The Nonlinear Field Space Theory

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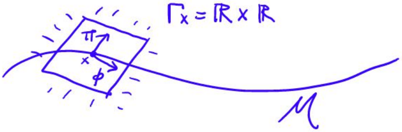
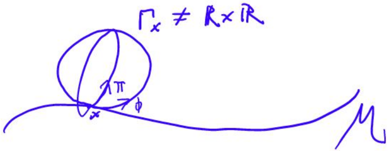
12 September, 2016, Szczecin

An idea

Standard Field Theory - Linear Field Space:



Nonlinear Field Space Theory[1]:



[1] J. M. & T. Trześniewski "The Nonlinear Field Space Theory", Physics Letters B **759** (2016) 424.

Relations/inpirations

- A compact field space is a natural way to implement the "Principle of finiteness" of physical theories, which once motivated the Born-Infeld theory (1938). Dynamical constraint on the field values.
- NFST is similar to the case of a relativistic particle, where the maximal speed of propagation is a result of the spacetime geometry, independently of the particular form of the Lagrange or Hamilton function.
- Lattice field theories $\rightarrow$ compact field spaces on discrete lattice.
- Non-linear sigma models (GellMann,1960; Witten,1984) - multi-component scalar field (but usually not field velocities or momenta) are constrained to lie on a Riemannian manifold.
- Relative Locality, curved particle momentum spaces.
- Loop Quantum Gravity, polymer quantization.

The scalar field

In the standard case the scalar field is a function $\phi : \mathcal{M} \to \mathcal{C}_\phi = \mathbb{R}$, where $\mathcal{M} = \Sigma \times \mathbb{R}$ is the spacetime manifold. In the canonical formulation the field ϕ is accompanied by the canonical momentum $\pi : \mathcal{M} \to \mathbb{R}$, obeying the Poisson bracket $\{\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)\} = \delta^{(3)}(\mathbf{x} - \mathbf{y})$. Then at every spacetime point the pair (ϕ, π) forms the phase space $\Gamma_{(\mathbf{x},t)} = T^*(\mathcal{C}_{\phi(\mathbf{x},t)}) = \mathbb{R} \times \mathbb{R}$ and the total phase space is given by $\prod_{(\mathbf{x},t)} \Gamma_{(\mathbf{x},t)}$.

The NFST may be constructed for field variables defined in the Fourier space rather than position space, which actually turns out to be more convenient. To this end we perform the Fourier transform of the field:

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}},$$

and similarly for the momentum $\pi(\mathbf{x}, t)$

The Fourier components are complex and, since the fields ϕ and π are real, satisfy the so-called reality conditions. With the use of a suitable canonical transformation they can be, however, redefined so that we will work only with real variables. This can be achieved in many different ways but the most convenient transformation is given by:

$$\tilde{\phi}_{\mathbf{k}} = \frac{e^{i\frac{\pi}{4}}\phi_{\mathbf{k}} + e^{-i\frac{\pi}{4}}\phi_{-\mathbf{k}}}{\sqrt{2}}, \quad \tilde{\pi}_{\mathbf{k}} = \frac{e^{i\frac{\pi}{4}}\pi_{\mathbf{k}} + e^{-i\frac{\pi}{4}}\pi_{-\mathbf{k}}}{\sqrt{2}},$$

where $\phi_{\mathbf{k}}, \pi_{\mathbf{k}} \in \mathbb{R}$ and $\{\phi_{\mathbf{k}}, \pi_{\mathbf{k}'}\} = \delta_{\mathbf{k},\mathbf{k}'}$.

Then, using the $\phi_{\mathbf{k}}$ and $\pi_{\mathbf{k}}$ variables, the standard Hamiltonian of a free massless scalar field:

$$H_\phi = \int_V d^3x \left(\frac{\pi^2}{2} + \frac{1}{2}\delta^{ab}\partial_a\phi\,\partial_b\phi \right),$$

can be Fourier-transformed into

$$H_\phi = \frac{1}{2}\sum_{\mathbf{k}} \left(\pi_{\mathbf{k}}^2 + k^2\phi_{\mathbf{k}}^2 \right),$$

where $k = \sqrt{\mathbf{k}\cdot\mathbf{k}}$.

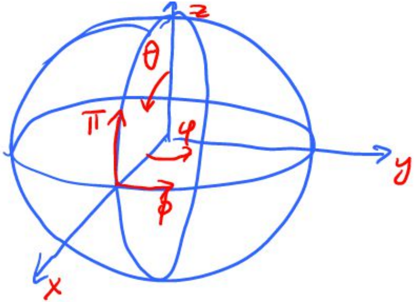
Spherical phase space

Parametrizing the spherical phase space $\Gamma_{\mathbf{k}} = S^2$ by the standard angular variables (φ, θ) we then obtain the symplectic form $\omega = J \sin\theta \, d\varphi \wedge d\theta$, where J is a free parameter.

In order to have the correct flat limit we choose

$$(-\pi, \pi] \ni \varphi = \frac{\phi_{\mathbf{k}}}{R}, \quad \text{and} \quad [0, \pi] \ni \theta = \frac{\pi}{2} - \frac{R\pi_{\mathbf{k}}}{J},$$

where R is a constant introduced for dimensional reasons.



With the above redefinition the ω form rewrites to:

$$\omega = \cos\left(\frac{\pi_{\mathbf{k}} R}{J}\right) d\pi_{\mathbf{k}} \wedge d\phi_{\mathbf{k}} \, .$$

Clearly, for canonical momenta such that $\pi_{\mathbf{k}} \ll \frac{J}{R}$ the $\phi_{\mathbf{k}}$ and $\pi_{\mathbf{k}}$ variables become Darboux coordinates with the standard symplectic form $\omega = d\pi_{\mathbf{k}} \wedge d\phi_{\mathbf{k}}$. Furthermore, if we have a symplectic form the Poisson tensor $\mathcal{P}^{ij}$ can be defined as $\mathcal{P}^{ij} = (\omega^{-1})^{ij}$, allowing us to calculate the Poisson bracket $\{f, g\} = \mathcal{P}^{ij}(\partial_i f)(\partial_j g)$. Hence the canonical Poisson bracket:

$$\{\phi_{\mathbf{k}}, \pi_{\mathbf{k}'}\} = \sec\left(\frac{\pi_{\mathbf{k}} R}{J}\right) \delta_{\mathbf{k},\mathbf{k}'} \, ,$$

which generalizes the standard one $\{\phi_{\mathbf{k}}, \pi_{\mathbf{k}'}\} = \delta_{\mathbf{k},\mathbf{k}'}$. The canonical bracket is, however, only locally well defined, because neither set of variables $(\phi_{\mathbf{k}}, \pi_{\mathbf{k}})$ nor (φ, θ) is globally given on S^2 – there is discontinuity at $\varphi = \pi$.

On the other hand, using $(\phi_{\mathbf{k}}, \pi_{\mathbf{k}})$ one can construct the well-known globally defined functions:

$$
\begin{aligned}
J_x &:= J\sin\theta\cos\varphi = J\cos\left(\frac{\pi_{\mathbf{k}}R}{J}\right)\cos\left(\frac{\phi_{\mathbf{k}}}{R}\right), \\
J_y &:= J\sin\theta\sin\varphi = J\cos\left(\frac{\pi_{\mathbf{k}}R}{J}\right)\sin\left(\frac{\phi_{\mathbf{k}}}{R}\right), \\
J_z &:= J\cos\theta = J\sin\left(\frac{\pi_{\mathbf{k}}R}{J}\right),
\end{aligned}
$$

which form the $\mathfrak{su}(2)$ Lie algebra $\{J_i, J_j\} = \epsilon_{ijk}J^k$.

On the corresponding Hilbert space $\mathcal{H}_J$ (with a given value of J) we write the $\mathfrak{su}(2)$ algebra as $[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}^k$. Then we have to take care of the issue of functional representations of states in $\mathcal{H}_J$, on which the operators $\hat{J}_i$ are acting.

Due to the non-product form of the considered phase space, the field configuration and momentum representations of a quantum state will be meaningful only locally. Therefore, in general we should instead define a quantum quasiprobability distribution (which is not necessarily a positive definite function) on the phase space, such as the Wigner function. With the use of a Wigner function $W(\varphi,\theta)$ the expectation value of an operator $\hat{A}$ can be given as the phase space average

$$\langle \hat{A} \rangle := \int_{S^2} d^2\Omega\, A(\varphi,\theta) W(\varphi,\theta).$$

The Wigner function for a pure state $|\Psi\rangle \in \mathcal{H}_J$ on the spherical phase space can be defined as

$$W(\varphi,\theta) := \mathrm{tr}(|\Psi\rangle\langle\Psi|\hat{w}(\varphi,\theta)),$$

where $\hat{w}(\varphi,\theta)$ denotes the Wigner operator.

The $\hat{J}_i$ operators can be expanded in powers of the operators $\hat{\phi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$. Such a procedure is valid for phase space quasiprobability distributions (such as the Wigner function) supported on sufficiently small values of $\phi_{\mathbf{k}}$ and $\pi_{\mathbf{k}}$ ($\phi_{\mathbf{k}} \ll R\frac{\pi}{2}$, $\pi_{\mathbf{k}} \ll \frac{J}{R}\frac{\pi}{2}$). We obtain

$$
\begin{aligned}
\hat{J}_x &= J\left(1 - \frac{1}{2R^2}\hat{\phi}_{\mathbf{k}}^2 - \frac{R^2}{2J^2}\hat{\pi}_{\mathbf{k}}^2 + \dots\right), \\
\hat{J}_y &= \frac{J}{R}\hat{\phi}_{\mathbf{k}} + \dots, \quad \hat{J}_z = R\hat{\pi}_{\mathbf{k}} + \dots,
\end{aligned}
$$

where dots denote higher powers of the $\hat{\phi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$ operators. In the leading order, the commutator $[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$ results in the following modified commutation relation:

$$
[\hat{\phi}_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}}] = i\hbar\left(\hat{\mathbb{I}} - \frac{1}{2R^2}\hat{\phi}_{\mathbf{k}}^2 - \frac{R^2}{2J^2}\hat{\pi}_{\mathbf{k}}^2 + \dots\right),
$$

where, due to the spectral theorem, for $O = \phi_{\mathbf{k}}, \pi_{\mathbf{k}}$ and $f(x) \in C^\infty$ the condition $\widehat{f(O)} = f(\hat{O})$ is satisfied.

One can, therefore, associate a nonlinear structure of the field phase space with a modification of the standard commutation relations. Furthermore, for the state in which $\langle \hat{\phi}_{\mathbf{k}} \rangle = 0 = \langle \hat{\pi}_{\mathbf{k}} \rangle$ the deformed commutation relation leads to the following generalized uncertainty principle:

$$\Delta\phi_{\mathbf{k}} \Delta\pi_{\mathbf{k}} \geq \frac{\hbar}{2} \left[1 - \frac{1}{2R^2} (\Delta\phi_{\mathbf{k}})^2 - \frac{R^2}{2J^2} (\Delta\pi_{\mathbf{k}})^2 \right].$$

Inspection of this inequality reveals that (neglecting higher order corrections) due to the spherical field phase space either the $\Delta\phi_{\mathbf{k}}$ or $\Delta\pi_{\mathbf{k}}$ uncertainty can be saturated to zero while the other uncertainty is kept constant. A similar effect was observed in (Bojowald and Kempf, 2012), where the periodic phase space of the form $\Gamma = \mathbb{R} \times S^1$ was studied.

Dynamics

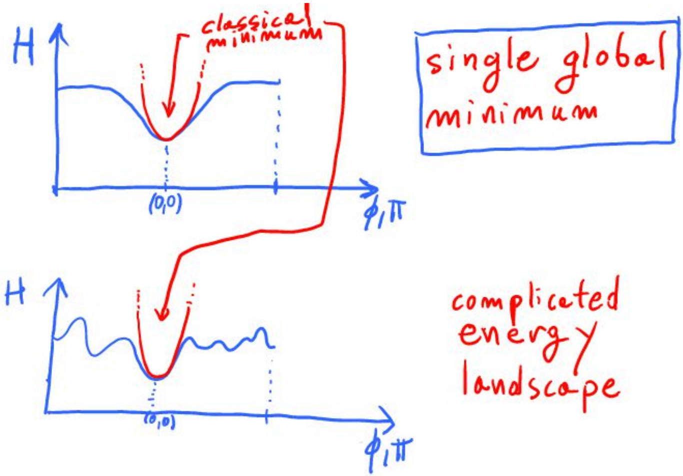
Having the kinematics defined we are now ready to introduce the (classical) dynamics of the considered NFST. To this end we have to find a Hamiltonian which is satisfying two requirements:

1. it is a globally defined function on the phase space,
2. it reduces to the classical Hamiltonian in the flat phase space limit (i.e. for $J \to \infty$).

In order to fulfill the condition (1) we can use globally defined variables J_i as the Hamilton function's building blocks.

Furthermore, the fact that in Nature one observes field excitations around $(\phi_{\mathbf{k}}, \pi_{\mathbf{k}}) = (0, 0)$ suggests that this point in the phase space should be the classical minimum of the Hamiltonian.

An ambiguity in reconstructing the global Hamiltonian:



In the global minimum $H_k \approx \frac{1}{2}\left(\pi_{\mathbf{k}}^2 + k^2\phi_{\mathbf{k}}^2\right)$.

Let us consider a spin (magnetic moment) immersed in the constant magnetic field **B**, which leads to a breakdown of the rotational invariance. Depending on the sign of the magnetic moment of a particle, the minimum energy state is associated with either parallel or anti-parallel alignment of the vectors **J** and **B**. Consequently, we have $H \propto \mathbf{J} \cdot \mathbf{B} = J_x B_x$. Analogously, we define the Hamiltonian for our model in the following way:

$$
\begin{aligned}
H_\phi &= \sum_{\mathbf{k}} H_{\mathbf{k}}\,, \quad \text{where} \\
H_{\mathbf{k}} &:= -Jk \cos\left(\frac{\pi_{\mathbf{k}}}{\sqrt{Jk}}\right) \cos\left(\sqrt{\frac{k}{J}}\phi_{\mathbf{k}}\right) \\
&= -Jk + \frac{1}{2}\left(\pi_{\mathbf{k}}^2 + k^2\phi_{\mathbf{k}}^2\right) - \frac{k}{4J}\phi_{\mathbf{k}}^2\pi_{\mathbf{k}}^2 \\
&- \frac{1}{24Jk}\left(\pi_{\mathbf{k}}^4 + k^4\phi_{\mathbf{k}}^4\right) + \mathcal{O}(J^{-2})\,,
\end{aligned}
$$

where the condition (2) is fixing $R = \sqrt{J/k}$.

The obtained Hamiltonian can be perturbatively diagonalized (at least up to the order J^{-1}) with the use of creation and annihilation operators. Due to the deformation of the canonical commutation relation, the expressions for the creation and annihilation operators $\hat{a}^{\dagger}_{\mathbf{k}}$, $\hat{a}_{\mathbf{k}}$ will differ from the usual ones. Furthermore, the $\hat{a}^{\dagger}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ fulfill the q-deformed version of their commutation relation: $\hat{a}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}} - q\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} = \hat{\mathbb{I}}$.

This allows us to express the field operators as follows:

$$\hat{\phi}_{\mathbf{k}} = \sqrt{\frac{\hbar}{2k}} \frac{\left(\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}}\right)}{\sqrt{1+\frac{\hbar}{2J}}}, \quad \hat{\pi}_{\mathbf{k}} = -i\sqrt{\frac{\hbar k}{2}} \frac{\left(\hat{a}_{\mathbf{k}} - \hat{a}^{\dagger}_{\mathbf{k}}\right)}{\sqrt{1+\frac{\hbar}{2J}}},$$

where the q-deformation factor:

$$q = \frac{1-\frac{\hbar}{2J}}{1+\frac{\hbar}{2J}} = 1 - \frac{\hbar}{J} + \mathcal{O}(J^{-2}).$$

The total Hilbert space of the system is $\mathcal{H} = \bigotimes_{\mathbf{k}} \mathcal{H}_{\mathbf{k}}$, where $\mathcal{H}_{\mathbf{k}} = \text{span}\,\{|0_{\mathbf{k}}\rangle, |1_{\mathbf{k}}\rangle, \dots, |n_{\text{max},\mathbf{k}}\rangle\}$. The actions of the $\hat{a}^{\dagger}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ operators on the $|n_{\mathbf{k}}\rangle$ basis states are found to have the form:

$$\hat{a}^{\dagger}_{\mathbf{k}}|n\rangle = \sqrt{\frac{1-q^{n+1}}{1-q}}|n+1\rangle\,, \quad \hat{a}_{\mathbf{k}}|n\rangle = \sqrt{\frac{1-q^{n}}{1-q}}|n-1\rangle\,,$$

giving the q-deformed expression for the occupation number operator $\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}|n_{\mathbf{k}}\rangle = \frac{1-q^n}{1-q}|n_{\mathbf{k}}\rangle$. Based on this, the Hamiltonian can be expanded as follows:

$$\begin{aligned} \hat{H}_{\mathbf{k}} &= -Jk\,\hat{\mathbb{I}} + \left(\frac{1}{2} - \frac{\hbar}{4J}\right) k\hbar\,\hat{\mathbb{I}} + k\hbar\left(1 - \frac{\hbar}{J}\right)\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} \\ &+ \frac{k\hbar}{24}\frac{\hbar}{J}\left(\hat{a}^{4}_{\mathbf{k}} + (\hat{a}^{\dagger}_{\mathbf{k}})^4 - 6(\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}})^2 - 6\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} - 6\hat{\mathbb{I}}\right) \\ &+ \mathcal{O}(J^{-2})\,. \end{aligned}$$

Propagator

Assuming statistical isotropy of the spatial field configurations, the two-point correlation function is given by

$$\langle 0|\hat{\phi}(\mathbf{x},t)\hat{\phi}(\mathbf{y},t')|0\rangle = \frac{1}{V}\sum_{\mathbf{k},n}|c_n|^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-i\Delta E_n(t-t')}$$
$$= \frac{1}{V}\sum_{\mathbf{k}}\int \frac{d\omega}{2\pi} D_{(\omega,\mathbf{k})} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-i\omega(t-t')},$$

where (for a given wave number) $\Delta E_n = E_n^{(1)} - E_0^{(1)}$ and, denoting $p^2 = -\omega^2 + k^2$, we calculate the propagator:

$$D_{(\omega,\mathbf{k})} = \frac{i\left(1-\frac{2}{J}\right)}{-\omega^2 + k^2\left(1-\frac{3}{J}\right) + i\epsilon} + \mathcal{O}(J^{-2})$$
$$= \frac{i}{-\omega^2+k^2} + \frac{i}{J}\frac{k^2+2\omega^2}{(-\omega^2+k^2)^2} + \mathcal{O}(J^{-2}).$$

Renormalized constants

From the propagator given as the single term one can deduce that the "renormalized" speed of light reads

$$c_{\text{ren}} = 1 - \frac{3}{2}\frac{\hbar}{J} + \mathcal{O}(J^{-2}).$$

Furthermore, the propagatorcan be used to predict the form of interaction potential between two point sources of the scalar field:

$$V(r) = 4\pi i \int \frac{d^3k}{(2\pi\hbar)^3} e^{i\mathbf{k}\cdot\mathbf{r}} D_{(0,\mathbf{k})} Q_0 = -\frac{Q_0}{r}\left(1 + \frac{\hbar}{J} + \mathcal{O}(J^{-2})\right),$$

where Q_0 is the charge of a field source. The difference with the standard case can be absorbed into "renormalized" charge

$$Q_{\text{ren}} = Q_0 \left(1 + \frac{\hbar}{J} + \mathcal{O}(J^{-2})\right).$$

Homogeneous, $\Gamma = S^2$ scalar field with the Hamiltonian $H_\phi \propto -J_x$.

A concrete example for Minkowski background space (appropriately shifted zero point energy):

$$\rho_* \geq \rho_\phi = \rho_* \left[1 - \cos\left(\frac{\pi_\phi}{\sqrt{\rho_*}}\right) \cos\left(\frac{m\phi}{\sqrt{\rho_*}}\right)\right]$$
$$= \frac{\pi_\phi^2}{2} + \frac{1}{2} m^2 \phi^2 + \mathcal{O}(1/\rho_*),$$

where ρ_* is a new energy density scale. The symplectic form:

$$\omega = \cos\left(\frac{\pi_\phi}{\sqrt{\rho_*}}\right) \pi_\phi \wedge \phi = \pi_\phi \wedge \phi + \mathcal{O}(1/\rho_*),$$

which leads to the following expression for the Poisson bracket:

$$\{\cdot, \cdot\} = \frac{1}{\cos\left(\frac{\pi_\phi}{\sqrt{\rho_*}}\right)} \left[\frac{\partial \cdot}{\partial \phi} \frac{\partial \cdot}{\partial \pi_\phi} - \frac{\partial \cdot}{\partial \pi_\phi} \frac{\partial \cdot}{\partial \phi}\right].$$

Introducing the scale factor dependence, we find:

$$\rho_* \geq \rho_\phi = \rho_* \left[1 - \cos\left(\frac{\pi_\phi}{q\sqrt{\rho_*}}\right)\cos\left(\frac{m\phi}{\sqrt{\rho_*}}\right)\right],$$

such that π_ϕ/q and ϕ are true scalars and $q \equiv a^3$. The symplectic form for the whole system (gravity+matter) can be now written as follows:

$$\omega = dp \wedge dq + \cos\left(\frac{\pi_\phi}{\sqrt{\rho_*}}\right)\pi_\phi \wedge \phi.$$

Applying the q and p variables the total Hamiltonian:

$$H_{tot} = Nq\left(-\frac{3}{4}\kappa p^2 + \rho_\phi\right).$$

With use of the primary constraint and the Hamilton equation $\dot{q} = \{q, H_{tot}\}$, for $N = 1$ gauge we recover the Friedmann equation:

$$\left(\frac{1}{3}\frac{\dot{q}}{q}\right)^2 \equiv H^2 = \frac{\kappa}{3}\rho_\phi = \frac{\kappa\rho_*}{3}\left[1 - \cos\left(\frac{\pi_\phi}{q\sqrt{\rho_*}}\right)\cos\left(\frac{m\phi}{\sqrt{\rho_*}}\right)\right].$$

The Ricci scalar for the model is equal:

$$\begin{aligned}\mathcal{R} &= 6(\dot{H}+2H^2) \\ &= 3\kappa\rho_* \left[\frac{4}{3} - \cos\left(\frac{m\phi}{\sqrt{\rho_*}}\right)\left(\frac{4}{3}\cos\left(\frac{\pi_\phi}{q\sqrt{\rho_*}}\right)\right.\right. \\ &+ \left.\left.\frac{\pi_\phi}{q\sqrt{\rho_*}}\sin\left(\frac{\pi_\phi}{q\sqrt{\rho_*}}\right)\right)\right],\end{aligned}$$

with $\frac{\pi_\phi}{q\sqrt{\rho_*}} \in \left[-\frac{\pi}{2q}, \frac{\pi}{2q}\right]$, together with $q \geq 1$, and $\frac{m\phi}{\sqrt{\rho_*}} \in (-\pi, \pi]$.

Value of the Ricci scalar is bounded from both sides:

$$-0.83\kappa\rho_* \leq \mathcal{R} \leq 8.83\kappa\rho_*.$$

One can conclude that the FRW singularity is avoided due to the bounded nature of the matter Hamiltonian function.

Summary

- NFTS - linear fields space is only an approximation.
- Compactness of the field space allows to implement "Principle of finiteness".
- Numerous interesting predictions, including: generalization of the uncertainty relations, algebra deformations, constraining of the maximal occupation number, shifting of the vacuum energy and renormalization of constants.
- For $J = \text{const}$, conformal invariance of Minkowski spacetime is preserved at the level of the field structure. In case of k-dependence of the J parameter, effects such as energy-dependence of the speed of propagation of field excitations are expected.
- Application in theoretical physics (e.g. quantum gravity) and condensed matter physics (e.g. continuous spin chains).
- Cosmological singularity avoidance due to bounded matter field functions.