Effects from canonical quantum gravity for slow-roll inflationary models

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Introduction

- Motivation
  - Quantum gravity lacks of experimental guidance.
  - Testable predictions.
  - A promising scenario: inflationary phase.
- Outline
  - Framework: geometrodynamical quantum gravity.
  - Perturbations of a homogeneous universe with a scalar matter field.
  - Construction of gauge invariant master variables.
  - Born-Oppenheimer approximation (expansion in Planck mass m_p).
    - Classical Hamilton-Jacobi equation.
    - Schrödinger equation.
    - Corrected Schrödinger equation.
  - Slow-roll approximation $\longrightarrow$ power spectra for perturbations.

Perturbation theory in General Relativity

- Idea: Background + perturbations $(g_{\mu\nu} + h_{\mu\nu})$.
- The action of General Relativity with a scalar matter field

$$\mathcal{S} = \int dt \int d^3x \left(\Pi^{ij} g_{ij,t} + \pi_\phi \phi_{,t} - H\right) ,$$

where the Hamiltonian is a linear combination of constraints,

$$H := N\,\mathcal{H}(\pi_\phi, \phi, \Pi^{ij}, g_{ij}) + N^k\,\mathcal{H}_k(\pi_\phi, \phi, \Pi^{ij}, g_{ij}) \,.$$

- First variation of the action $\delta S = 0 \rightarrow$ background equations of motion. In particular,

$$\mathcal{H} = 0, \qquad \mathcal{H}_k = 0.$$

Perturbation theory in General Relativity

- Second variation of the action $\delta^2 S \to$ action functional for the linear perturbations.

$$\frac{1}{2}\,\delta^2\mathcal{S} = \int dx^4 \Big\{ p^{ij} h_{ij,t} + \delta\pi_\phi\,\delta\phi_{,t} - \delta N\,\delta\mathcal{H} - \delta[N^i]\,\delta[\mathcal{H}_i] - \frac{N}{2}\,\delta^2[\mathcal{H}] - \frac{N^i}{2}\,\delta^2[\mathcal{H}_i] \Big\}.$$

- Constraint equations for the perturbations

$$\delta\mathcal{H} = 0, \qquad \delta\mathcal{H}_i = 0.$$

Quantization

- A complete action for all degrees of freedom:

$$S_{total} := S + \frac{1}{2}\delta^2 S$$

$$H_{\rm total} := N\left(\mathcal{H} + \frac{1}{2}\,\delta^2\mathcal{H}\right) + N^i\left(\mathcal{H}_i + \frac{1}{2}\,\delta^2\mathcal{H}_i\right) + \delta N\,\delta\mathcal{H} + \delta N^i\,\delta\mathcal{H}_i\,.$$

- The 8 constraints of the system:

$$\left(\mathcal{H} + \frac{1}{2}\,\delta^2\mathcal{H}\right) = 0, \qquad \left(\mathcal{H}_i + \frac{1}{2}\,\delta^2\mathcal{H}_i\right) = 0,$$

$$\delta\mathcal{H} = 0, \qquad \delta\mathcal{H}_i = 0.$$

- Different possibilities:
  - Quantization à la Dirac. (Halliwell & Hawking, 1985)
  - Fix a gauge.
  - Construction of gauge-invariant master variables.

Construction of gauge-invariant master variables

- Assume vacuum to explain the general procedure.
- Notation: the six components $(h_{ij}, p^{ij}) \to (h_I, p^J)$, with $I, J = 1, \ldots, 6$, and $\{h_I(x), p^J(y)\} = \delta_I{}^J \delta^3(x-y)$.
- The four linearized constraints are the generators: $\delta\mathcal{H}(h_I, p^J)$, and $\delta\mathcal{H}_k(h_I, p^J)$.
- Canonical transformation to a new set of variables ($\tilde{h}_I$, $\tilde{p}^J$), so that $\delta\mathcal{H} = \tilde{p}^4$ and $\delta\mathcal{H}_K = \tilde{p}^K$, for $K = 1, 2, 3$.
  - $(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4)$: gauge-dependent.
  - $(\tilde{p}^1, \tilde{p}^2, \tilde{p}^3, \tilde{p}^4)$: gauge-invariant, but constrained to vanish.
  - $(\tilde{h}_5, \tilde{p}^5)$ and $(\tilde{h}_6, \tilde{p}^6)$: master gauge-invariant variables.
- Examples
  - Regge-Wheeler and Zerilli functions.
  - Mukhanov-Sasaki variable.

Homogeneous and isotropic background

- Consider Friedmann-Robertson-Walker metric with conformal time η:

$$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2)$$

- In this homogeneous case $N^i = 0 \Rightarrow$ the only constraint left will be $\left(\mathcal{H} + \frac{1}{2}\,\delta^2\mathcal{H}\right)$.
- The perturbation tensor $h_{\mu\nu}$ is decomposed in scalar, vector, and tensor components.
- 3 master gauge-invariant variables:
  - Scalar sector: Mukhanov-Sasaki v.
  - Tensorial sector: The two polarizations $v^{(\lambda)} = ah^{(\lambda)}$, with $\lambda = +, \times$.
- The constraint: $\left(\mathcal{H} + \frac{1}{2}\,\delta^2\mathcal{H}\right) = \mathcal{H}_0 + \mathcal{H}_S + \mathcal{H}_T,$
  $\mathcal{H}_0$ the Hamiltonian of the background minisuperspace model; and $\mathcal{H}_S$ and $\mathcal{H}_T$ Hamiltonians of parametric harmonic oscillators.

Homogeneous and isotropic background

- The background Hamiltonian

$$^{0}\mathcal{H} = -\frac{2}{m_p^2}\,\pi_a^2 + \frac{1}{2a^2}\,\pi_\phi^2 + m_p^2 V(\tilde{\phi})\,,$$

  with a Planck mass $m_P := 3\pi/(2G)$, and $\tilde{\phi} := \phi/m_P$.

- Quantization via the variable $\alpha := \log(a/a_0)$. Background Wheeler-DeWitt equation:

$$\frac{1}{2}\left[\frac{e^{-2\alpha}}{m_p^2}\,\frac{\partial^2}{\partial\alpha^2} - e^{-2\alpha}\,\frac{\partial^2}{\partial\tilde{\phi}^2} + m_p^2 V(\tilde{\phi})\right]\Psi_0(\alpha,\tilde{\phi}) = 0\,.$$

Homogeneous and isotropic background

- The scalar sector

$$\int d^3x\,\mathcal{H}_S = \frac{1}{2}\int d^3x\,\left(\pi^2 + \delta^{ij}\partial_i v \partial_j v - \frac{z''}{z}\right),$$

where π is the conjugate momentum of v and $z := a^2\phi'/a'$.

- Performing a Fourier transformation:

$$\int d^3x\,\mathcal{H}_S = \frac{1}{2}\int d^3\mathbf{k}\,\left(\pi_\mathbf{k}\overline{\pi}_\mathbf{k} + \omega_S^2(\eta) v_\mathbf{k}\overline{v}_\mathbf{k}\right),$$

with $\omega_S^2(\eta) = k^2 - \frac{z''}{z}$.

- Tensor modes same Hamiltonian with $\omega_T^2(\eta) = k^2 - \frac{a''}{a}$
- Standard quantization procedure: $[\hat{v}_\mathbf{k}, \hat{\pi}_\mathbf{p}] = i\,\delta\,(\mathbf{k}-\mathbf{p})\,.$
  $\hat{v}_\mathbf{k}\Psi = v_\mathbf{k}\Psi\,, \quad \hat{\pi}_\mathbf{k}\Psi = -\,i\,\frac{\partial\Psi}{\partial v_\mathbf{k}}.$

Master Wheeler-DeWitt equation

- Product ansatz: $\Psi = \Psi_0(\alpha,\phi)\prod_{\mathbf{k}} \Psi^S_{\mathbf{k}}(\alpha,\phi,v_{\mathbf{k}})\Psi^T_{\mathbf{k}}(\alpha,\phi,v_{\mathbf{k}})$
- Equation for each $\mathbf{k}$, S and T

$$\left\{e^{-2\alpha}\left[\frac{1}{m_p^2}\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\phi^2}\right]+m_p^2\,V(\phi)+\left[-\frac{\partial^2}{\partial v_{\mathbf{k}}^2}+\omega^2_{S,T}(\eta)\,v_{\mathbf{k}}^2\right]\right\}\Psi^{S,T}_{\mathbf{k}}(\alpha,\phi,v_{\mathbf{k}})=0\,,$$

with

$$\omega_S^2(\eta) = k^2-\frac{z''}{z}, \qquad z := a^2\phi'/a',$$

$$\omega_T^2(\eta) = k^2-\frac{a''}{a}.$$

Semiclassical approximation

- Ansatz: $\Psi_{\mathbf{k}}(\alpha, \phi, v_{\mathbf{k}}) = e^{i\, S(\alpha,\phi,v_{\mathbf{k}})}$
- Power expansion: $S = m_p^2 S_0 + S_1 + m_p^{-2} S_2 + \ldots$
- $\mathcal{O}(m_p^4)$: Independence of the perturbations: $\frac{\partial}{\partial v_{\mathbf{k}}} S_0(\alpha, \phi, v_{\mathbf{k}}) = 0\,.$
- $\mathcal{O}(m_p^2)$: Hamilton-Jacobi eq. $\longleftrightarrow$ classical dynamics.
- $\mathcal{O}(m_p^0)$: Schrödinger eq. $\longleftrightarrow$ QFT on curved spacetimes

$$i\,\frac{\partial}{\partial \eta}\,\psi_{\mathbf{k}}^{(0)} = \mathcal{H}_{\mathbf{k}}\psi_{\mathbf{k}}^{(0)},$$

with $\mathcal{H}_{\mathbf{k}} := -\frac{1}{2}\frac{\partial^2}{\partial v^2} + \frac{1}{2}\omega_{\mathbf{k}}^2 v_{\mathbf{k}}^2$

Semiclassical approximation

- $\mathcal{O}(m_p^{-2})$: Corrected Schrödinger eq. $\longleftrightarrow$ quantum-gravitational corrections.

$$i\,\frac{\partial}{\partial\eta}\,\psi_{\mathbf{k}}^{(1)} = \mathcal{H}_{\mathbf{k}}\psi_{\mathbf{k}}^{(1)} - \frac{\psi_{\mathbf{k}}^{(1)}}{2\,m_{\mathrm{P}}^2\,\psi_{\mathbf{k}}^{(0)}}\left[\frac{\left(\mathcal{H}_{\mathbf{k}}\right)^2}{V}\,\psi_{\mathbf{k}}^{(0)} + i\,\frac{\partial}{\partial\eta}\left(\frac{\mathcal{H}_{\mathbf{k}}}{V}\right)\psi_{\mathbf{k}}^{(0)}\right].$$

- Issue with imaginary terms.
  - Non-unitarity. Ambiguous framework.
  - Inconsistencies in the equations.
  - Exponentially amplified oscillations (towards the past).
  - Not allow for a Bunch-Davies-like initial data.

Gaussian ansatz

- Gaussian ansatz for both $\psi_{\mathbf{k}}^{(0)}$ and $\psi_{\mathbf{k}}^{(1)}$:

$$\psi_{\mathbf{k}}(\eta, v_{\mathbf{k}}) = N(\eta)\, e^{-\frac{1}{2}\,\Omega(\eta)\, v^2}.$$

- Power spectra

$$\mathcal{P}^{(0)}(k) \propto \frac{1}{\Re\mathfrak{e}(\Omega_{(0)})}.$$

- The corrected power spectra:

$$\mathcal{P}^{(1)}(k) = \mathcal{P}^{(0)}(k)\Big\{1 + \Delta\Big\},$$

with

$$\Delta := \lim_{\eta\to 0} \frac{\Re\mathfrak{e}\,\Omega_{(0)} - \Re\mathfrak{e}\,\Omega_{(1)}}{\Re\mathfrak{e}\,\Omega_{(0)}}.$$

Gaussian ansatz

- Equation for $\Omega_{(i)}$, with $i = 0, 1$:

$$i\,\Omega'_{(i)} = \Omega^2_{(i)} - \omega^2_{(i)}\,,$$

where

$$\begin{aligned}\omega^2_{(0)} &:= \omega^2,\\ \omega^2_{(1)} &:= \omega^2 - \frac{1}{2m_{\rm P}^2 V}\,\Re\mathfrak{e}\left\{\left[3\Omega_{(0)} - i\,(\ln V)'\right]\left[\omega^2 - \Omega^2_{(0)}\right] + 2\,i\,\omega\,\omega'\right\}.\end{aligned}$$

- Transformation

$$\Omega_{(i)} = -i\,\frac{y'_{(i)}(\eta)}{y_{(i)}(\eta)} \quad\Longrightarrow\quad y''_{(i)} + \omega^2_{(i)}\,y_{(i)} = 0\,.$$

- For $\Omega_{(1)}$ consider the linearization: $\widetilde{\Omega}_{(1)} := \Omega_{(1)} - \Omega_{(0)}$.

Initial data ($\eta \to -\infty$)

- Bunch-Davies vacuum: each mode oscillating with a constant frequency k

$$y_{(0)} = e^{ik\eta} \Longrightarrow \mathfrak{Re}(\Omega_{(0)}) = k\,,\ \mathfrak{Im}(\Omega_{(0)}) = 0.$$

- Natural initial data for the corrected case:

$$y_{(1)} = e^{i\beta\eta} \Longrightarrow \mathfrak{Re}(\Omega_{(1)})^2 = \mathfrak{Re}(\omega^2_{(1)})\,,\ \mathfrak{Im}(\Omega_{(1)}) = 0.$$

Slow-roll approximation

- The slow-roll parameters:

$$\epsilon := -\frac{\dot{H}}{H^2}\,,$$

$$\delta := \epsilon - \frac{\dot{\epsilon}}{2H\epsilon}\,,$$

and their combination $\gamma := 2\epsilon - \delta$.

- The frequencies:

$$\omega_S^2 = k^2 - \frac{2+3\gamma}{\eta^2} + \mathcal{O}(2)\,,$$

$$\omega_T^2 = k^2 - \frac{2+3\epsilon}{\eta^2} + \mathcal{O}(2)\,.$$

- $\gamma = \epsilon$ converts the scalar equation into the tensorial one.

Slow-roll approximation: uncorrected power spectra

- The equation for $\Omega_{(0)}$ is analytically solved with Bunch-Davies initial data

$$\Omega_{(0)} = k(\Omega_{dS}^{(0)}(\xi) + \gamma \Omega_{\gamma}^{(0)}(\xi)),$$

  with $\xi := -k\eta$.

- The usual (uncorrected) power spectra are obtained:

$$\mathcal{P}_{\mathrm{S}}^{(0)}(k) = \frac{G\, H_k^2}{\pi\, \epsilon}\left[1 - 2\epsilon + \gamma(4 - 2\gamma_{\mathsf{E}} - 2\ln(2))\right].$$

Slow-roll approximation: corrected power spectra

- The corrected scalar frequencies:

$$\omega_{(1)}^2 = k^2 - \frac{2+3\gamma}{\eta^2} + \left(\frac{H_k^2}{m_p^2\, k}\right)\left(\omega_{dS}^2 + \epsilon\,\omega_\epsilon^2 + \gamma\,\omega_\gamma^2\right) + \mathcal{O}(2).$$

- ω_{dS}, ω_ϵ, and ω_γ only depend on time $\xi := -k\eta$.
- We construct a similar decomposition for $\widetilde{\Omega}_{(1)}$:

$$\widetilde{\Omega}_{(1)} := \left(\frac{H_k}{m_p\, k}\right)^2 \left(\Omega_{dS}^{(1)} + \epsilon\,\Omega_\epsilon^{(1)} + \gamma\,\Omega_\gamma^{(1)}\right),$$

- Evolution equations for $\Omega_{dS}^{(1)}$, $\Omega_\epsilon^{(1)}$, and $\Omega_\gamma^{(1)}$ only depend on ξ.
- Initial data ($\xi \to \infty$): $\Omega_{dS}^{(1)} = \frac{1}{4}$, $\Omega_\gamma^{(1)} = \frac{3}{8}$ and $\Omega_\epsilon^{(1)} = -\frac{23}{12} + \frac{1}{2}\ln(\xi)$.

Slow-roll approximation: corrected power spectra

- The dependence on different parameters of the correction to the power spectra is analytically obtained:

$$\Delta_S = \frac{H_k^2}{k^3 m_p^2}\Big[\alpha_{dS} + \epsilon\,\alpha_\epsilon + \gamma\,\alpha_\gamma\Big].$$

- The definition of different numerical factors:

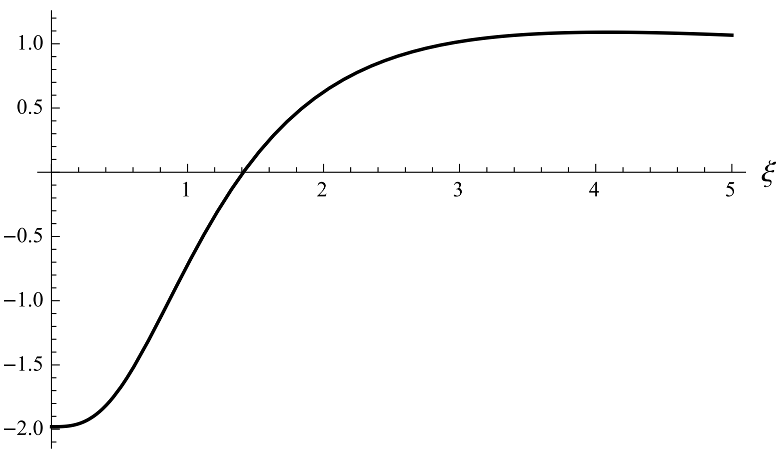
$$\begin{aligned}
\alpha_{dS} &= -\lim_{\xi\to 0}\left(\frac{\Re\mathfrak{e}\Omega_{dS}^{(1)}}{\Re\mathfrak{e}\Omega_{dS}^{(0)}}\right),\\
\alpha_{\epsilon} &= -\lim_{\xi\to 0}\left(\frac{\Re\mathfrak{e}\Omega_{\epsilon}^{(1)}}{\Re\mathfrak{e}\Omega_{dS}^{(0)}}\right),\\
\alpha_{\gamma} &= \lim_{\xi\to 0}\left(\frac{\Re\mathfrak{e}\Omega_{\gamma}^{(0)}\Re\mathfrak{e}\Omega_{dS}^{(1)} - \Re\mathfrak{e}\Omega_{dS}^{(0)}\Re\mathfrak{e}\Omega_{\gamma}^{(1)}}{\left(\Re\mathfrak{e}\Omega_{dS}^{(0)}\right)^2}\right).
\end{aligned}$$

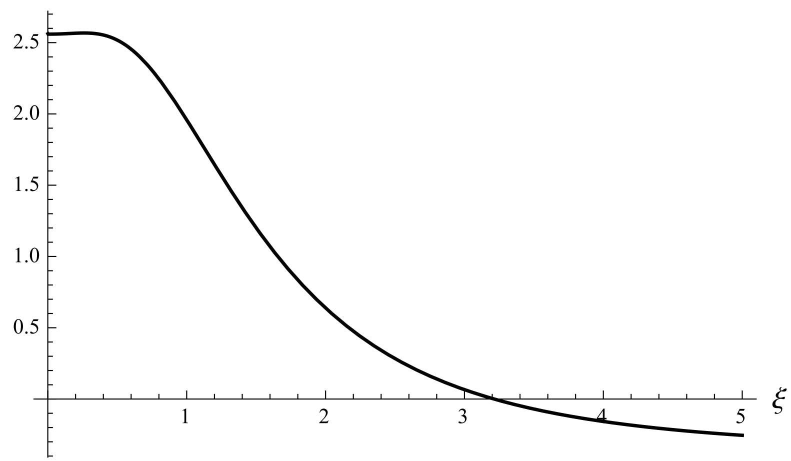
Slow-roll approximation: corrected power spectra

- $\Omega_{dS}^{(1)}$ can be analytically obtained

$$\alpha_{dS} = \frac{1}{4e^2}\left[3e^4\text{Ei}(-2) + 9\text{Ei}(2) - e^2\right] \approx 0.988.$$

- The evolution of the expressions that define α_ϵ (left) and α_γ (right) as a function of ξ:





- Numerical values: $\alpha_\epsilon \approx -1.98$, and $\alpha_\gamma \approx 2.56$.

Conclusions

- Perturbation theory on a FRW background with an inflaton field.
- Construction of master gauge invariant variables.
- Quantum geometrodynamical framework $\rightarrow$ Wheeler-DeWitt eq.
- Born-Oppenheimer type of approximation with m_p as expansion parameter.
- Corrected Schrödinger equation.
- Corrections to the power spectra for both tensor and scalar modes in the slow-roll regime.
- Enhancement of the power spectra (more relevant for large scales).

Evolution of $\Re\mathfrak{e}\Omega_\epsilon^{(1)}$ and $\Re\mathfrak{e}\Omega_\gamma^{(1)}$

The evolution of $\Re\mathfrak{e}\Omega_\epsilon^{(1)}$ and $\Re\mathfrak{e}\Omega_\gamma^{(1)}$ (continuous black lines) with their asymptotic values (red dashed lines):