Complete Hamiltonian analysis

of cosmological perturbations at all orders

Debottam Nandi
Co-author & supervisor: Dr. S. Shankaranarayanan

Indian Institute of Science Education and Research Thiruvananthapuram
Trivandrum - 695016, Kerala, India

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1. Complete Hamiltonian analysis of cosmological perturbations at all orders. Debottam Nandi & S. Shankaranarayanan, *JCAP* **1606** (2016), no. 06 038, [arXiv:1512.02539].
2. Complete Hamiltonian analysis of cosmological perturbations at all orders II: Non-canonical scalar field. Debottam Nandi & S. Shankaranarayanan, submitted to *JCAP*, [arXiv:1606.05747].

VARCOSMOFUN'16, Szczecin, Poland

Introduction

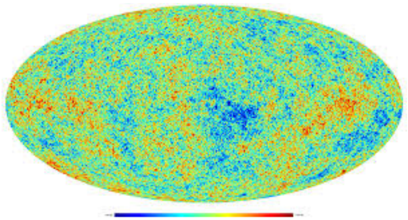


Figure: CMB from Planck's data

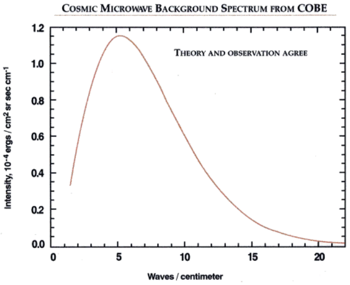


Figure: CMB blackbody radiation

- Cosmic Microwave Background (CMB) data (photon shower) with uniform temperature 2.73K.
- Early universe $\rightarrow$ homogeneous and isotropic.
- Friedman-Robertson-Walker model (FRW): $ds^2 = -dt^2 + a^2(t)\,d\bar{x}^2 \;=\; a^2(\eta)\left[-d\eta^2 + d\bar{x}^2\right]$, where t is cosmic time, η is conformal time and a is the scale factor.

- Problem $\rightarrow$ Horizon problem and flatness problem.
- Matter or Radiation dominated era $\rightarrow$ can not solve the puzzle.
- Solution $\rightarrow$ Inflation can solve these problems $\rightarrow$ $\ddot{a} > 0$ $\rightarrow$ accelerated universe.
- Universe is filled with inflaton $\rightarrow$ Scalar field(s).

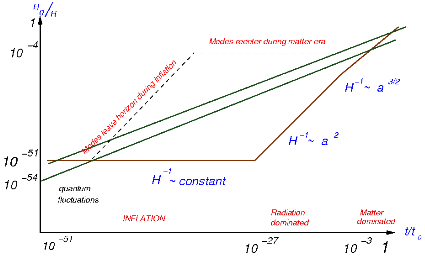


Figure: Hubble radius vs. Cosmic time

- Quantum fluctuations freeze after crossing Horizon $\rightarrow$ Classical perturbations
- Inhomogeneity $\rightarrow$ 1 in 10^5 $\rightarrow$ Perturbation theory can be trusted.
- Linear order perturbation theory $\rightarrow$ perturbed field equations are linear in $(\delta g_{\mu\nu}, \delta\varphi)$ $\rightarrow$ solution Gaussian in nature.

$$< \mathcal{R}\mathcal{R}' > \neq 0, \quad < \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3 >= 0$$

$$\uparrow \qquad\qquad \uparrow$$

Power-spectrum Non-gaussianity

Motivation

- Higher-order perturbations $\rightarrow$ non-zero non-gaussianity $\rightarrow$ More information to know about the universe.
- Quantization: Path-integral formalism is relatively difficult to apply for gravity models than canonical quantization $\rightarrow$ We need Hamiltonian formulation of cosmological perturbation theory.
- All observables are related to n-point correlation functions and interaction Hamiltonian is needed to compute higher-order correlation functions.
- In general, slow-roll approximations are used to evaluate interaction Hamiltonian. In our work, we provide an approach which is independent of any approximations.

Problems and difficulties with previous method

- Previous approach is based on Chen *et al.*[2006]. Perturbed Lagrangian:

$$\mathcal{L} = f_0\dot{\alpha}^2 + j_2 + g_0\dot{\alpha}^3 + g_1\dot{\alpha}^2 + g_2\dot{\alpha} + j_3 + h_0\dot{\alpha}^4 + h_1\dot{\alpha}^3 + h_2\dot{\alpha}^2 + h_3\dot{\alpha} + j_4 \tag{1}$$

- Subscripts $0, 1, 2, 3 \cdots$ are orders of perturbations. α is first order perturbed quantity (curvature perturbation, ζ).
- Difficulty: It is easy to apply for reduced form, however, difficult for constrained system, e.g., Gravity.

- Momentum π in terms of $\dot{\alpha}$.
- Inverse relation using order-by-order approximations.
- Hamiltonian in phase-space.

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \tag{2}$$

$$\dot{\alpha} = \frac{\pi}{2\,f_0} + c_2 \pi^2 + c_3 \pi^3 \tag{3}$$

$$c_2 = -\frac{3g_0}{8f_0^3} - \frac{g_1}{2\,f_0^2 \pi} - \frac{g_2}{2/,\, f_0 \pi^2} \tag{4}$$

$$\begin{aligned} c_3 = &-\frac{1}{2\,f_0}\left(\frac{3\,g_0 c_2}{f_0} + \frac{h_0}{2\,f_0^3}\right) \\ &- \frac{1}{2\,f_0 \pi}\left(2\,g_1 c_2 + \frac{3\,h_1}{4\,f_0^2}\right) - \frac{h_2}{2\,f_0^2 \pi^2} - \frac{h_3}{2\,f_0 \pi^3} \end{aligned} \tag{5}$$

$$\mathcal{H} = \pi \dot{\alpha} - \mathcal{L}. \tag{6}$$

- Problem: momentum π corresponding to α is written as a polynomial of $\dot{\varphi}$ and vice-versa using order-by-order approximation $\rightarrow$ Higher-order correlations??
- Also, both momentum and α are perturbed quantities, hence the polynomial relations are not consistent.

Interaction Hamiltonian

- After obtaining Hamiltonian in terms of (π, α), we need to invert back the Hamiltonian in terms of configuration-space variable $(\dot{\alpha}, \alpha)$.
- Problem: This time, only linear-order relation is being used: $\dot{\alpha} = \frac{\pi}{2f_0} \Rightarrow \pi = 2f_0\alpha$.

$$\mathcal{H}_3^{int} = -g_0\dot{\alpha}^3 - g_1\dot{\alpha}^2 - g_2\dot{\alpha} - j_3 \tag{7}$$

$$\begin{aligned}\mathcal{H}_4^{int} &= \left(\frac{9\,g_0^2}{4\,f_0} - h_0\right)\dot{\alpha}^4 + \left(\frac{3g_0g_1}{f_0} - h_1\right)\dot{\alpha}^3 \\ &+ \left(\frac{3g_0g_2}{2f_0} + \frac{g_1^2}{f_0} - h_2\right)\dot{\alpha}^2 + \left(\frac{g_1g_2}{f_0} - h_3\right)\dot{\alpha} \\ &+ \frac{g_2^2}{4f_0} - j_4\end{aligned} \tag{8}$$

- Third order interaction Hamiltonian: $\mathcal{H}_3^{\text{int}} = -\mathcal{L}_3$ which is true for any system.

Other problems and difficulties

- Not easy to extend the formalism to higher-order perturbations.
- Not easy to extend the formalism to obtain mixed-mode cross-correlations.
- Consistent Hamiltonian formalism $\rightarrow$ Can solve all the key issues.
- Hamiltonian formulation of gauge-invariant cosmological perturbation $\rightarrow$ How to resolve Gauge-issue??

Gauge-fixing and Gauge issue in Hamiltonian formulation

- Gauge-fixing: 1×3 decomposition.

$$\delta g_{\mu\nu} = \begin{pmatrix} -2\,a^2\phi & a^2 B_{|i} \\ a^2\,B_{|i} & -2a^2\psi\delta_{ij} + 2E_{|ij} \end{pmatrix},$$

$$\varphi = \bar{\varphi} + \delta\varphi$$

- five scalar variables ϕ, B, ψ, E & $\delta\varphi$ $\rightarrow$ two gauge freedom $\rightarrow$ fix two $\rightarrow$ remaining becomes Gauge-invariant variables.
- Ex: flat-slicing gauge: $\delta g_{ij} = 0 \Rightarrow \psi = E = 0 \rightarrow \delta\varphi$ coincides with gauge-invariant curvature perturbation.
- Equations of motion of three variables $\phi, B, \delta\varphi \rightarrow$ Gauge-invariant equations.
- How to interpret gauge-invariance in Hamiltonian formalism???
- Langlois [1994] $\rightarrow$ Gauge invariance by using Hamilton-Jacobi approach to canonical scalar field $\rightarrow$ difficult to extend the approach to any other scalar field at any higher order of perturbations $\rightarrow$ *problem remains the same.*

Models

1. Simple model: $\mathcal{S}_{Simple} = \int dt \left[\frac{1}{2y} \left((d_t x)^2 + (d_t y)^2 \right) - \frac{1}{4} \left(x^4 + y^4 \right) \right]$ to resolve gauge-issue.
2. Canonical scalar field:
   $\mathcal{S}_{Canonical} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right]$.
3. Specific Galilean scalar field:
   $\mathcal{S}_{Galilean} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R - g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \Box \varphi - V(\varphi) \right]$.
4. Generalized non-canonical scalar field:
   $\mathcal{S}_{NC} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R + P(X, \varphi) \right], \quad X \equiv \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$.

Simple model

- Action for the model: $\mathcal{S}_{Simple} = \int dt \left[\frac{1}{2y} \left((d_t x)^2 + (d_t y)^2 \right) - \frac{1}{4} \left(x^4 + y^4 \right) \right]$
- Equations of motion:

$$d_t \left(\frac{d_t x}{y} \right) + x^3 = 0 \tag{9}$$

$$d_t \left(\frac{d_t y}{y} \right) = -\frac{1}{2y^2} \left((d_t x)^2 + (d_t y)^2 \right) - y^3 \tag{10}$$

- Momenta: $\pi_x = \frac{d_t x}{y}, \quad \pi_y = \frac{d_t y}{y},$.
- Hamiltonian: $\mathcal{H} = \frac{1}{2} y \left(\pi_x^2 + \pi_y^2 \right) + \frac{1}{4} \left(x^4 + y^4 \right)$.
- Perturbation theory: x unperturbed, all other quantities are perturbed.

$$x = x_0, \quad y = y_0 + \epsilon y_1$$

$$\pi_x = \pi_{x0} + \epsilon \pi_{x1}, \quad \pi_y = \pi_{y0} + \epsilon \pi_{y1}$$

- Perturbed equations of motion:

$$d_t\left(\frac{d_t x_0}{y_0}\right) + x_0^3 = 0 \tag{11}$$

$$d_t\left(\frac{d_t y_0}{y_0}\right) = -\frac{1}{2{y_0}^2}\left((d_t x_0)^2 + (d_t y_0)^2\right) - y_0^3 \tag{12}$$

$$d_t\left(\frac{d_t y_1}{y_0} - \frac{d_t y_0}{{y_0}^2} y_1\right) = \frac{1}{{y_0}^3}\left((d_t x_0)^2 + (d_t y_0)^2\right) y_1 - \frac{d_t y_0}{{y_0}^2} d_t y_1 - 3{y_0}^2 y_1 . \tag{13}$$

- $\pi_x = \frac{d_t x}{y} \to \pi_{x1} = -\frac{y_1}{y_0}\pi_{x0} \neq 0$
- Perturbed momenta of unperturbed variables do not vanish.

Proposed Hamiltonian approach

- $\mathcal{H} = \mathcal{H}_0 + \epsilon\mathcal{H}_1 + \epsilon^2\mathcal{H}_2 + \epsilon^3\mathcal{H}_3 + \epsilon^4\mathcal{H}_4 + \ldots$
- Zeroth order Hamiltonian: $\mathcal{H}_0 = \frac{1}{2}\, y_0 \left({\pi_{x0}}^2 + {\pi_{y0}}^2\right) + \frac{1}{4}\, \left({x_0}^4 + {y_0}^4\right)$
- Second order Hamiltonian: $\mathcal{H}_2 = y_1 \left(\pi_{x0}\pi_{x1} + \pi_{y0}\pi_{y1}\right) + \frac{1}{2}\, y_0 \left({\pi_{x1}}^2 + {\pi_{y1}}^2\right) + \frac{3}{2}\, {y_0}^2 {y_1}^2.$
- The above Hamiltonian is not sufficient $\longrightarrow$ Need extra condition $\rightarrow \frac{\delta\mathcal{H}_2}{\delta\pi_{x1}} = 0 \rightarrow y_1\pi_{x0} + y_0\pi_{x1} = 0 \rightarrow$ Consistent $\longrightarrow$ It is complete.
- Note that: extra condition is only needed if one/some quantities are unperturbed and rest are perturbed quantities.
- Third and fourth order interaction Hamiltonian:

$$\mathcal{H}_3 = \frac{1}{2}\, y_1 \left({\pi_{x1}}^2 + {\pi_{y1}}^2\right) + {y_1}^3 y_0 \tag{14}$$

$$\mathcal{H}_4 = \frac{1}{4} {y_1}^4 \tag{15}$$

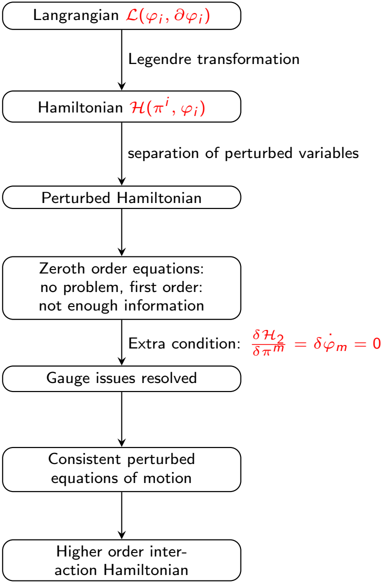
- Inverting to configuration-space using Hamilton's equations:

$$\mathcal{H}_3 = \frac{1}{2}\, (\partial_t x_0\,)^2 {y_0}^{(-4)} {y_1}^3 + \frac{1}{2}\, (\partial_t y_0\,)^2 {y_0}^{(-4)} {y_1}^3 - \partial_t y_0\, \partial_t y_1\, {y_0}^{(-3)} {y_1}^2 + \frac{1}{2}\, (\partial_t y_1\,)^2 {y_0}^{(-2)} y_1 + {y_1}^3 y_0$$

$$\mathcal{H}_4 = \frac{1}{4} {y_1}^4 \tag{16}$$

- Recovering the known result: third order interaction Hamiltonian $\mathcal{H}_3 = -\mathcal{L}_3$.
- Hamiltonian formulation is consistent with Lagrangian formulation.

Flow-chart


Langrangian $\mathcal{L}(\varphi_i, \partial\varphi_i)$
Legendre transformation
Hamiltonian $\mathcal{H}(\pi^i, \varphi_i)$
separation of perturbed variables
Perturbed Hamiltonian
Zeroth order equations: no problem, first order: not enough information
Extra condition: $\frac{\delta \mathcal{H}_2}{\delta \pi^m} = \delta\dot{\varphi}_m = 0$
Gauge issues resolved
Consistent perturbed equations of motion
Higher order interaction Hamiltonian

Canonical scalar field

- Action: $\mathcal{S}_{Canonical} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right]$.
- Line-element: $ds^2 = (-N^2 + N^i N_i)dt^2 + 2N_i dx^i dt + \gamma_{ij}dx^i dx^j$, N and N^i are called Lapse function and shift vector, respectively, ${}^{(3)}R$ is the Ricci scalar for 3-metric γ_{ij}.
- Hamiltonian density:

$$\mathcal{H}_{Can} = \gamma_{jk}\partial_i N^k\,\pi^{ij} + N^k\partial_k\gamma_{ij}\,\pi^{ij} + \gamma_{ik}\partial_j N^k\,\pi^{ij} - N\gamma_{ij}\gamma_{kl}\kappa\pi^{kl}\pi^{ij}\gamma^{-\frac{1}{2}} + 2\,N\gamma_{ik}\,\gamma_{jl}\,\kappa\pi^{kl}\pi^{ij}\gamma^{-\frac{1}{2}} + \frac{1}{2}\,N{\pi_\varphi}^2\gamma^{-\frac{1}{2}} + N^i\pi_\varphi\partial_i\varphi - \frac{1}{2}\,N^{(3)}R\gamma^{\frac{1}{2}}\kappa^{-1} + \frac{1}{2}\,N\gamma^{ij}\partial_i\varphi\,\partial_j\varphi\,\gamma^{\frac{1}{2}} + NV\gamma^{\frac{1}{2}} = 0. \tag{17}$$

- Zeroth order Hamiltonian $\rightarrow$ Consistent.
- In order to obtain gauge-invariant equations of motion $\rightarrow$ Perturbation in flat-slicing gauge $\Rightarrow \delta\gamma_{ij} = 0$ click here

- Similarly like the simple model, in Flat-slicing gauge we need an extra condition: $\frac{\delta \mathcal{H}_2^C}{\delta \pi_1^{ij}} = 0 \rightarrow$ solves the gauge-issue in flat-slicing gauge $\rightarrow$ all equations of motion are consistent.
- Mixed mode: In case of scalar-tensor mode at linear order $\rightarrow$ we do not need extra condition as $\delta\gamma_{ij} = a^2\, h_{ij} \neq 0,\;\, h_{ij}$ is tensor-perturbation.
- In exactly the same way, it can be extended to higher-order perturbations.

Interaction Hamiltonian for scalar perturbations

- Third order interaction Hamiltonian in phase-space:

$$\mathcal{H}_3^C(\pi,\varphi) = -\frac{1}{2}N_1\partial_i {N_1}^i\,\partial_j {N_1}^j\,{N_0}^{(-2)}\kappa^{-1}a^3 + 2\,\delta_{ij}\partial_k {N_1}^k\,{\pi_0}^{ij}{N_0}^{(-2)}{N_1}^2a^2 - \frac{1}{4}N_1\partial_i {N_1}^j\,\partial_j {N_1}^i\,{N_0}^{(-2)}\kappa^{-1}a^3 + \frac{1}{4}N_1\delta_{ij}\delta^{lk}\partial_k {N_1}^i\,\partial_l {N_1}^j\,{N_0}^{(-2)}\kappa^{-1}a^3 + \frac{1}{2}\delta_{ij}\partial_k {N_1}^i\,{\pi_0}^{kj}{N_0}^{(-2)}{N_1}^2a^2 + \frac{1}{2}\delta_{ij}\partial_k {N_1}^i\,{\pi_0}^{jk}{N_0}^{(-2)}{N_1}^2a^2 - 2\,\delta_{ij}\partial_k {N_1}^k\,{\pi_0}^{ij}{N_0}^{(-2)}{N_1}^2a^2 + \frac{1}{2}\delta_{ij}\partial_k {N_1}^j\,{\pi_0}^{ki}{N_0}^{(-2)}{N_1}^2a^2 + \frac{1}{2}\delta_{ij}\partial_k {N_1}^j\,{\pi_0}^{ik}{N_0}^{(-2)}{N_1}^2a^2 + 2\,\delta_{ij}\delta_{kl}\kappa{\pi_0}^{ik}{\pi_0}^{jl}{N_0}^{(-2)}{N_1}^3a + \frac{1}{2}N_1{\pi_{\varphi 1}}^2a^{(-3)} + {N_1}^i\pi_{\varphi 1}\partial_i\varphi_1 + \delta_{ij}\delta_{kl}\kappa{\pi_0}^{ij}{\pi_0}^{lk}{N_0}^{(-2)}{N_1}^3a + \frac{1}{2}N_1\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,a + \frac{1}{2}N_1V_{\varphi\varphi}\,{\varphi_1}^2a^3 + \frac{1}{6}N_0V_{\varphi\varphi\varphi}\,{\varphi_1}^3a^3. \tag{18}$$

- Fourth order interaction Hamiltonian in phase-space:

$$\mathcal{H}_4^C = \frac{1}{6}N_1V_{\varphi\varphi\varphi}\,{\varphi_1}^3a^3 + \frac{1}{24}N_0V_{\varphi\varphi\varphi\varphi}\,{\varphi_1}^4a^3 \tag{19}$$

Result - Canonical scalar field

- We have successfully constructed Hamiltonian formulation for canonical scalar field.
- There is no approximations in our method.
- The system is not reduced as in the previous method $\rightarrow$ applicable to constrained systems.
- Can easily be extended to Higher order perturbations.
- Can easily be extended to obtain higher-order correlation functions.
- Can easily be extended to obtain mixed-mode correlations.
- Solves the Gauge-issue.

Galilean scalar field

- Specific Galilean model:

$$\mathcal{S}_{Galilean} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}R - g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\Box\varphi - V(\varphi)\right]$$
$$= \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\, S - V(\varphi)\right] + \int d^4x\,\lambda\,(S - \Box\varphi) \tag{20}$$

- To linearize the derivative, we have introduced two extra variables: λ, S. Hence, in phase-space 4 extra variables.
- Three primary constraint: $\pi_i, \pi_S = 0; \pi_N + \frac{\lambda}{N}\pi_\lambda = 0$
- Hamiltonian density:

$$\mathcal{H}_D = -S\lambda + NV\gamma^{\frac{1}{2}} + N^i\pi_\lambda\partial_i\lambda + N^i\pi_\varphi\partial_i\varphi - \pi_\lambda\pi_\varphi N^2 + \pi_\lambda\lambda\partial_i N^i + 2\gamma_{ij}\partial_k N^i\,\pi^{jk} + \gamma^{ij}\lambda\partial_i N\,\partial_j\varphi\, N^{-1}$$
$$-\gamma^{ij}\partial_i\lambda\,\partial_j\varphi - \lambda\partial_i\gamma^{ij}\,\partial_j\varphi - \frac{1}{2}N^{(3)}R\gamma^{\frac{1}{2}}\kappa^{-1} - SN^3\pi_\lambda{}^2\gamma^{\frac{1}{2}} - \frac{3}{4}N\kappa\pi_\lambda{}^2\gamma^{-\frac{1}{2}}\lambda^2 - N^i\pi_\lambda\lambda\partial_i N\,N^{-1} +$$
$$N^i\partial_i\gamma_{lm}\,\pi^{lm} - N^i\partial_l\gamma_{im}\,\pi^{lm} + N^i\partial_m\gamma_{il}\,\pi^{lm} - \gamma^{ij}\gamma^{kl}\lambda\partial_i\gamma_{jk}\,\partial_l\varphi + \frac{1}{2}\gamma^{ij}\gamma^{kl}\lambda\partial_i\gamma_{kl}\,\partial_j\varphi +$$
$$NS\gamma^{ij}\partial_i\varphi\,\partial_j\varphi\,\gamma^{\frac{1}{2}} - N\pi_\lambda\gamma_{ij}\kappa\lambda\pi^{ij}\gamma^{-\frac{1}{2}} + 2\,N\gamma_{ij}\gamma_{kl}\kappa\pi^{ik}\pi^{jl}\gamma^{-\frac{1}{2}} - N\gamma^{ij}\gamma^{kl}\kappa\pi^{ij}\pi^{kl}\gamma^{-\frac{1}{2}} +$$
$$\xi\left(\pi_N + \frac{\lambda}{N}\pi_\lambda\right). \tag{21}$$

Counting degrees of freedom

Dirac constraints:

- Primary constraint: $\Phi_p = \Phi_p(q_i, p_i) = 0$
- Conservation of primary constraint $\rightarrow$ Secondary constraint: $\Phi_s \equiv \{\Phi_p, \mathcal{H} + \xi\Phi_p\} \equiv \{\Phi_p, \mathcal{H}_D\} \approx 0.$
- Conservation of secondary constraint $\rightarrow$ Tertiary constraint and so on.
- First-class constraint and second-class constraint.
- Degrees of freedom in phase-space: (Total phase-space variable - 2 $\times$ number of first-class constraints - number of second-class constraints)
- Degrees of freedom in configuration-space $= \frac{1}{2}\times$ phase-space constraints.

- Zeroth order phase-space $\rightarrow$ $(a,\, N,\, \varphi_0,\, \lambda_0,\, S_0)$ $\rightarrow$ 10 phase-space variables.
- 2 first-class constraints: Φ^2_p, Φ^2_s, 4 second-class constraints, degrees of freedom in configuration-space: $\frac{1}{2} \times (10 - 2 \times 2 - 4) = 1$ $\rightarrow$ No extra degress of freedom. click here
- Similarly, also for first order perturbation or full theory, it can be shown that Galilean theory has no extra degrees of freedom.

Result - Galilean scalar field

- Since we are using consistent Hamiltonian formulations, it can also deal with higher-order derivative systems.
- Hamiltonian of Galilean/Lagrangian with higher order derivative contains extra variables in the phase-space but no extra degrees of freedom.

Non-canonical scalar field

- Action: $\mathcal{S}_{NC} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}R + P(X,\varphi)\right], \quad X \equiv \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi.$
- Momentum corresponding to φ:

$$\pi_\varphi \equiv \frac{\partial \mathcal{L}_{NC}}{\partial \varphi'} = -\sqrt{\gamma}P_X\sqrt{-2X+Y}, \quad Y \equiv \gamma^{ij}\partial_i\varphi\partial_j\varphi \tag{22}$$

- $\dot{\varphi}$ cannot be written in terms of π_φ for generalized non-canonical scalar field.
- We do not know how to invert π_φ for generalized non-canonical scalar field $\rightarrow$ No generalized form of Hamiltonian $\rightarrow$ our approach cannot be extended to non-canonical scalar field. click here

- Introducing new phase-space variable:
  $G(\pi_\varphi, \gamma, Y, \varphi) = \tilde{G}(X, \gamma, Y, \varphi) \equiv P - P_X\,(2X - Y)$
- Ex: Canonical scalar field: $G(\gamma, \pi_\varphi, Y, \varphi) = -\frac{1}{2}\frac{\pi_\varphi^2}{\gamma} - \frac{1}{2}\,Y - V(\varphi)$
  Tachyonic field: $G(\gamma, \pi_\varphi, Y, \varphi) = -\frac{1}{\sqrt{\gamma}}\sqrt{1+Y}\sqrt{\pi_\varphi^2 + \gamma\,V^2}$.
- Hamiltonian density:

$$\mathcal{H} = 2\gamma_{ij}\partial_k N^j\,\pi^{ik} + N^i\partial_i\gamma_{jk}\,\pi^{jk} - \frac{N\kappa}{\sqrt{\gamma}}\left(\gamma_{ij}\gamma_{kl} - 2\gamma_{ik}\gamma_{jl}\right)\pi^{ij}\pi^{kl}$$
$$- \frac{N\sqrt{\gamma}}{2\kappa}\,{}^{(3)}R - N\sqrt{\gamma}\;G(\pi_\varphi, \gamma, Y, \varphi) + N^i\pi_\varphi\partial_i\varphi \qquad (23)$$

- Since, we get the generalized form of the Hamiltonian, now we can extend our proposed approach to non-canonical scalar field.

- Second order Hamiltonian becomes

$$\mathcal{H}_2 = \delta_{ij}\partial_k {N_1}^j\,({\pi_1}^{ik} + \pi_1^{ki})a^2 - N_0\kappa a\,(\delta_{ij}\delta_{kl} - 2\delta_{ik}\delta_{jl}){\pi_1}^{ij}{\pi_1}^{kl} - 2\,N_1\kappa a\,(\delta_{ij}\delta_{kl} - 2\delta_{ik}\delta_{jl}){\pi_0}^{ij}{\pi_1}^{kl}$$
$$- G_\varphi N_1 a^3 \varphi_1 - G_{\pi_\varphi} N_1\,\pi_{\varphi 1}\,a^3 - \frac{1}{2}\,G_{\varphi\varphi} N_0\,{\varphi_1}^2 a^3 - \frac{1}{2}\,G_{\pi_\varphi \pi_\varphi} N_0\,{\pi_{\varphi 1}}^2 a^3 - G_{\varphi\pi_\varphi} N_0 \pi_{\varphi 1} a^3 \varphi_1$$
$$- G_Y N_0 \delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,a + {N_1}^i \pi_{0\varphi}\partial_i\varphi_1 \qquad (24)$$

- Speed of sound in phase-space: $c_s^2 \equiv -\frac{\text{coeff. of } \varphi_1''}{\text{coeff. of } \nabla^2\varphi_1} = 2\,N_0^2\,a^4\,G_{\pi_\varphi\pi_\varphi}\,G_Y.$
- It can be verified that, using a specific model like canonical or tachyonic model, Hamiltonian formulation of non-canonical scalar field is consistent.
- How to invert G and its derivatives $P(X,\varphi)$ for generalized non-canonical scalar field?

Inversion

- Any function, $F \equiv F(\pi_\varphi, \gamma, Y, \varphi)$

$$
\begin{aligned}
F &= F(\pi_\varphi, \gamma, Y, \varphi) \Rightarrow dF = F_{\pi_\varphi} d\pi_\varphi + F_\gamma d\gamma + F_Y dY + F_\varphi d\varphi \\
\pi_\varphi &= \pi_\varphi(X, \gamma, Y, \varphi) \Rightarrow d\pi_\varphi = \frac{\partial \pi_\varphi}{\partial X} dX + \frac{\partial \pi_\varphi}{\partial \gamma} d\gamma + \frac{\partial \pi_\varphi}{\partial Y} dY + \frac{\partial \pi_\varphi}{\partial \varphi} d\varphi
\end{aligned}
$$

- Inversion formulae:

$$
F_{\pi_\varphi} = \frac{\tilde{F}_X}{\frac{\partial \pi_\varphi}{\partial X}}, \quad F_\gamma = \tilde{F}_\gamma - F_{\pi_\varphi} \frac{\partial \pi_\varphi}{\partial \gamma} \tag{25}
$$

$$
F_Y = \tilde{F}_Y - F_{\pi_\varphi} \frac{\partial \pi_\varphi}{\partial Y}, \quad F_\varphi = \tilde{F}_\varphi - F_{\pi_\varphi} \frac{\partial \pi_\varphi}{\partial \varphi} \tag{26}
$$

- In our case, for arbitrary non-canonical scalar field, we do not know the exact form of $G(\pi_\varphi, \gamma, Y, \varphi)$ but we know $\tilde{G}(X, \gamma, Y, \varphi) \equiv P - P_X (2X - Y)$

$$
G_{\pi_\varphi} = -\frac{\sqrt{-2X}}{a^3}, \quad G_{\pi_\varphi \pi_\varphi} = \frac{1}{a^6 (P_X + 2XP_{XX})}, \; G_{\pi_\varphi \pi_\varphi \pi_\varphi} = -\frac{\sqrt{-2X}\,(3P_{XX} + 2XP_{XXX})}{a^9 (P_X + 2XP_{XX})^3}
$$

$$
G_\varphi = P_\varphi, \quad G_{\varphi \pi_\varphi} = \frac{\sqrt{-2X} P_{X\varphi}}{a^3 (P_X + 2XP_{XX})}, \; G_{\varphi\varphi} = P_{\varphi\varphi} - \frac{2X\, P_{X\varphi}^2}{(P_X + 2XP_{XX})} \tag{27}
$$

- Speed of sound in conformal time:

$$c_s^2 = \frac{P_X}{P_X + 2XP_{XX}}$$

Interaction Hamiltonian

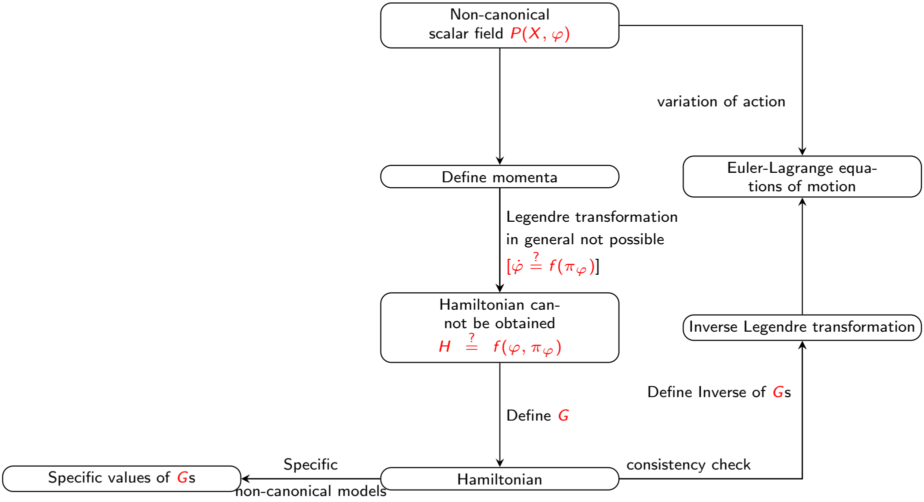
- Third order interaction Hamiltonian in phase-space:

$$\begin{aligned}\mathcal{H}_3 \quad = \quad & -N_1\delta_{ij}\delta_{kl}\kappa{\pi_1}^{ij}{\pi_1}^{kl}a + 2\,N_1\delta_{ij}\delta_{kl}\kappa{\pi_1}^{ik}{\pi_1}^{jl}a - \frac{1}{2}\,G_{\varphi\varphi}N_1{\varphi_1}^2a^3 - \frac{1}{2}\,G_{\pi_\varphi\pi_\varphi}N_1{\pi_{\varphi 1}}^2a^3 - \\ & G_{\varphi\pi_\varphi}N_1\pi_{\varphi 1}a^3\varphi_1 - G_Y N_1\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,a - G_{Y\pi_\varphi}N_0\pi_{\varphi 1}\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,a - \\ & G_{\varphi Y}N_0\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,\varphi_1 a - \frac{1}{6}\,G_{\pi_\varphi\pi_\varphi\pi_\varphi}N_0{\pi_{\varphi 1}}^3a^3 - \frac{1}{6}\,G_{\varphi\varphi\varphi}N_0{\varphi_1}^3a^3 - \\ & \frac{1}{2}\,G_{\varphi\pi_\varphi\pi_\varphi}N_0{\pi_{\varphi 1}}^2a^3\varphi_1 - \frac{1}{2}\,G_{\varphi\varphi\pi_\varphi}N_0\pi_{\varphi 1}{\varphi_1}^2a^3 + {N_1}^i\pi_{\varphi 1}\partial_i\varphi_1 \end{aligned} \tag{28}$$

- Fourth order interaction Hamiltonian in phase-space:

$$\begin{aligned}\mathcal{H}_4 \quad = \quad & -\frac{1}{2}\,G_{YY}N_0\delta^{ij}\delta^{kl}\partial_i\varphi_1\,\partial_j\varphi_1\,\partial_k\varphi_1\,\partial_l\varphi_1\,a^{-1} - G_{\pi_\varphi Y}N_1\pi_{\varphi 1}\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,a - \\ & G_{\varphi Y}N_1\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,\varphi_1 a - \frac{1}{6}\,G_{\pi_\varphi\pi_\varphi\pi_\varphi}N_1{\pi_{\varphi 1}}^3a^3 - \frac{1}{6}\,G_{\varphi\varphi\varphi}N_1{\varphi_1}^3a^3 - \\ & \frac{1}{2}\,G_{\varphi\pi_\varphi\pi_\varphi}N_1{\pi_{\varphi 1}}^2a^3\varphi_1 - \frac{1}{2}\,G_{\pi_\varphi\pi_\varphi Y}N_0\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,{\pi_{\varphi 1}}^2a - \frac{1}{2}\,G_{\varphi\varphi\pi_\varphi}N_1\pi_{\varphi 1}{\varphi_1}^2a^3 - \\ & \frac{1}{2}\,G_{\varphi\varphi Y}N_0\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,\varphi_1^2 a - G_{\varphi\pi_\varphi Y}N_0\pi_{\varphi 1}\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,\varphi_1 a - \\ & \frac{1}{24}\,G_{\pi_\varphi\pi_\varphi\pi_\varphi\pi_\varphi}N_0{\pi_{\varphi 1}}^4a^3 - \frac{1}{24}\,G_{\varphi\varphi\varphi\varphi}N_0{\varphi_1}^4a^3 - \frac{1}{6}\,G_{\varphi\pi_\varphi\pi_\varphi\pi_\varphi}N_0{\pi_{\varphi 1}}^3a^3\varphi_1 - \\ & \frac{1}{6}\,G_{\varphi\varphi\varphi\pi_\varphi}N_0\pi_{\varphi 1}{\varphi_1}^3a^3 - \frac{1}{4}\,G_{\varphi\varphi\pi_\varphi\pi_\varphi}N_0{\pi_{\varphi 1}}^2{\varphi_1}^2a^3 \end{aligned} \tag{29}$$

Flow-chart



Non-canonical scalar field $P(X, \varphi)$
variation of action
Define momenta
Euler-Lagrange equations of motion
Legendre transformation in general not possible
$[\dot{\varphi} \stackrel{?}{=} f(\pi_\varphi)]$
Hamiltonian cannot be obtained
$H \stackrel{?}{=} f(\varphi, \pi_\varphi)$
Inverse Legendre transformation
Define Inverse of Gs
Define G
Specific values of Gs
Specific non-canonical models
Hamiltonian
consistency check

Result - Non-canonical scalar field

- In case of non-canonical scalar field, by introducing a new phase-space variable, we obtained generalized Hamiltonian of non-canonical system.
- we have obtained a new expression of sound speed in terms of phase-space variables and similarly we can obtain other background quantities like slow-roll parameters, Hubble's factor and others in terms of phase-space variables.
- In case of non-canonical scalar field, we have obtained a simple mechanism to invert general non-canonical terms from phase-space to configuration space and vice-versa.
- In the same way, we can extend our approach to generalized scalar-tensor theories like Hordenski's model also.

Summary and Extension

- This approach is model independent.
- This can not only solves the difficulties/problems with the previous method but also gives us more information like the counting degrees of freedom etc.
- We are currently working on the effect of higher-order perturbations in the interaction Hamiltonian $\rightarrow$ significant corrections in Bi-spectrum and Tri-spectrum.

Thank you!

Canonical scalar field

- Action: $\mathcal{S}_{Canonical} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right]$.
- Line-element: $ds^2 = (-N^2 + N^iN_i)dt^2 + 2N_i dx^i dt + \gamma_{ij}dx^i dx^j$, N and N^i are called Lapse function and shift vector, respectively, ${}^{(3)}R$ is the Ricci scalar for 3-metric γ_{ij}.
- Hamiltonian density:

$$\mathcal{H} = \gamma_{jk}\partial_i N^k\,\pi^{ij} + N^k\partial_k\gamma_{ij}\,\pi^{ij} + \gamma_{ik}\partial_j N^k\,\pi^{ij} - N\gamma_{ij}\gamma_{kl}\kappa\pi^{kl}\pi^{ij}\gamma^{-\frac{1}{2}} + 2\,N\gamma_{ik}\,\gamma_{jl}\,\kappa\pi^{kl}\pi^{ij}\gamma^{-\frac{1}{2}} + \frac{1}{2}\,N{\pi_\varphi}^2\gamma^{-\frac{1}{2}} + N^i\pi_\varphi\partial_i\varphi - \frac{1}{2}\,N^{(3)}R\gamma^{\frac{1}{2}}\kappa^{-1} + \frac{1}{2}\,N\gamma^{ij}\partial_i\varphi\,\partial_j\varphi\,\gamma^{\frac{1}{2}} + NV\gamma^{\frac{1}{2}} = 0. \tag{30}$$

- Perturbation in Flat-slicing gauge:

$N = N_0 + \epsilon N_1,\, N^i = \epsilon N_1^i,\, \varphi = \varphi_0 + \epsilon\varphi_1,\, \pi^{ij} = {\pi_0}^{ij} + \epsilon\pi_1^{ij},\, \pi_\varphi = \pi_{\varphi 0} + \epsilon\pi_{\varphi 1},\, \gamma_{ij} = \gamma_{0ij} = a^2\delta_{ij}$.

- Zeroth order Hamiltonian: $\mathcal{H}_0^C = N_0\left(-\frac{1}{12}\,\kappa{\pi_a}^2a^{-1} + \frac{1}{2}\,{\pi_{\varphi 0}}^2a^{(-3)} + Va^3\right) = 0$, ${\pi_0}^{ij} = \frac{1}{6a}\pi_a\delta^{ij}$.
- Second order Hamiltonian:

$$\mathcal{H}_2^C = \delta_{ij}\partial_k{N_1}^j\,{\pi_1}^{ik}a^2 + \delta_{ij}\partial_k{N_1}^j\,{\pi_1}^{ik}a^2 - N_0\delta_{ij}\delta_{kl}\kappa{\pi_1}^{ij}{\pi_1}^{kl}a - 2\,N_1\delta_{ij}\delta_{kl}\kappa{\pi_0}^{ij}{\pi_1}^{kl}a + 2\,N_0\delta_{ij}\delta_{kl}\kappa{\pi_1}^{ik}{\pi_1}^{jl}a + 4\,N_1\delta_{ij}\delta_{kl}\kappa{\pi_0}^{ik}{\pi_1}^{jl}a + \frac{1}{2}\,N_0{\pi_{\varphi 1}}^2a^{(-3)} + N_1\pi_{\varphi 0}\pi_{\varphi 1}a^{(-3)} + {N_1}^i\pi_{\varphi 0}\partial_i\varphi_1 + \frac{1}{2}\,N_0\delta^{ij}\partial_i\varphi_1\,\partial_j\varphi_1\,a + N_1V_\varphi\,a^3\varphi_1 + \frac{1}{2}\,N_0V_{\varphi\varphi}\,{\varphi_1}^2a^3. \tag{31}$$

return page

- background phase space contains 10 variable (a, N, φ_0, λ_0, S_0 and corresponding momenta).
- There are two primary constrained equations:

$$\Phi_p^1 \equiv \pi_{S0} = 0, \;\; \Phi_p^2 \equiv \pi_{N0} + \lambda_0 \pi_{\lambda 0} N_0^{-1} = 0 \tag{32}$$

- Conservation of primary constraints gives rise to secondary constraints:

$$\Phi_s^1 \equiv \{\Phi_p^1, \mathcal{H}_{D0}\} \approx 0 \qquad \Rightarrow \lambda_0 + {N_0}^3 {\pi_{\lambda 0}}^2 a^3 \approx 0 \tag{33}$$

$$\begin{aligned} \Phi_s^2 \equiv \{\Phi_p^2, \mathcal{H}_{D0}\} \approx 0 \qquad & \Rightarrow -V_0 a^3 + \pi_{\lambda 0}\pi_{\varphi 0} N_0 + \frac{3}{4a^3}\kappa {\pi_{\lambda 0}}^2 {\lambda_0}^2 + \frac{1}{12\,a}\kappa \pi_a^2 \\ & + \frac{\kappa}{2\,a^2} N_0 \pi_a \pi_{\lambda 0} \lambda_0 = 0 \end{aligned} \tag{34}$$

- Further, conservation of secondary constraint (33) leads to tertiary constraint and we also get quartenary constraint:

$$\Phi_t \equiv \{\Phi_s^1, \mathcal{H}_{D0}\} \approx 0 \tag{35}$$

$$\Rightarrow \pi_{\varphi 0} - \kappa N_0^2 {\pi_{\lambda 0}}^2 a - 3\kappa N_0^2 {\pi_{\lambda 0}}^3 \lambda_0 = 0 \tag{36}$$

$$\begin{aligned} \Phi_q \equiv \{\Phi_t, \mathcal{H}_{D0}\} \approx 0 \Rightarrow & \, S_0 \left(-2\kappa N_0^2 \pi_{\lambda 0} a + 15\kappa N_0^5 {\pi_{\lambda 0}}^5 \pi_a a\right) - N_0 V_\varphi - \frac{5}{2}\kappa^2 N_0^3 {\pi_{\lambda 0}}^3 \lambda_0 a^{-2} \\ & + \frac{5}{6}\kappa^2 N_0^3 {\pi_{\lambda 0}}^2 \pi_a a^{-1} + 18\kappa^2 N_0^6 {\pi_{\lambda 0}}^6 \lambda_0 + 6\kappa^2 N_0^6 {\pi_{\lambda 0}}^6 \pi_a a = 0 \end{aligned} \tag{37}$$

Return page

Non-canonical scalar field

- Momentum corresponding to γ_{ij}:

$$\pi^{ij} \equiv \frac{\partial \mathcal{L}_{NC}}{\partial \gamma'_{ij}} = \frac{\sqrt{\gamma}}{2\kappa}\left(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl}\right)K_{kl} \quad (38)$$

$$K_{ij} = \frac{\kappa}{\sqrt{\gamma}}\left(\gamma_{ij}\gamma_{mn} - 2\gamma_{im}\gamma_{jn}\right)\pi^{mn} \quad (39)$$

- Hamiltonian density:

$$\mathcal{H}_{NC} = 2\gamma_{ij}\partial_k N^j \pi^{ik} + N^i \partial_i \gamma_{jk}\pi^{jk} - \frac{N\kappa}{\sqrt{\gamma}}\left(\gamma_{ij}\gamma_{kl} - 2\gamma_{ik}\gamma_{jl}\right)\pi^{ij}\pi^{kl} - \frac{N\sqrt{\gamma}}{2\kappa}\,{}^{(3)}R - N\sqrt{\gamma}\left(P - P_X\left(2X - Y\right)\right) + N^i \pi_\varphi \partial_i \varphi = 0 \quad (40)$$

- Hamiltonian constraint

$$\mathcal{H}_N \equiv \frac{\delta \mathcal{H}_{NC}}{\delta N} = -\frac{\kappa}{\sqrt{\gamma}}\left(\gamma_{ij}\gamma_{kl} - 2\gamma_{ik}\gamma_{jl}\right)\pi^{ij}\pi^{kl} - \frac{\sqrt{\gamma}}{2\kappa}\,{}^{(3)}R - \sqrt{\gamma}\,G(\pi_\varphi, \gamma, Y, \varphi) \quad (41)$$

- Momentum constraint

$$\mathcal{H}_i \equiv \frac{\delta \mathcal{H}_{NC}}{\delta N^i} = -2\partial\left(\gamma_{im}\pi^{mn}\right) + \pi^{kl}\partial_i \gamma_{kl} + \pi_\varphi \partial_i \varphi \quad (42)$$

return page