Synthetic Approach to the Singularity Problem

Michael Heller, Jerzy Król
Copernicus Center for Interdisciplinary Studies
Cracow, Poland
Institute of Physics, University of Silesia

Full text see:
arXiv:1607.08264[gr-qc]

It is obvious that general relativity in its standard formulation is done in the `category of sets and functions between sets` (SET category), just as all macroscopic physical theories (usually without explicitly specifying this assumption), and most often in its subcategory of smooth manifolds and smooth maps between them.

However, when we approach a level of “very small” quantities – as, for example, in our search for quantum gravity and in the singularity problem – the situation can drastically change.

`Topos formulation of quantum mechanics` can serve as a motivation for applying similar approach to some unsolved problems of general relativity.

INTRODUCTION

It is obvious that general relativity in its standard formulation is done in the `category of sets and functions between sets` (SET category), just as all macroscopic physical theories (usually without explicitly specifying this assumption), and most often in its subcategory of smooth manifolds and smooth maps between them.

However, when we approach a level of "very small" quantities – as, for example, in our search for quantum gravity and in the singularity problem – the situation can drastically change.

`Topos formulation of quantum mechanics` can serve as a motivation for applying similar approach to some unsolved problems of general relativity.

Category theorists have elaborated a categorical version of differential geometry, the so-called **Synthetic Differential Geometry** (SDG) which almost exactly parallels the usual differential geometry employed in relativistic calculations. The *essential difference* consists in the fact that **in SDG infinitesimals appear which substantially enrich the usual real line.** Owing to this fact geometry acquires a tool to penetrate infinitesimally small portions of a given manifold (a manifold's "germs", to be defined below) which in the usual approach are invisible (in SET they simply do not exist). This creates an invaluable opportunity for physical applications.

Category theorists have elaborated a categorical version of differential geometry, the so-called **Synthetic Differential Geometry** (SDG) which almost exactly parallels the usual differential geometry employed in relativistic calculations. The *essential difference* consists in the fact that **in SDG infinitesimals appear which substantially enrich the usual real line**. Owing to this fact geometry acquires a tool to penetrate infinitesimally small portions of a given manifold (a manifold's "germs", to be defined below) which in the usual approach are invisible (in SET they simply do not exist). This creates an invaluable opportunity for physical applications.

However, to make use of this opportunity we must `change from SET to a suitable category`. And our working hypothesis is that ***the fundamental level (below Planck's threshold) is structured by some other (than SET) category***. Of course, some correspondence between SET and this category should exist.

The SDG Strategy

We begin with enriching the usual real line $\mathbb{R}$ by assuming the existence of infinitesimals, such as

$$D := \{x \in \mathbb{R} | x^2 = 0\},$$

$x \in D$ is so small (but not necessarily equal to zero) that $x^2 = 0$. The real line, enriched in this way, will be denoted by R.

Example. We want to compute the derivative of the function $f(x) = x^2$ at $x = c$.

For $d \in D$ we have

$$f(c+d) = (c+d)^2 = c^2 + 2cd + d^2 = c^2 + (2c)d.$$

The linear part of the latter expression can be identified with the derivative of $f(x)$ at c; therefore

$$f'(c) = 2c.$$

Let us notice: *It is a pure algebra! No concept of any limit!*

Example. We want to compute the derivative of the function $f(x) = x^2$ at $x = c$.
For $d \in D$ we have

$$f(c+d) = (c+d)^2 = c^2 + 2cd + d^2 = c^2 + (2c)d.$$

The linear part of the latter expression can be identified with the derivative of $f(x)$ at c; therefore

$$f'(c) = 2c.$$

Let us notice: *It is a pure algebra! No concept of any limit!*

Example. We want to compute the derivative of the function $f(x) = x^2$ at $x = c$.
For $d \in D$ we have

$$f(c+d) = (c+d)^2 = c^2 + 2cd + d^2 = c^2 + (2c)d.$$

The linear part of the latter expression can be identified with the derivative of $f(x)$ at c; therefore

$$f'(c) = 2c.$$

***Let us notice:** It is a pure algebra! No concept of any limit!*

Example. We want to compute the derivative of the function $f(x) = x^2$ at $x = c$.
For $d \in D$ we have

$$f(c + d) = (c + d)^2 = c^2 + 2cd + d^2 = c^2 + (2c)d.$$

The linear part of the latter expression can be identified with the derivative of $f(x)$ at c; therefore

$$f'(c) = 2c.$$

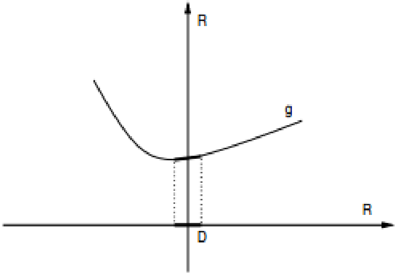
Let us notice: *It is a pure algebra! No concept of any limit!*

Axiom

We change this example into the rule by assuming
Axiom 1. For any $g : D \to R$, there exists a unique $b \in R$ such that

$$\forall d \in D, g(d) = g(0) + d \cdot b.$$

We can define the derivative of any function $f : R \to R$ at c to be $f'(c) = b$.
Let us notice that Axiom 1 states that the graph of g coincides with a fragment of the straight line through $(0, g(0))$ and the slope b.



WORK/Fig1.PNG

The important consequence of Axiom 1 is that **every function has a derivative.**

In fact, in SDG all functions are differentiable.

If this is so, all our problems with differentiability and extendibility in dealing with space-time singularities are eliminated with one stroke. But a high price is to be paid for such a step.

The important consequence of Axiom 1 is that **every function has a derivative**.
In fact, in SDG all functions are differentiable.
If this is so, all our problems with differentiability and extendibility in dealing with space-time singularities are eliminated with one stroke. But a high price is to be paid for such a step.

Excluded middle is excluded

Let us consider the function

$$g(d) = \begin{cases} 1 & \text{if } d \neq 0 \\ 0 & \text{if } d = 0 \end{cases}$$

From Axiom 1 we have: $D \neq \{0\}$. Therefore, by the law of excluded middle, there exists $d_0 \neq 0$ in D and, on the strength of Axiom 1,

$$g(d_0) = g(0) + d \cdot b$$

which lead to $1 = g(d_0) = d \cdot b$, and after squaring we get $1 = 0$.

There is only one strategy to save infinitesimals – to block the law of excluded middle. And indeed the SDG is founded on this strategy.

It works on the basis of weakening classical logic to the **inuitionistic logic** (in which the law of excluded middle is not valid).

Of course, one cannot change logic at will. This would lead to the complete mental anarchy. A modified logic requires `a correct structural environment` that would not only justify but also ENFORCE the correct modification of logic. In the case of SDG this environment is provided by **suitable topoi**.

There is only one strategy to save infinitesimals – to block the law of excluded middle. And indeed the SDG is founded on this strategy.
It works on the basis of weakening classical logic to the **inuitionistic logic** (in which the law of excluded middle is not valid).
Of course, one cannot change logic at will. This would lead to the complete mental anarchy. A modified logic requires `a correct structural environment` that would not only justify but also ENFORCE the correct modification of logic. In the case of SDG this environment is provided by **suitable topoi**.

There is only one strategy to save infinitesimals – to block the law of excluded middle. And indeed the SDG is founded on this strategy.
It works on the basis of weakening classical logic to the **inuitionistic logic** (in which the law of excluded middle is not valid).
Of course, one cannot change logic at will. This would lead to the complete mental anarchy. A modified logic requires `a correct structural environment` that would not only justify but also ENFORCE the correct modification of logic. In the case of SDG this environment is provided by **suitable topoi**.

There are a few possibilities to choose a suitable topos. As the simplest topos of this kind we propose (at least for dydactic purposes) the topos $\mathcal{G}$, called the topos of germ determined ideals (it is a Grothendieck topos).

We have the following diagram

$$\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{G} \overset{\Gamma}{\to} \mathrm{SET}$$

The category $\mathbb{M}$ of manifolds and smooth maps sits in $\mathcal{G}$ fully and faithfully, and the functor Γ, called the global sections functor, is defined by $\Gamma(F) = F(1)$ where 1 is the terminal object of the category.

Categorical Environment of Singularities

There are a few possibilities to choose a suitable topos. As the simplest topos of this kind we propose (at least for dydactic purposes) the topos $\mathcal{G}$, called the topos of germ determined ideals (it is a Grothendieck topos).
We have the following diagram

$$\mathbb{M} \overset{s}{\hookrightarrow} \mathcal{G} \overset{\Gamma}{\to} \mathrm{SET}$$

The category $\mathbb{M}$ of manifolds and smooth maps sits in $\mathcal{G}$ fully and faithfully, and the functor Γ, called the global sections functor, is defined by $\Gamma(F) = F(1)$ where 1 is the terminal object of the category.

Infinitesimal Curvature

The usual C^∞-manifold concept has its SDG generalisation as the **formal manifold** concept. `It is just a smooth manifold equipped with an infinitesimal extension`. In the following, we consider formal manifolds in the category $\mathcal{G}$.

In a previous work (arXiv:1605.03099[math.DG]),

- we have constructed an **infinitesimal** version of n-dimensional formal manifold, i.e. a formal manifold the local maps of which have infinitesimal domains, and
- we have shown that the curvature tensor $\mathcal{R}$ of any locally infinitesimal formal manifold, assumes only infinitesimal values.

This result has important consequences for the singularity problem. We illustrate them with the help of the following simplified model.

Infinitesimal Curvature

The usual C^∞-manifold concept has its SDG generalisation as the **formal manifold** concept. `It is just a smooth manifold equipped with an infinitesimal extension`. In the following, we consider formal manifolds in the category $\mathcal{G}$.

In a previous work (arXiv:1605.03099[math.DG]),

- we have constructed an **infinitesimal** version of n-dimensional formal manifold, i.e. a formal manifold the local maps of which have infinitesimal domains, and
- we have shown that the curvature tensor $\mathcal{R}$ of any locally infinitesimal formal manifold, assumes only infinitesimal values.

This result has important consequences for the singularity problem. We illustrate them with the help of the following simplified model.

Infinitesimal Curvature

The usual C^∞-manifold concept has its SDG generalisation as the **formal manifold** concept. `It is just a smooth manifold equipped with an infinitesimal extension`. In the following, we consider formal manifolds in the category $\mathcal{G}$.

In a previous work (arXiv:1605.03099[math.DG]),

- we have constructed an **infinitesimal** version of n-dimensional formal manifold, i.e. a formal manifold the local maps of which have infinitesimal domains, and
- we have shown that the curvature tensor $\mathcal{R}$ of any locally infinitesimal formal manifold, assumes only infinitesimal values.

This result has important consequences for the singularity problem. We illustrate them with the help of the following simplified model.

Example: A Simple Model

Let us consider a model for an evolving universe given by

$$S^3 \times \mathbb{R} \subset \mathbb{R}^4$$

where $\mathbb{R}$ can be interpreted as a cosmic time, and S^3 as a 3-dimensional instantaneous time section.

Let us farther assume that the diameter ρ_{S^3} of S^3 shrinks to zero size (i.e., to a point in $\mathbb{R}^4$) which we call "singularity" and situate it at, say, $x_0 = 0$, $x_0 \in \mathbb{R}$.

Topologically, we have a cone over S^3 with the vertex at the singularity. We should remember that it is a simple cone singularity rather than a curvature singularity met in standard cosmological models. This is how the evolution looks like **in the category SET** (which the standard relativistic cosmology tacitly assumes).

Example: A Simple Model

Let us consider a model for an evolving universe given by

$$S^3 \times \mathbb{R} \subset \mathbb{R}^4$$

where $\mathbb{R}$ can be interpreted as a cosmic time, and S^3 as a 3-dimensional instantaneous time section.
Let us farther assume that the diameter ρ_{S^3} of S^3 shrinks to zero size (i.e., to a point in $\mathbb{R}^4$) which we call "singularity" and situate it at, say, $x_0 = 0,\ x_0 \in \mathbb{R}$.
Topologically, we have a cone over S^3 with the vertex at the singularity. We should remember that it is a simple cone singularity rather than a curvature singularity met in standard cosmological models. This is how the evolution looks like **in the category SET** (which the standard relativistic cosmology tacitly assumes).

Example: A Simple Model

Let us consider a model for an evolving universe given by

$$S^3 \times \mathbb{R} \subset \mathbb{R}^4$$

where $\mathbb{R}$ can be interpreted as a cosmic time, and S^3 as a 3-dimensional instantaneous time section.
Let us farther assume that the diameter ρ_{S^3} of S^3 shrinks to zero size (i.e., to a point in $\mathbb{R}^4$) which we call "singularity" and situate it at, say, $x_0 = 0,\ x_0 \in \mathbb{R}$.
Topologically, we have a cone over S^3 with the vertex at the singularity. We should remember that it is a simple cone singularity rather than a curvature singularity met in standard cosmological models. This is how the evolution looks like **in the category SET** (which the standard relativistic cosmology tacitly assumes).

How this evolution can be described when regarded as happening **inside the topos** $\mathcal{G}$? We have

$$(S^3_{\mathcal{G}} \times_{\mathcal{G}} R) \hookrightarrow R^4.$$

Let us notice that $S^3_{\mathcal{G}} \neq S^3$ because now $S^3_{\mathcal{G}}$ is enriched by infinitesimals. Since in $\mathcal{G}$ infinitesimals appear, we can call it a FUNDAMENTAL OR MICRO LEVEL, whereas SET will represent the MACRO LEVEL.

$S^3_{\mathcal{G}} \times_{\mathcal{G}} R$ becomes a locally infinitesimal formal manifold. If $S^3_{\mathcal{G}}$ contracts, its 3-curvature grows, but when its radius reaches infinitesimal size, the components of the curvature become infinitesimal (if not zero). In this way, the conic singularity is avoided (and the evolution can be prolonged beyond 0 of R).

Passing through...

In spite of its attractiveness, the simplified model presented in the preceding section has one serious disadvantage. It does not tell us how the curvature tending to infinity suddenly becomes infinitesimal (or whether there is no obstacle preventing such an outcome).

To address this problem, we must choose another topos whose inner environment is sensitive enough to see what happens on approaching the singularity.

The topos we will work with is called the **Basel topos**, denoted by $\mathcal{B}$. It has has much more complex structure than the topos $\mathcal{G}$, but many properties of $\mathcal{G}$ are incorporated in $\mathcal{B}$. In particular, the category $\mathbb{M}$ of manifolds also sits fully and faithfully in $\mathcal{B}$.

If time runs out, go to the last slide.

Passing through...

In spite of its attractiveness, the simplified model presented in the preceding section has one serious disadvantage. It does not tell us how the curvature tending to infinity suddenly becomes infinitesimal (or whether there is no obstacle preventing such an outcome).

To address this problem, we must choose another topos whose inner environment is sensitive enough to see what happens on approaching the singularity.

The topos we will work with is called the **Basel topos**, denoted by $\mathcal{B}$. It has has much more complex structure than the topos $\mathcal{G}$, but many properties of $\mathcal{G}$ are incorporated in $\mathcal{B}$. In particular, the category $\mathbb{M}$ of manifolds also sits fully and faithfully in $\mathcal{B}$.

If time runs out, go to the last slide.

Passing through...

In spite of its attractiveness, the simplified model presented in the preceding section has one serious disadvantage. It does not tell us how the curvature tending to infinity suddenly becomes infinitesimal (or whether there is no obstacle preventing such an outcome).

To address this problem, we must choose another topos whose inner environment is sensitive enough to see what happens on approaching the singularity.

The topos we will work with is called the **Basel topos**, denoted by $\mathcal{B}$. It has has much more complex structure than the topos $\mathcal{G}$, but many properties of $\mathcal{G}$ are incorporated in $\mathcal{B}$. In particular, the category $\mathbb{M}$ of manifolds also sits fully and faithfully in $\mathcal{B}$.

If time runs out, go to the last slide.

Basel Topos

In $\mathcal{B}$ there exist the natural numbers N, called also the **smooth natural numbers**, which are `end-extension of the standard natural numbers` $\mathbb{N}$, i.e. $\mathbb{N} \subsetneq N$.

Let us now consider a real function

$$f : \mathbb{R} \to \mathbb{R}$$

in SET which has divergent behaviour near $0 \in \mathbb{R}$, i.e. $\lim_{x \to 0^+} f(x) = +\infty$. In our example of a 3-sphere with its radius shrinking to zero at $0 \in \mathbb{R}$, the scalar curvature of the sphere is simply given by $f(x) = \frac{1}{x}$. In general, we can have a function of arbitrary fast divergence.

We have demonstarted that the smooth topos $\mathcal{B}$ gives us a tool allowing for smoothly overpassing such divergences, and we have *constructed* an example of such a behaviour..

Basel Topos

In $\mathcal{B}$ there exist the natural numbers N, called also the **smooth natural numbers**, which are `end-extension of the standard natural numbers` $\mathbb{N}$, i.e. $\mathbb{N} \subsetneq N$.
Let us now consider a real function

$$f : \mathbb{R} \to \mathbb{R}$$

in SET which has divergent behaviour near $0 \in \mathbb{R}$, i.e. $\lim_{x \to 0^+} f(x) = +\infty$. In our example of a 3-sphere with its radius shrinking to zero at $0 \in \mathbb{R}$, the scalar curvature of the sphere is simply given by $f(x) = \frac{1}{x}$. In general, we can have a function of arbitrary fast divergence.
We have demonstarted that the smooth topos $\mathcal{B}$ gives us a tool allowing for smoothly overpassing such divergences, and we have *constructed* an example of such a behaviour..

Basel Topos

In $\mathcal{B}$ there exist the natural numbers N, called also the **smooth natural numbers**, which are `end-extension of the standard natural numbers` $\mathbb{N}$, i.e. $\mathbb{N} \subsetneq N$.
Let us now consider a real function

$$f : \mathbb{R} \to \mathbb{R}$$

in SET which has divergent behaviour near $0 \in \mathbb{R}$, i.e. $\lim_{x \to 0^+} f(x) = +\infty$. In our example of a 3-sphere with its radius shrinking to zero at $0 \in \mathbb{R}$, the scalar curvature of the sphere is simply given by $f(x) = \frac{1}{x}$. In general, we can have a function of arbitrary fast divergence.
We have demonstarted that the smooth topos $\mathcal{B}$ gives us a tool allowing for smoothly overpassing such divergences, and we have *constructed* an example of such a behaviour..

Concluding Remark

New tools provided by SDG, non available in the standard approach, give us an opportunity to investigate what happens on “infinitesimally small neighbourhoods” and when various processes “go to infinity”. The first candidate in this line of research, taken up in the present work, is the singularity problem.

As the next step one could try categorical and SDG methods in the search for quantum gravity theory, especially that the very intensive search with the help of other methods has so far given rather modest results.

Concluding Remark

New tools provided by SDG, non available in the standard approach, give us an opportunity to investigate what happens on “infinitesimally small neighbourhoods” and when various processes “go to infinity”. The first candidate in this line of research, taken up in the present work, is the singularity problem.

As the next step one could try categorical and SDG methods in the search for quantum gravity theory, especially that the very intensive search with the help of other methods has so far given rather modest results.

However, there is a price one has to pay for all these advantages: when changing from SET to another topos, one must be ready `to switch from classical to intuitionistic logic`.

This is a radical step. Our brain has evolved through a long interaction with its macroscopic environment, the logical structure of which is shaped by the internal logic of the topos SET, i.e. classical logic.

However, the logic of the entire universe, on all its levels, is not obliged to conform to the logic of our brains.

However, there is a price one has to pay for all these advantages: when changing from SET to another topos, one must be ready `to switch from classical to intuitionistic logic`.

This is a radical step. Our brain has evolved through a long interaction with its macroscopic environment, the logical structure of which is shaped by the internal logic of the topos SET, i.e. classical logic.

However, the logic of the entire universe, on all its levels, is not obliged to conform to the logic of our brains.

However, there is a price one has to pay for all these advantages: when changing from SET to another topos, one must be ready `to switch from classical to intuitionistic logic`.

This is a radical step. Our brain has evolved through a long interaction with its macroscopic environment, the logical structure of which is shaped by the internal logic of the topos SET, i.e. classical logic.

However, the logic of the entire universe, on all its levels, is not obliged to conform to the logic of our brains.