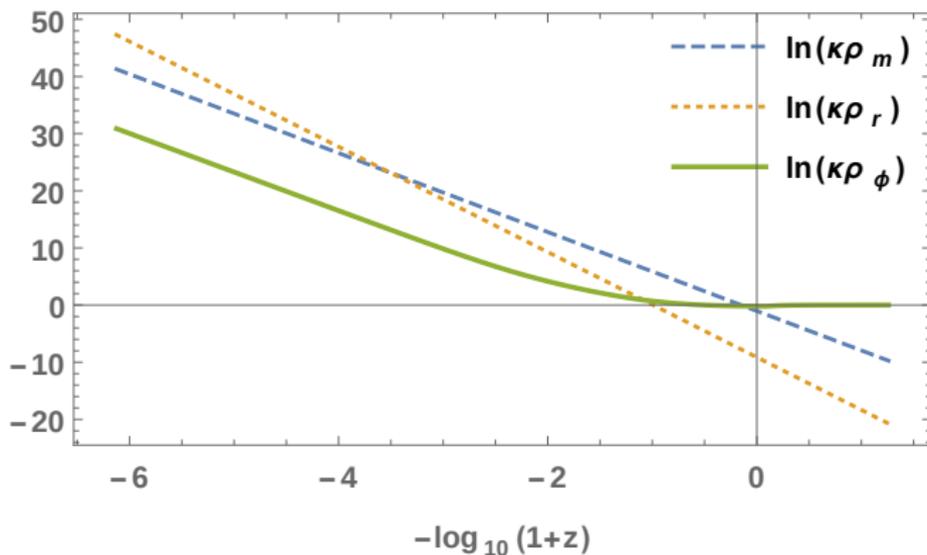
Interesting but unfortunately, models with realistic initial conditions are not driven to the critical point.



## 4. Extension with $X_0$ , $X_1$ and $X_2$

Considering

$$X_2(\psi) = \alpha\psi^n, \quad X_1(\psi) = -\alpha\psi^n + \frac{\beta}{\psi^m}, \quad X_0(\psi) = -\frac{\beta}{\psi^m}$$

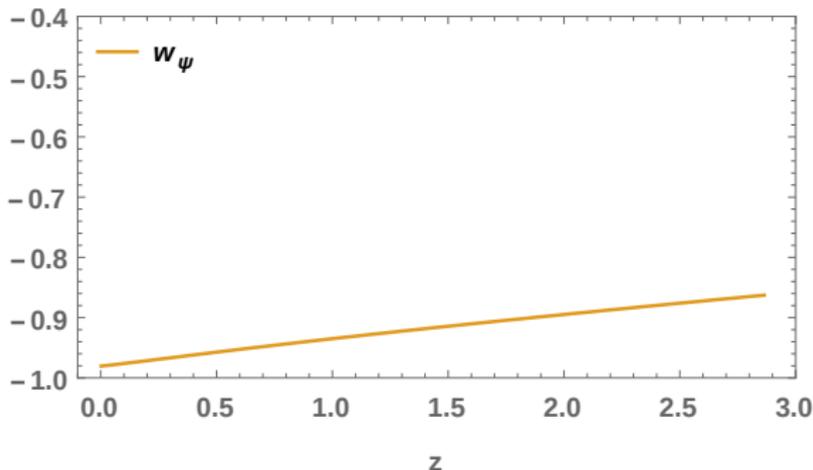


## 4. Extension with $X_0$ , $X_1$ and $X_2$

Considering

$$X_2(\psi) = \alpha\psi^n, \quad X_1(\psi) = -\alpha\psi^n + \frac{\beta}{\psi^m}, \quad X_0(\psi) = -\frac{\beta}{\psi^m}$$

We can obtain a model with  $w_\psi = w_0 + w_a(1 - a)$  s.t.  
 $w_0 = -0.98$  and  $w_a = 0.04$



# Summary of non-linear models

- 1  $X_3 = \psi^n$  is the dominant contribution  
Does not allow for  $w_{\text{eff}}$  corresponding to radiation and matter domination.
- 2  $X_2 = \psi^n$  is the dominant contribution  
 $w_{\text{eff}}$  is too small today when compared with observational limits or,  $\Omega_\psi$  is too large in the early Universe.
- 3  $X_0$  and  $X_1$  are the sole contributions  
Models with realistic initial conditions are not driven to the critical point.
- 4 Non-trivial combination of  $X_0$ ,  $X_1$  and  $X_2$   
Can obtain a range of parameter space compatible with observational limits.

# More to do

- Region of viable parameter space;
- Need to study of perturbations;
- Look for combination of linear and non-linear models.