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Do gravitational waves carry energy-momentum and angular momentum?

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Abstract. We show that gravitational waves which possess a non-vanishing Riemann tensor $R_{iklm} \neq 0$ always carry energy-momentum and angular momentum. Our proof uses canonical superenergy and supermomentum tensors for the gravitational field.

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1 Introduction

In General Relativity (**GR**) the gravitational field Γ^i_{kl} does not possess any energy-momentum tensor. Instead, it only possesses the so-called "energy-momentum pseudotensors". In fact, this is a consequence of the Einstein Equivalence Principle (**EEP**)[1]. Because of that, many authors [1–5] put in doubt the reality of the energy-momentum and the angular momentum transfer by gravitational waves. As the main argument, some of these authors use the fact that for the majority of exact solutions of the vacuum Einstein field equations which represent gravitational waves, energy-momentum pseudotensors *globally vanish* in certain coordinate systems. In consequence, these pseudotensors give "no gravitational energy and no gravitational energy flux". Some other authors [1, 3] argue that the vanishing of the components ${}_gt^{0k}$ (or ${}_gt_0{}^k$) of the gravitational energy-momentum pseudotensor ${}_gt^{ik}$ (or ${}_gt_i{}^k$) may be treated as a coordinate condition coupled to the Einstein equations and yield (in special coordinates) "global vanishing of the pure gravitational energy and the pure gravitational energy flux".

However, *such conclusions are physically incorrect*. Firstly, these authors neglect an important role of the four-velocity $\vec{v}$ $(\vec{v} \cdot \vec{v} = 1)$ of an observer **O** in the definition of the energy density ϵ and the energy flux P^i of the field. Namely, the correct definitions of the energy density ϵ and its flux P^i for such an ob-

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[1] Following R. Penrose [6], many authors have considered the so-called *quasilocal energy-momentum* in **GR** [7–14]. However, recent investigations [15–18] have shown that these quantities are by no means better than the old energy-momentum complexes and pseudotensors.

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server are

$$\epsilon = T^{ik} v_i v_k = T_{ik} v^i v^k \,, \tag{1}$$

$$P^i = (\delta^i_k - v^i v_k)\, T^{kl} v_l \,, \tag{2}$$

in contrast to

$$\epsilon = T^{00} \,, \qquad P^\alpha = T^{0\alpha} \,. \tag{3}$$

From now on we use the notation in which $i, k, l, ... = 0, 1, 2, 3$; $\alpha, \beta, \gamma, \delta, ... = 1, 2, 3$, and T^{ik} are the components of the energy-momentum tensor. In fact T^{00} and $T^{0\alpha}$ give the energy density ϵ and the energy flux P^i for an observer **O** provided in the global coordinates applied, one has[2] $v^i = \frac{\delta^i_0}{\sqrt{g_{00}}}$, $v_k = \frac{g_{k0}}{\sqrt{g_{00}}}$. This is also true for the gravitational energy-momentum pseudotensor $_g t^{ik}$ (or $_g t_i{}^k$). In consequence, even if globally in a chosen system of coordinates $_g t^{0k} = 0$ (or $_g t_0{}^k = 0$), then, as one can see easily, *not for all observers* $\epsilon = P^i = 0$, it rather depends on the four-velocity $\vec{v}$ of the observer **O**. Moreover, a coordinate condition of the kind $_g t^{0k} = 0$ (or $_g t_0{}^k = 0$) *is not sufficient for the whole variety of the gravitational energy-momentum pseudotensors*. In fact, we need a different coordinate condition for every gravitational energy-momentum pseudotensor.

Secondly, and most importantly, these authors forget that gravitational energy-momentum (and gravitational angular momentum) pseudotensors, as functions of the Levi-Civita connection coefficients,[3] describe the energy-momentum of the *total gravitational field*, which is a combination of the real gravitational field (for which $R_{iklm} \neq 0$) and the inertial force field (for which $R_{iklm} = 0$). The inertial force field is generated by a suitably chosen system of coordinates. This is also a consequence of the **EEP**.

Thus, if in some coordinate systems $_g t^{ik}$ or $(_g t_i{}^k)$ (or only $_g t^{0k}$ or $_g t_0{}^k$) globally vanish, *this does not mean* that in these systems there is no pure gravitational energy and energy flux.[4] Simply, it just means that in such coordinate systems *the energy and the energy flux of the real gravitational field cancel* with the energy and the energy flux of the inertial force field.[5] The energy and the energy-momentum flux (as well as the angular momentum) of the real gravitational field which has $R_{iklm} \neq 0$ *always exist and do not vanish*. In order to show this, one can use *the canonical superenergy (and the canonical angular supermomentum) tensors*[6] for the gravitational field [19–23].

The canonical superenergy tensor $_g S_i{}^k$ and the canonical angular supermomentum tensor $_g S^{ikl} = (-)_g S^{kil}$ are constructed in such a way that they *extract covariant information* about the real gravitational field which is hidden in the canonical en-

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[2] This means that the observer **O** is at rest with respect to a chosen coordinate system.

[3] Physically, this connection plays the role of the total gravitational field strengths.

[4] Even if we confine ourselves to those observers which are at rest in a chosen global coordinate system and for whom $_g t^{00}$ (or $_g t_0{}^0$) is the "energy density" and $_g t^{0\alpha}$ (or $_g t_0{}^\alpha$) gives the "energy flux".

[5] Energy-momentum of the inertial force field gives contribution to energy-momentum pseudotensors.

[6] Or other superenergy and angular supermomentum tensors.

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ergy-momentum and the canonical angular momentum pseudotensors. These superenergy and angular supermomentum tensors are obtained from suitable gravitational pseudotensors *by some kind of averaging* and they are functions of the curvature tensor and its covariant derivatives only. Thus, they *truly describe only the real gravitational field*, which has $R^i{}_{klm} \neq 0$.

The paper is organized as follows. In Section II we shortly remind ourselves of canonical superenergy and supermomentum tensors for the gravitational field in **GR**. In Section III we briefly review other definitions of superenergy tensors, and in Section IV we apply the canonical superenergy and supermomentum tensors to the analysis of the gravitational waves. Finally, in Section V we give some concluding remarks.

2 Canonical superenergy tensor and canonical angular supermomentum tensor

We briefly remind here a general definition of the superenergy tensor $S_a{}^b(P)$ of the gravitational field and the matter field.

In the normal coordinates **NC(P)** (see e.g. [24–26]), we define [19–23]

$$S_{(a)}{}^{(b)}(P) := (-) \lim_{\Omega \to P} \frac{\int\limits_{\Omega} \left[T_{(a)}{}^{(b)}(y) - T_{(a)}{}^{(b)}(P) \right] \mathrm{d}\Omega}{1/2 \int\limits_{\Omega} \sigma(P; y)\, \mathrm{d}\Omega}, \tag{4}$$

where

$$T_{(a)}{}^{(b)}(y) := T_i{}^k(y)\, e^i_{(a)}(y)\, e^{(b)}_k(y), \tag{5}$$

$$T_{(a)}{}^{(b)}(P) := T_i{}^k(P)\, e^i_{(a)} e^{(b)}_k(P) = T_a{}^b(P) \tag{6}$$

are the so-called *physical or tetrad components* of the pseudotensor (or tensor) field $T_i{}^k(y)$ which describes an energy-momentum, $\{y^i\}$ are the normal coordinates, $e^i_{(a)}(y), e^{(b)}_k(y)$ are an orthonormal tetrad $e^i{}_{(a)}(P) = \delta^i_a$ and its dual $e^{(a)}_k(P) = \delta^a_k$ parallelly propagated along geodesics through P (P = an origin of the **NC(P)**), and

$$e^i_{(a)}(y)\, e^{(b)}_i(y) = \delta^b_a. \tag{7}$$

We take as Ω a sufficiently small ball (centered at **P**)

$$y^{0^2} + y^{1^2} + y^{2^2} + y^{3^2} \leq R^2, \tag{8}$$

which for an auxiliary positive-definite metric $h^{ik} := 2v^i v^k - g^{ik}$, can be given as

$$h_{ik} y^i y^k \leq R^2. \tag{9}$$

An observer **O** is at rest at the beginning **P** of the used normal coordinates **NC(P)** and its four-velocity is v^i.

Following Synge [27] we have introduced the two-point *world function* $\sigma(P; y)$

$$\sigma(P; y) \doteq \tfrac{1}{2} \left(y^{0^2} - y^{1^2} - y^{2^2} - y^{3^2} \right). \tag{10}$$

The symbol $\doteq$ means that an equation is valid only in special coordinates.

The *world function* $\sigma(P; y)$ can covariantly be defined by the eikonal-like equa-tion

$$g^{ik}\,\partial_i\sigma\,\partial_k\sigma = 2\sigma\,, \tag{11}$$

toge-ther with $\sigma(P, P) = 0,\ \partial_i\sigma(P, P) = 0$.

The ball Ω can be also given by the inequality

$$h^{ik}\,\partial_i\sigma\,\partial_k\sigma \le R^2\,. \tag{12}$$

Since the tetrad components and normal components are equal at **P**, we will write e the components of any quantity attached to **P** without tetrad brackets, e.g., we will write $S_a{}^b(P)$ instead of $S_{(a)}{}^{(b)}(P)$ and so on.

If $T_i{}^k$ are the components of a symmetric energy-momentum tensor of matter, then we get from (4)

$${}_mS_a{}^b(P) = \delta^{mn}\nabla_{(m}\nabla_{n)}\hat{T}_a{}^b\,. \tag{13}$$

Hat over a quantity denotes a value at **P**.

Using the four-velocity vector $\vec{v}$ of a fictitious observer **O** being at rest at **P** and the local metric $\hat{g}^{ab} \mathrel{\hat=} \eta^{ab}$, one can write (13) in a covariant way as

$${}_mS_a{}^b(P; v^l) = (2\hat{v}^l\hat{v}^m - \hat{g}^{lm})\,\nabla_{(l}\nabla_{m)}\hat{T}_a{}^b\,. \tag{14}$$

The last formula gives *the canonical superenergy tensor for matter*.

For the gravitational field $\Gamma^i_{kl} = \{{}^{\,i}_{kl}\}$, substitution of the canonical Einstein en--momentum pseudotensor for the gravitational field

$$\begin{aligned}{}_Et_i{}^k = \frac{c^4}{16\pi G}\Big\{&\delta^k_i g^{ms}(\Gamma^l_{mr}\Gamma^r_{sl} - \Gamma^r_{ms}\Gamma^l_{rl})\\ &+ g^{ms}{}_{,i}\Big[\Gamma^k_{ms} - \frac{1}{2}\,(\Gamma^k_{tp}g^{tp} - \Gamma^l_{tl}g^{kt})\,g_{ms}\\ &- \frac{1}{2}\,(\delta^k_s\Gamma^l_{ml} + \delta^k_m\Gamma^l_{sl})\Big]\Big\},\end{aligned} \tag{15}$$

as $T_i{}^k$ in (4) gives

$${}_gS_a{}^b(P; v^l) = (2\hat{v}^l\hat{v}^m - \hat{g}^{lm})\;{}_E\hat{W}_a{}^b{}_{lm}\,, \tag{16}$$

where

$$\begin{aligned}{}_EW_a{}^b{}_{lm} = \frac{2\alpha}{9}\Big[&B^b{}_{alm} + P^b{}_{alm} - \frac{1}{2}\,\delta^b_a R^{ijk}{}_m(R_{ijkl} + R_{ikjl})\\ &+ 2\delta^b_a\chi^2 E_{(l|g}E^g{}_{|m)} - 3\chi^2 E_{a(l|}E^b{}_{|m)} + 2\chi R^b{}_{(ag|(l)}E^g{}_{|m)}\Big],\end{aligned} \tag{17}$$

$$\alpha = \frac{c^4}{16\pi G} = \frac{1}{2\chi}\,, \tag{18}$$

and

$$E_i^k := T_i^k - \tfrac{1}{2}\,\delta_i^k T \tag{19}$$

is the modified energy-momentum of matter. On the other hand,

$$B^b{}_{alm} := 2R^{bik}{}_{(l|}R_{aik|m)} - \tfrac{1}{2}\,\delta^b_a R^{ijk}{}_l R_{ijkm} \tag{20}$$

are the components of the *Bel-Robinson* tensor (see eg. [30, 59] and references cited therein) while

$$P^b{}_{alm} := 2R^{bik}{}_{(l|}R_{aki|m)} - \tfrac{1}{2}\,\delta^b_a R^{ijk}{}_l R_{ikjm}\,. \tag{21}$$

We call ${}_gS_a{}^b(P; v^l)$ the *canonical superenergy tensor* for the gravitational field.[7] In vacuum ${}_gS_a{}^b(P; v^l)$ takes a simpler form

$${}_gS_a{}^b(P; v^l) = \frac{8\alpha}{9}\,(2\hat{v}^l\hat{v}^m - \hat{g}^{lm})\left[\hat{R}^{b(ik)}{}_{(l|}\hat{R}_{aik|m)} - \frac{1}{2}\,\delta^b_a\hat{R}^{i(kp)}{}_{(l|}\hat{R}_{ikp|m)}\right] \tag{22}$$

and the quadratic form ${}_gS_{ab}(P; v^l)\,v^a v^b$, where $v^a v_a = 1$, *is positive-definite.*

We define the canonical angular supermomentum tensor in analogy to the canonical superenergy tensor. Namely, we define an average

$$S^{(a)(b)(c)}(P) = S^{abc}(P) := (-)\lim_{\Omega\to P}\frac{\int\limits_{\Omega}\left[M^{(a)(b)(c)}(y) - M^{(a)(b)(c)}(P)\right]\mathrm{d}\Omega}{1/2\int\limits_{\Omega}\sigma(P;y)\,\mathrm{d}\Omega}\,, \tag{23}$$

where, as formerly,

$$M^{(a)(b)(c)}(y) := M^{ikl}(y)\,e^{(a)}_i(y)\,e^{(b)}_k(y)\,e^{(c)}_l(y)\,, \tag{24}$$

$$\begin{aligned} M^{(a)(b)(c)}(P) &:= M^{ikl}(P)\,e^{(a)}_i(P)\,e^{(b)}_k(P)\,e^{(c)}_l(P) \\ &= M^{ikl}(P)\,\delta^a_i\delta^b_k\delta^c_l = M^{abc}(P) \end{aligned} \tag{25}$$

are *physical* (or tetrad) components of the field $M^{ikl}(y) = (-)\,M^{kil}(y)$ describing the angular momentum.

As $M^{ikl}(y) = (-)\,M^{kil}(y)$ for matter, we take the *material part* of the *Bergmann-Thomson angular momentum complex* in **GR** [28, 29] (See also Appendix) ${}_{BT}M^{ikl} = (-)\,{}_{BT}M^{kil} : {}_{BT}M^{ikl}{}_{,l} = 0$.

This material parts reads as

$$M^{ikl}(y) = \sqrt{|g|}\,(y^iT^{kl} - y^kT^{il})\,, \tag{26}$$

where $T^{ik} = T^{ki}$ are the components of the symmetric energy-momentum tensor of matter[8] and $\{y^i\}$ denote normal coordinates.

The Eq. (26) gives us the *total angular momentum densities* for matter (orbital and spin) because the dynamical tensor $T^{ik} = T^{ki}$ is obtained from the canonical tensor by means of the Belinfante symmetrization procedure (and, therefore, includes the material spin).

For the gravitational field, as ${}_gM^{ikl}(y)$, we prefer the expression most closely related to the Einstein canonical energy-momentum complex (See Appendix). We

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[7] We must emphasize that the canonical superenergy tensor ${}_gS_a{}^b(P; v^L)$ is originally a function of the curvature components R_{iklm} and tensor Ricci components R_{ik}. We have eliminated all the Ricci components by using of the Einstein equations $R_{ik} = \chi E_{ik}$.

[8] This tensor is the source in the Einstein equations.

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will call this expression *canonical*, too. Namely, as ${}_gM^{ikl}(y)$ we take the formula given by Bergmann and Thomson [28, 29] (see also Appendix)

$$ {}_gM^{ikl}(y) = {}_F U^{i[kl]}(y) - {}_F U^{k[il]}(y) + \sqrt{|g|}\,(y^i_{BT} t^{kl} - y^k_{BT} t^{il}) , \tag{27} $$

where

$$ {}_F U^{i[kl]} := g^{im}\, {}_F U_m{}^{[kl]} , \tag{28} $$

and ${}_F U_m{}^{[kl]}$ are the *Freud superpotentials*, and

$$ {}_{BT}t^{kl} := g_E^{ki} t_i{}^l + \frac{g^{mk}{}_{,p}}{\sqrt{|g|}}\; {}_F U_m{}^{[lp]} \tag{29} $$

is the *Bergmann-Thomson gravitational energy-momentum pseudotensor* [28, 29].

The Eq. (27) is the *gravitational part* of the *Bergmann-Thomson angular momentum complex* ${}_{BT}M^{ikl} := M^{ikl} + {}_gM^{ikl}$ which was first introduced in [28].

One can interpret Eq. (27) as the sum of a *spinorial part*

$$ S^{ikl} := {}_F U^{i[kl]} - {}_F U^{k[il]} \tag{30} $$

and an *orbital part*

$$ O^{ikl} := \sqrt{|g|}\,(y^i_{BT} t^{kl} - y^k_{BT} t^{il}) \tag{31} $$

of the gravitational angular momentum densities.

Substituting (26) into (23) we get the *canonical angular supermomentum tensor* for *matter*

$$ {}_mS^{abc}(P; v^l) = 2[(2\hat{v}^a \hat{v}^p - \hat{g}^{ap})\, \nabla_p \hat{T}^{bc} - (2\hat{v}^b \hat{v}^p - \hat{g}^{bp})\, \nabla_p \hat{T}^{ac}] . \tag{32} $$

On the other hand, substitution of (27) into (23) gives the *gravitational canonical angular supermomentum tensor*[9]

$$ \begin{aligned} {}_gS^{abc}(P; v^l) = \alpha(2\hat{v}^p \hat{v}^t - \hat{g}^{pt})\, &[\chi(\hat{g}^{ac}\hat{g}^{br} - \hat{g}^{bc}\hat{g}^{ar})\, \nabla_{(t}\hat{E}_{pr)} \\ &+ 2\hat{g}^{ar}\nabla_{(t}\hat{R}^{(b}{}_p{}^{c)}{}_{r)} - 2\hat{g}^{br}\nabla_{(t}\hat{R}^{(a}{}_p{}^{c)}{}_{r)} \\ &+ 2/3\hat{g}^{bc}(\nabla_r \hat{R}^r{}_{(t}{}^a{}_{p)} - \chi\nabla_{(p}\hat{E}^a_{t)}) \\ &- 2/3\hat{g}^{ac}(\nabla_r \hat{R}^r{}_{(t}{}^b{}_{p)} - \chi\nabla_{(p}\hat{E}^b_{t)})] . \end{aligned} \tag{33} $$

In vacuum, $T_{ik} = 0 \equiv E_{ik} := T_{ik} - 1/2 g_{ik} T = 0$, and the gravitational canonical angular supermomentum tensor ${}_gS^{abc}(P; v^l) = (-)\,{}_gS^{bac}(P; v^l)$ simplifies to

$$ {}_gS^{abc}(P; v^l) = 2\alpha(2\hat{v}^p \hat{v}^t - \hat{g}^{pt})\, [\hat{g}^{ar}\nabla_{(p}\hat{R}^{(b}{}_t{}^{c)}{}_{r)} - \hat{g}^{br}\nabla_{(p}\hat{R}^{(a}{}_t{}^{c)}{}_{r)}] . \tag{34} $$

It is interesting that the orbital part $0^{ikl} = \sqrt{|g|}\,(y^i_{BT} t^{kl} - y^k_{BT} t^{il})$ of the ${}_gM^{ikl}$ gives *no contribution* to the tensor ${}_gS^{abc}(P; v^l)$. Only a spinorial part $S^{ikl} := {}_F U^{i[kl]} - {}_F U^{k[il]}$ *gives a non-zero contribution* to this tensor. Also, notice that the canonical angular supermomentum tensors ${}_gS^{abc}(P; v^l)$ and ${}_mS^{abc}(P; v^l)$, gravitation and matter, *need not any radius vector* to be defined.

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9 We have also eliminated in (33) the components of the Ricci tensor with the help of the stein equations.

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given. Of course, one can also make our averaging just along a worldline of a fiducial observer **O** (then, we use *Fermi normal coordinates*). In this case our averaging becomes similar to the averaging used by Mashhoon et al. However, there are important differences. Firstly, our construction does not use the parameter ε and uses a four-dimensional domain for averaging. In consequence, it is *covariant and universal*, and because we do not use parameter ε, it gives superenergy and angular supermomentum tensors instead of the energy-momentum and the angular momentum tensors. Secondly, our starting objects are *energy-momentum and angular momentum tensors and pseudotensors*; and not the Riemann curvature tensor. Thus, our formalism is closely related to the field theoretical formalism of the canonical energy-momentum and canonical angular momentum in general relativity.

We consider the construction that was made by Mashhoon et al. (Faraday's tensor, Maxwell tensor for the field R_{iklm}) a bit formal and too much alike the formalism of the electromagnetic field which is different from gravitational field. Also, it seems that an averaging similar to Pirani's averaging (and our or Mashhoon's generalization of this averaging) in a natural way brings us to the superenergy level. Thus, we think that a mathematical trick with the parameter ε used by Mashhoon et al. in order to get an energy-momentum tensor after averaging is somewhat artificial. For example, Mashhoon's "energy density" requires an arbitrary dimensional parameter L, and it has bad transformational properties. In consequence, we prefer our definitions of the superenergy and angular supermomentum tensors.

4 Canonical superenergy and canonical angular supermomentum tensors of gravitational waves

By a direct calculation one can easily check that the canonical superenergy tensor ${}_gS_i{}^k(P;v^l)$ and the canonical angular supermomentum tensor ${}_gS^{ikl}(P;v^m)$ give *positive-definite* superenergy density, a *non-vanishing* superenergy flux, and *non-vanishing* angular supermomentum flux for every known solution of the vacuum Einstein field equations which represents a gravitational wave with $R_{iklm} \neq 0$. For example, for the linearly polarized gravitational wave which propagates in positive direction of the z-axis [in holonomic coordinates (t,x,y,z)] [42]

$$\begin{aligned} ds^2 &= dt^2 - L^2(e^{2\beta}\,dx^2 + e^{-2\beta}\,dy^2) - dz^2 \\ &= du\,dv - L^2(e^{2\beta}\,dx^2 + e^{-2\beta}\,dy^2)\,, \end{aligned} \tag{35}$$

where

$$\begin{aligned} &L = L(u)\,, \qquad \beta = \beta(u)\,, \qquad u := t - z, \\ &L'' + (\beta')^2 = 0\,, \\ &\beta' := \frac{d\beta}{du}\,, \qquad L' := \frac{dL}{du}\,, \qquad L'' := \frac{d^2\beta}{du^2}\,, \end{aligned} \tag{36}$$

in the Lorentzian coframe

$$\vartheta^0 = dt\,, \qquad \vartheta^1 = Le^{\beta}\,dx\,, \qquad \vartheta^2 = Le^{-\beta}\,dy\,, \qquad \vartheta^3 = dz\,, \tag{37}$$

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we have [21] *positive-definite* ${}_gS_0{}^0$ component

$$ {}_gS_0{}^0 = \frac{2\alpha}{9}\,(7A^2 + 7B^2 + 6C^2) > 0\,, \tag{38}$$

where

$$ \begin{aligned} A &:= \frac{L''}{L} + 2\,\frac{L'\beta'}{L} + \beta'' + \beta'^2\,, \\ B &:= \frac{L''}{L} - 2\,\frac{L'\beta'}{L} - \beta'' + \beta'^2\,, \\ C &:= \left(\frac{L'}{L}\right)^2 - \beta'^2\,. \end{aligned} \tag{39}$$

In this coframe we also have

$$ {}_gS_3{}^0 = (-)\,{}_gS_0{}^3 = (-)\,\tfrac{20}{9}\,\alpha B^2 \neq 0\,. \tag{40}$$

From the above results we conclude that the *gravitational superenergy density* ${}^g\epsilon_s$ defined covariantly as

$$ {}^g\epsilon_s := {}_gS_i{}^k v^i v_k (\doteq {}_gS_0{}^0) \tag{41}$$

is a *positive-definite scalar*.

The gravitational Poynting's (super)vector P^i defined as

$$ P^i := (\delta^i_k - v^i v_k)\,{}_gS_l{}^k(P; v^a)\,v^l\,, \tag{42}$$

has the following components in the Lorentzian coframe

$$ P^0 \doteq P^1 \doteq P^2 \doteq 0\,, \qquad P^3 \doteq {}_gS_0{}^3 \doteq \tfrac{20}{9}\,\alpha B^2\,. \tag{43}$$

For the same gravitational wave (35)–(36) we also have in the *null coreper* [22]

$$ \begin{aligned} {}_gS^{120} &= (-)\,{}_gS^{210} = 4/3\alpha v^1 v^2 F'\,, \\ {}_gS^{122} &= (-)\,{}_gS^{212} = 4/3\alpha v^{1^2} F'\,, \\ {}_gS^{130} &= (-)\,{}_gS^{310} = (-)\,4/3\alpha v^1 v^3 F'\,, \\ {}_gS^{133} &= (-)\,{}_gS^{313} = (-)\,4/3\alpha v^{1^2} F'\,, \\ {}_gS^{010} &= (-)\,{}_gS^{100} = 4/3\alpha F'\,(v^{3^2} - v^{2^2})\,, \\ {}_gS^{012} &= (-)\,{}_gS^{102} = (-)\,4/3\alpha v^1 v^2 F'\,, \\ {}_gS^{013} &= (-)\,{}_gS^{103} = 4/3\alpha v^1 v^3 F'\,, \\ {}_gS^{020} &= (-)\,{}_gS^{200} = 4\alpha v^0 v^2 F'\,, \\ {}_gS^{022} &= (-)\,{}_gS^{202} = 2\alpha F'(2v^0 v^1 - 1)\,, \\ {}_gS^{030} &= (-)\,{}_gS^{300} = (-)\,4\alpha v^0 v^3 F'\,, \\ {}_gS^{033} &= {}_gS^{303} = 2\alpha F'(1 - 2v^0 v^1)\,. \end{aligned} \tag{44}$$

The other components of ${}_gS^{abc}(P; v^l)$ vanish in this coreper.

In the above formulas $F' := \frac{dF}{du}$ and $F = F(t - z) =: F(u)$ is an arbitrary function different from zero, v^i $(i = 0, 1, 2, 3)$ is the four-velocity of an observer **O** which is at rest at the beginning **P** of the **NC(P)** adapted to the holonomic coordinates (t, x, y, z), i.e., $v^i = \delta^i_0$ in these normal coordinates.

It is clear from the above results that the plane gravitational waves *possess and carry* the superenergy and the angular supermomentum. This is, of course, true in any admissible coordinates and was proved in our papers [19, 21–23]. The analogical result we have also *for any other* real gravitational wave. W

Also, it results from the above fact that every gravitational wave, which has $R_{iklm} \neq 0$, *must also carry* the gravitational energy-momentum and the gravitational angular momentum. If not, then there would be a *contradiction* between an "energy level" and a "superenergy level", because our canonical superenergy and angular supermomentum tensors originated as a kind of averaging of the suitable pseudotensors. Notice also in this context that, as it follows from the Definition (4), that the $\epsilon(\Omega) := {}_gS_{ik}(P; v^l)\, v^i v^k \int\limits_{\Omega} \sigma(P; y)\, d\Omega$ gives the approximate (relative with respect to **P**) energy contained in a sufficiently small domain Ω and $P^i(\Omega) := 1/2(\delta^i_k - v^i v_k)\, {}_gS_l{}^k(P; v^m)\, v^l \int\limits_{\Omega} \sigma(P; y)\, d\Omega$ gives Poynting's vector of this (relative with respect to **P**) energy contained in Ω. $\epsilon(\Omega)$ and $P^i(\Omega)$ *do not vanish* in any admissible coordinates for a gravitational wave which has $R_{iklm} \neq 0$. V

Thus, the non-zero superenergy density and its non-zero flux for a gravitational wave really *demand a non-zero gravitational energy and its non-zero flux* for such a wave. All this is in agreement with the results obtained years ago by Mashhoon et al., [34–36] by the application of the **GEM** stress-energy tensor for gravitoelectromagnetic field.

On the "energy-momentum level", when we only use pseudotensors, the very important fact that any gravitational wave which has $R_{iklm} \neq 0$ *always transfer the energy-momentum* is camouflaged in some coordinates by energy-momentum of the inertial forces field. The analogous considerations based on the Definition (23) are valid also for the angular momentum.

asymptotically

5 Concluding remarks

Global quantities only

If one wants to get correct information about an energy-momentum and an angular momentum of the real gravitational field by the application pseudotensors, then one has to use the pseudotensors in very special coordinates only (see e.g. [43–45]). Namely, one must use pseudotensors in coordinates in which Γ^i_{kl} describe only the real gravitational field. The examples of such coordinates are given, for example, by *global Bondi-Sachs coordinates* for a closed system [46] or, in general, by normal coordinates **NC(P)** [24–26, 47, 48].

In order to get information about the gravitational energy-momentum and the gravitational angular momentum in arbitrary admissible coordinates one must use covariant expressions which depend on the curvature tensor. Our canonical superenergy and angular supermomentum tensors are exactly the quantities of such a kind. In application to gravitational radiation these quasilocal quantities unambiguously show, that any gravitational waves which have $R_{iklm} \neq 0$ always transfer the energy-momentum and the angular momentum. Thus, the conclusions given in the references like [1, 3, 5] (see also [58]), where authors use pseudotensors only are incorrect.

II. AN INFORMATION ABOUT CONSIDERED GRAVITATIONAL WAVES (Ad. Supplement) (W1)

Linearly polarized, plane gravitational wave in the coordinates (U, V, X, Y)

$$ds^2 = 2\overbrace{(Y^2 - X^2)\frac{F(U)}{2}}^{H(U)}dU^2 + 2dUdV - dX^2 - dY^2 \qquad (8)$$

$F = F(U)$ is an arbitrary function.

Plane-fronted gravitational wave with parallel rays (p-p wave) is a generalization of the plane wave having the following line element in the coordinates (U, V, X, Y)

$$ds^2 = 2H(X, Y, U)\,dU^2 + 2dUdV - dX^2 - dY^2 \qquad (9)$$

where

$$\Delta H := \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)H = 0. \leftarrow \text{Einstein equation}$$

The vector tangent to the V-lines is null and covariantly constant.

The Einstein-Rosen (cylindrical) gravitational wave has the following line element in cylindrical coordinates $x^0 = ct$, $x^1 = \rho$, $x^2 = \varphi$, $x^3 = z$:

$$ds^2 = e^{2(\gamma - \psi)}(c^2dt^2 - d\rho^2) - \rho^2 e^{-2\psi}d\varphi^2 - e^{2\psi}dz^2. \qquad (10)$$

The metric functions $\gamma(x^0, x^1)$, $\psi(x^0, x^1)$ satisfy the following system of partial, differential equations (=Einstein equations)

$$\begin{aligned}
&\psi_{,11} + \frac{1}{\rho}\psi_{,1} - \psi_{,00} = 0, \\
&\gamma_{,1} = \rho\left[(\psi_{,1})^2 + (\psi_{,0})^2\right], \qquad (11)\\
&\gamma_{,0} = 2\rho\,\psi_{,0}\psi_{,1}.
\end{aligned}$$

III. The Results of Calculations of the Gravitational Superenergy Density

W2

$$ {}_g\epsilon_s := {}_gS_i^{\ k} u^i u_k \qquad (12) $$

and Components, P^i, of the Gravitational Poynting's Supervector Defined by

$$ P^i := (\delta^i_k - u^i u_k)\, {}_gS_l^{\ k}(P; u^n) u^l \qquad (13). $$

A. Plane wave and plane-fronted wave

We get in the so-called null coreper

$$ \vartheta^0 = H\,dU + dV,\ \vartheta^1 = dV,\ \vartheta^2 = dX,\ \vartheta^3 = dY: \qquad (14) $$

$$ {}_gS_1^{\ 0} = \frac{16\alpha}{9}\left[(H_{,xx})^2 + 2(H_{,xy})^2 + (H_{,yy})^2\right] > 0, \qquad (15) $$

and the other components of the canonical superenergy tensor ${}_gS_i^{\ k}(P; u^l)$ vanish.

From this we obtain that the canonical superenergy density of the plane (or plane-fronted) gravitational wave in the null coreper

$$ {}_g\epsilon_S = {}_gS_i^{\ k} u^i u_k \overset{*}{=} {}_gS_1^{\ 0}(u^1)^2 \overset{*}{=} {}_gS_1^{\ 0}(u_0)^2 > 0 \qquad (16) $$

is a positive-definite scalar.

The spatial Poynting's supervector $P^i = (\delta^i_k - u^i u_k) S_l^{\ k} u^l$ possesses the all set of its components different from zero in the case:

$$ \begin{aligned} P^0 &\overset{*}{=} (1 - u^0 u_0)\, {}_gS_1^{\ 0} u^1, \\ P^1 &\overset{*}{=} (-)\, {}_gS_1^{\ 0} (u^1)^2 u_0, \\ P^2 &\overset{*}{=} (-)\, {}_gS_1^{\ 0} u^1 u^2 u_0, \\ P^3 &\overset{*}{=} (-)\, {}_gS_1^{\ 0} u^1 u^3 u_0, \end{aligned} \qquad (17) $$

So, the plane and plane-fronted gravitational waves have positive-definite canonical superenergy densities and transfer superenergy flux. (6)

(W3)

Remark: If we calculate the components ${}_E t_i{}^k$ of the canonical energy-momentum pseudotensor for plane (or plane-fronted) wave, then we will obtain ${}_E t_i{}^k \stackrel{*}{=} 0$ (in the both, null or natural coreper). (globally).

So, these waves have null canonical "energy densities" and do not transfer any energy-momentum in these coordinates. (Following the canonical en-momentum complex, ${}_E K_i{}^k{}_{,l}$)

B. Cylindrical wave

We get in natural coreper $(dx^0, d\rho, d\varphi, dz)$:

$$
\begin{aligned}
{}_g S_0{}^0 = \frac{4\alpha}{g} e^{-4(\gamma-\psi)} \Big\{ & \left[\gamma_{,00} - \gamma_{,11} - \frac{1}{\rho}\psi_{,1}\right]^2 + \left[\psi_{,11} - \gamma_{,0}\psi_{,0} + 2\psi_{,1}^2 - \psi_{,0}^2 - \gamma_{,1}\psi_{,1}\right]^2 + \left[\gamma_{,0}\left(\psi_{,1} - \psi_{,0} - \frac{1}{2\rho}\right) - 2\psi_{,0}^2 - \psi_{,00}\right]^2 \\
& + 3\left[\psi_{,0}^2 + \psi_{,00} + \frac{1}{\rho}\gamma_{,1} - \gamma_{,0}\psi_{,0} - \gamma_{,1}\psi_{,1}\right]^2 \\
& + 3\left[\gamma_{,0}\psi_{,0} - \psi_{,0}^2 + \gamma_{,1}\psi_{,1} - 2\psi_{,1}^2 - \psi_{,11}\right]^2 \Big\} > 0, \quad (18)
\end{aligned}
$$

and

$$
{}_g S_0{}^1 \neq 0, \quad {}_g S_1{}^1 \neq 0, \quad {}_g S_2{}^2 \neq 0, \quad {}_g S_3{}^3 \neq 0. \quad (19)
$$

The rest of the components of the tensor ${}_g S_i{}^k(P; u^l)$ vanishes in this case.

${}_g\epsilon_s \stackrel{*}{=} {}_g S_0{}^0 > 0.$

The Einstein-Rosen cylindrical wave possesses a positive-definite canonical superenergy density, and transfer a superenergy flux $\longrightarrow P^1 \stackrel{*}{=} {}_g S_0{}^1 u^0 = e^{\psi-\gamma}\, {}_g S_0{}^1,$

$P^0 \stackrel{*}{=} 0, \; P^2 \stackrel{*}{=} 0, \; P^3 \stackrel{*}{=} 0.$

(452A)

V

$$\epsilon(\Omega_3) := (-)\, {}_gS_{ik}(P; v^l)\, v^i v^k \int_{\Omega_3} \sigma(P;y)\, d^3\Omega$$

$$P^i(\Omega_3) := (-)\, \left(\delta^i_k - v^i v_k\right) {}_gS_l{}^k(P; v^m)\, v^l \int_{\Omega_3} \sigma(P;y)\, d^3\Omega$$

$\epsilon(\Omega_3)$ and $P^i_i(\Omega_3)$ do not vanish in any admissible coordinates for a gravitational wave with $R_{iklm} \neq 0$.

Ω_3 is a sufficiently small ball centered at the point P (and observer O).

Other usefull expressions $\quad [\epsilon(\Omega_3)] = \text{Joul}$

$$\epsilon(P) := \oint_{S_2} {}_g\epsilon_s(P)\, d^2S \simeq {}_g\epsilon_s(P) \oint_{S_2} d^2S = 4\pi R^2\, {}_g\epsilon_s(P)$$

$$P^i(P) := \oint_{S_2} {}_gP^i(P)\, d^2S \simeq {}_gP^i(P) \oint_{S_2} d^2S = 4\pi R^2\, {}_gP^i(P).$$

$S_2 : \gamma_{\alpha\beta} X^\alpha X^\beta = R^2 \leftarrow$ a sufficiently small sphera in instantaneous local 3-space of the observer O. This sphere surrounds observer O.

$$d^2S = \sqrt{\gamma_{\alpha\beta}\, \sigma^\alpha \sigma^\beta}\,, \quad (\alpha, \beta = 1, 2, 3) \qquad [\epsilon(P)] = \frac{J}{m^3}$$

$$\sigma^\alpha = \gamma^{\alpha\beta} \sigma_\beta = \frac{1}{2} \gamma^{\alpha\beta} \epsilon_{\beta\gamma\delta}\, dx^\gamma \wedge dx^\delta$$

$$\epsilon_{123} = \sqrt{\gamma} \qquad \gamma = \det(\gamma_{\alpha\beta})$$

II. Using of the averaged relative energy-momentum tensors to analyze energy and momentum of the Friedman and more general homogeneous universes.

General definition of the averaged relative energy-momentum tensor [For details, see gr-qc/0510114 [18] Foundations of Physics, Nr. 3 (2007) and CQG, 22 (2005) 4051 [19]]:

In a normal coordinates **NC(P)** we define:

$$\langle T_a{}^b(P)\rangle := \lim_{\varepsilon\to 0}\frac{\int_\Omega [T_{(a)}{}^{(b)}(y) - T_{(a)}{}^{(b)}(P)]\,d^4\Omega}{\frac{\varepsilon^2}{2}\int_\Omega d^4\Omega}, \quad (1).$$

where

$$T_{(a)}{}^{(b)}(y) := T_i{}^k(y)\, e^i{}_{(a)}(y)\, e_k{}^{(b)}(y),$$

$$T_{(a)}{}^{(b)}(P) := T_i{}^k(P)\, e^i{}_{(a)}(P)\, e_k{}^{(b)}(P) = T_a{}^b(P)$$

are the tetrad (or physical) components of a tensor or a pseudotensor $\mathbf{T}_a{}^b$ which describes of an energy-momentum distribution, **y**, is the collection of the normal coordinates, **NC(P)**, at a given point **P**; $e^i{}_{(a)}(y)$, $e_k{}^{(b)}(y)$ denote an orthonormal tetrad field and its dual respectively.

$$e^{i}_{(a)}(P) = \delta^{i}_{a}, \quad e_{k}{}^{(a)}(P) = \delta^{a}_{k},$$

$$e^{i}_{(a)}(y)\, e_{i}{}^{(b)}(y) = \delta^{b}_{a}$$

and they are parallelly propagated along geodesics through P.

For a sufficiently small domain Ω which surrounds P we require

$$\int_{\Omega} y^{i}\, d^{4}\Omega = 0, \qquad \int_{\Omega} y^{i} y^{k}\, d^{4}\Omega = \delta^{ik} M,$$

where
$$M = \int_{\Omega} (y^{0})^{2} d^{4}\Omega = \int_{\Omega} (y^{1})^{2} d^{4}\Omega = \int_{\Omega} (y^{2})^{2} d^{4}\Omega$$
$$= \int_{\Omega} (y^{3})^{2} d^{4}\Omega$$

is a common value of the moments of inertia of the domain Ω with respect to the subspaces $y^{i} = 0$ $(i = 0, 1, 2, 3)$.

As Ω we take a small analytic ball defined by

$$(y^{0})^{2} + (y^{1})^{2} + (y^{2})^{2} + (y^{3})^{2} \leq \underset{\sim}{\varepsilon^{2} L^{2}} \; (= R^{2}).$$

Here $\varepsilon > 0$ means a [redacted] parameter, [redacted] and $L > 0$ is a fundamental length.

(AP3) 11

Trick with putting $R = \varepsilon L$ was discovered by B. Mashhoon. This trick leads to proper dimensionality of the averaged quantities.

At **P** (= beginning of the **NC(P)**) we have equality of the tetrad and normal components, e.g., $T_{(a)}{}^{(b)}(P) = T_a{}^b(P)$.

We will write the components of any quantity at **P** without tetrad brackets, e.g., $T_a{}^b(P)$ instead of $T_{(a)}{}^{(b)}(P)$ and so on.

For the matter energy-momentum tensor ${}_mT_a{}^b(y)$ the averaging formula **(1)** gives

$$\langle {}_mT_a{}^b(P; v^l)\rangle = \left(2\hat{v}^l\hat{v}^m - \hat{g}^{lm}\right)\nabla_{(l}\nabla_{m)}\hat{T}_a{}^b \times$$

$$\times \frac{L^2}{6} = {}_mS_a{}^b(P; v^l)\frac{L^2}{6}. \qquad \mathbf{(2)}.$$

v^l = 4-velocity components of an observer **O** which is at rest at **P** and g^{lm} is the local inverse metric. Hat "$\wedge$" over a quantity denotes its value at **P**. $L>0$ is a fundamental length. ${}_mS_a{}^b(P; v^l)$ means the components of the canonical superenergy tensor of matter.

For the gravitational field one gets (12) the following tensor (if one uses ${}_{E}t_a{}^b$ in (AP4) the averaging process)

$$\left\langle {}_{g}t_a{}^b(P;v^l)\right\rangle = \frac{\alpha}{27}\left(2\hat{v}^l\hat{v}^m - \hat{g}^{lm}\right)\Big[\hat{B}^b{}_{\cdot alm} + \hat{P}^b{}_{\cdot alm} - \frac{1}{2}\delta_a{}^b\hat{R}^{ijk}{}_{\cdot\cdot\cdot m}\left(\hat{R}_{ijkl} + \hat{R}_{ikjl}\right) + 2\beta^2\delta_a^b\hat{E}_{(l|g}\hat{E}^g{}_{\cdot|m)} - 3\beta^2\hat{E}_{a(l|}\hat{E}^b{}_{\cdot|m)} + 2\beta\hat{R}^b{}_{\cdot(ag)(l|}\hat{E}^g{}_{\cdot|m)}\Big]L^2 = {}_{g}S_a{}^b(P;v^l)\cdot\frac{L^2}{6}. \quad (3)$$

$\alpha = \frac{1}{16\pi} = \frac{1}{2\beta}$ $\quad (G = c = 1)$,

$$E_i{}^k := T_i{}^k - \frac{1}{2}\delta_i^k T_a{}^a.$$

↑ Modyfied energy-momentum tensor of matter.

$B^b{}_{\cdot alm}$ mean components of the Bel-Robinson tensor and $P^b{}_{\cdot alm}$ are the components of a tensor which is very closely related to the Bel-Robinson tensor ($P^b{}_{\cdot alm}$ has almost the same analytic form as $B^b{}_{\cdot alm}$ and the same symmetries).

The averaged energy-momentum tensors (P) (15)

$\langle {}_mT_a{}^b(P;v^l)\rangle$ and $\langle {}_gt_a{}^b(P;v^l)\rangle$ **AP5**

can be considered as the tensors of the averaged relative energy-momentum and they are very closely related to the superenergy tensors.

One can try to establish the fundamental length, $L>0$ with the help of the Loop quantum gravity (But this is not necessary, e.g., one can use GUT).

Let us define:

$${}_g\epsilon := \langle {}_gt_a{}^b(P;v^l)\rangle v^a v_b \leftarrow \text{averaged gravitational relative energy density}$$

$${}_m\epsilon := \langle {}_mT_a{}^b(P;v^l)\rangle v^a v_b \leftarrow \text{averaged matter relative energy density}$$

and the total averaged relative energy density

$$\epsilon := {}_g\epsilon + {}_m\epsilon. \qquad \boxed{[{}_g\epsilon] = [{}_m\epsilon] = [\epsilon] = \frac{J}{m^3}}$$

Here (v^a) mean the components of the 4-velocity of an observer O which is studying gravitational and matter fields.

In Friedman universes: $O \equiv$ fundamental observer $\Downarrow$ $v^a = \overset{\circ}{g}{}^a$

Then for Friedman universes we have:

(1) ${}_g\epsilon$ and ${}_m\epsilon$ are positive-definite $\Downarrow$ ϵ is also > 0 for the all Friedman universes.

Other my papers used during preparation of this Lecture:

APG

1. Acta Physica Polonica, B11 (1980) 255.
2. Reports on Mathematical Physics, 40 (1997) 485
3. CQG, 22 (2005) 4051.
4. Found. of Physics, 37 (2007) 341.