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Introduction to Basics on radiation probing and imaging using x-ray detectors

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Part 2

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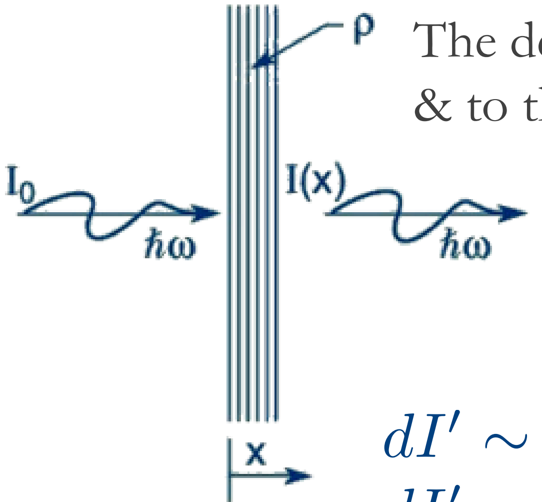
Interaction x-rays with matter

Macroscopic observation

The decay in intensity ~ intensity & to the thickness of an absorber



Homogeneous differential equation

$$dI' \sim -I' \cdot \rho(x') \cdot dx'$$

$$dI' = -I' \cdot \frac{\mu}{\rho}(x') \cdot \rho(x') \cdot dx'$$

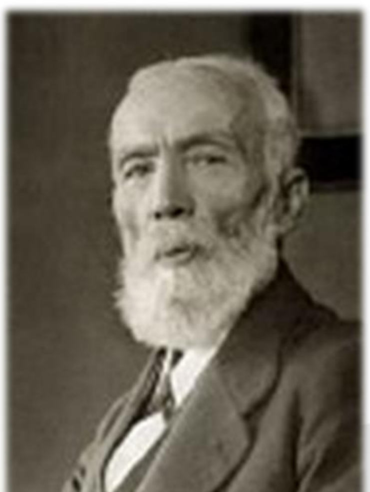
$$\int_{I_0}^{I} \frac{dI'}{I'} = -\int_{0}^{x} \frac{\mu}{\rho}(x') \cdot \rho(x') \cdot dx'$$

$$\ln\left(\frac{I}{I_0}\right) = -\frac{\mu}{\rho}(x) \cdot \rho(x) \cdot x$$

$$I(x) = I_0 \cdot e^{-\frac{\mu}{\rho}(x) \cdot \rho(x) \cdot x}$$

Röntgen 1895

Lamb - Beer's law



Іва́н Па́влович Пулю́й
Ivan Pavlovich Puluj 1894

Interaction x-rays with matter

Particle Wave Dualism

$$I = \overrightarrow{E} \cdot \overrightarrow{E^*}$$

Wave field

$$I \sim N$$

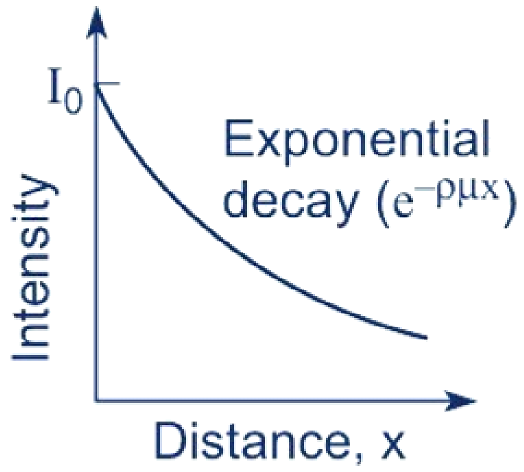
Numbers of measured photons

$$N \sim \varphi$$

Flux (photons/time)
fluence (photons/ area)

$$\dim\left[\frac{\mu}{\rho}\right] = \frac{cm^2}{gr}$$

Interaction x-rays with matter



Lamb-Beer Law for one material and one photon energy

$$I\left(x\right) = I_0 \cdot e^{-\left(\frac{\mu}{\rho}\right)\cdot\rho(x)\cdot x}$$

$$\frac{I(x)}{I_0} = e^{-\mu\cdot x}$$

$$\frac{I(x)}{I_0} = e^{-\mu\cdot x} = e^{-\frac{x}{\lambda}}$$

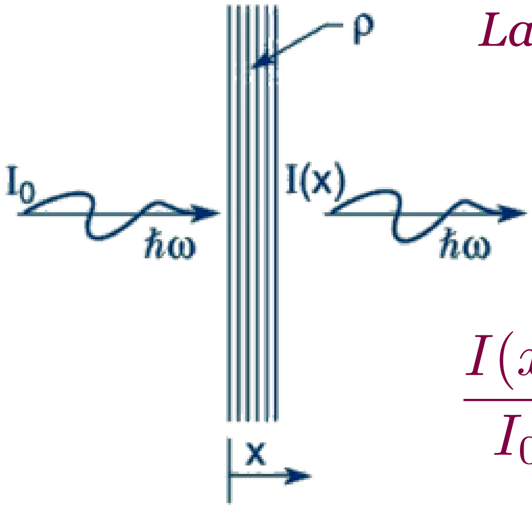
I(x) -> 0 for x -> ∞

Most absorption in the first layers

λ is called "mean free path"

$$-ln\left(\frac{1}{2}\right) = \frac{x}{\lambda} \qquad \lambda \cdot ln\left(2\right) = x_{1/2}$$

Interaction x-rays with matter



Lamb-Beer Law for one material and one photon energy

$$I(x) = I_0 \cdot e^{-\left(\frac{\mu}{\rho}\right)\cdot\rho(x)\cdot x}$$

$$\frac{I(x)}{I_0} = e^{-\left(\frac{\mu}{\rho}\right)\cdot\ \rho\cdot x}$$

is called "**transmission**"
This is the fraction of photons not having any interaction in the sample

$$1 - \frac{I(x)}{I_0} = 1 - e^{-\left(\frac{\mu}{\rho}\right)\cdot\ \rho\cdot x}$$

is called "**absorption**"
This is the fraction of photons having an interaction in the sample

Interaction x-rays with matter

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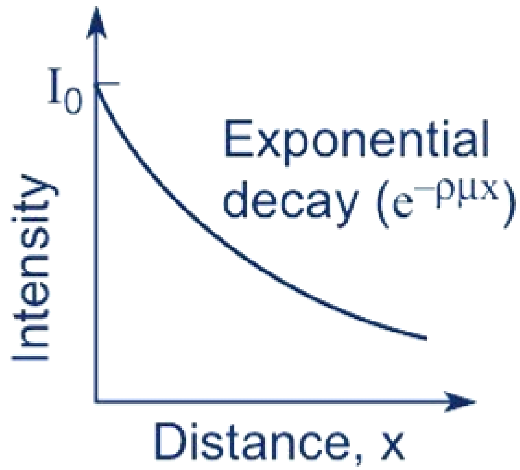

Lamb-Beer Law for one material and one photon energy

$$I\,(x) = I_0 \cdot e^{-\left(\frac{\mu}{\rho}\right)\cdot\rho(x)\cdot x}$$

μ = linear absorption coefficient

$\frac{\mu}{\rho}$ = mass absorption coefficient

For i materials and one photon energy

$$I(x) = I_0 \cdot e^{-\sum_i \left(\frac{\mu}{\rho}\right)_i \cdot \rho_i(x_i)\cdot\Delta x_i}$$

$$P = -\ln\left(\frac{I\,(x)}{I_0}\right) = \sum_i \left(\frac{\mu}{\rho}\right)_i \cdot \rho_i(x_i)\cdot\Delta x_i$$

P is called *projection* or *ray sum*

For i materials and γ photon energies

$$I(E_\gamma, z) = \sum_{\gamma=1}^{K} P(E_\gamma)\cdot e^{-\sum_{i=1}^{N}\mu(E_\gamma,\Delta z_i)\cdot\Delta z_i}\cdot\Delta E$$

$$I(E_j, z) = P(E_j)\cdot\Delta E\cdot e^{-\mu(E_j,\Delta z_1)\cdot\Delta z_1}\cdot e^{-\mu(E_j,\Delta z_2)\cdot\Delta z_2}\cdot\ldots\cdot e^{-\mu(E_j,\Delta z_N)\cdot\Delta z_N}$$

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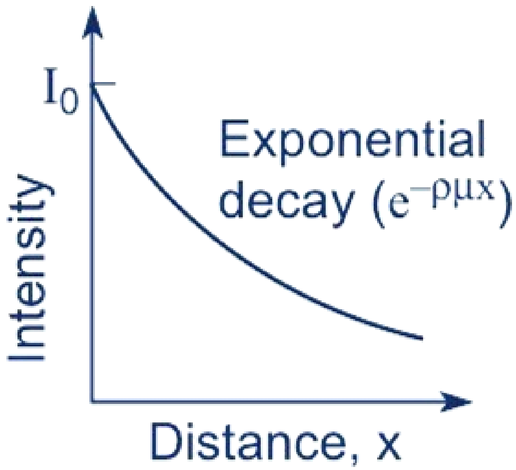


$$T(E_j,z) = -\frac{I(E_j,z)}{P(E_j)\cdot \Delta E} = \mu(E_j,\Delta z_1)\cdot \Delta z_1 + \mu(E_j,\Delta z_2)\cdot \Delta z_2 + \dots + \mu(E_j,\Delta z_N)\cdot \Delta z_N$$

$$\begin{pmatrix} T(E_1,z) \\ . \\ T(E_N,z) \end{pmatrix} = \begin{pmatrix} \mu(E_1,\Delta z_1) & \cdots & \mu(E_N,\Delta z_N) \\ \vdots & \ddots & \vdots \\ \mu(E_N,\Delta z_1) & \cdots & \mu(E_N,\Delta z_N) \end{pmatrix} \cdot \begin{pmatrix} \Delta z_1 \\ . \\ \Delta z_N \end{pmatrix}$$

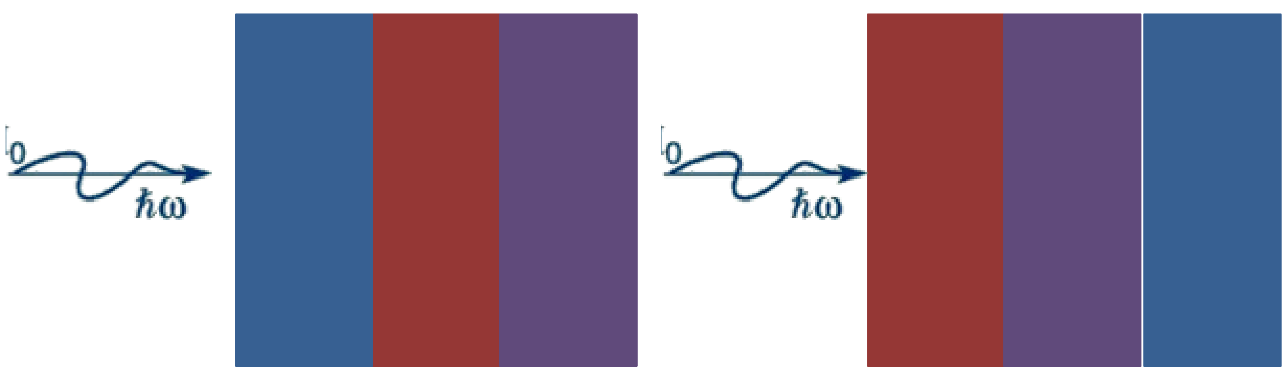
Spectral imaging

Interaction x-rays with matter

For i materials and one photon energy

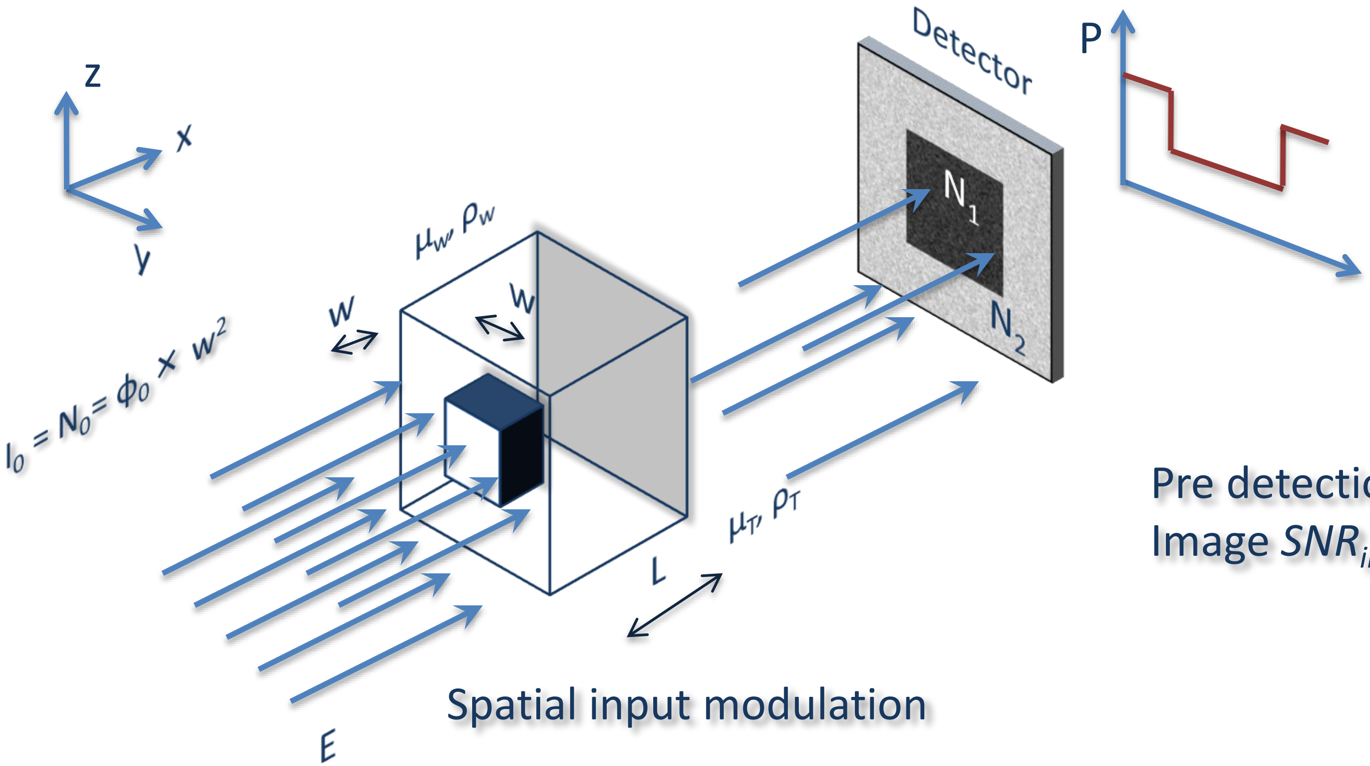
$$I(x) = I_0 \cdot e^{-\sum_i \left(\frac{\mu}{\rho}\right)_i \cdot \rho_i(x_i) \cdot \Delta x_i}$$

$$P = -\ln\left(\frac{I(x)}{I_0}\right) = \sum_i \left(\frac{\mu}{\rho}\right)_i \cdot \rho_i(x_i) \cdot \Delta x_i$$



For x-rays the order of materials is exchangeable without changing the projection – Principle of Superposition

Signal to noise



Spatial input modulation

Pre detection Image SNR_{in}

$$N_1 = \varphi_0 \, w^2 e^{-\mu_T L} e^{-(\mu_w - \mu_T) w}$$

$$N_2 = \varphi_0 \, w^2 e^{-\mu_T L}$$

$$C = \frac{\mu_w - \mu_t}{\mu_t} = \frac{N_2 - N_1}{N_2} = \frac{\Delta N}{N_2}$$

Signal to noise

Photon statistics – Poisson noise

<N> mean value
√N noise

$$s=\sqrt{\left(\frac{\partial \Delta N}{\partial N_1}\right)^2\cdot\left(\Delta(\Delta N)\right)^2+\left(\frac{\partial \Delta N}{\partial N_2}\right)^2\cdot\left(\Delta(\Delta N)\right)^2};\quad \Delta(\Delta N)=\sqrt{\Delta N}$$

$$=\sqrt{\varphi_0\, w^2 e^{-\mu_T L}\left(1+e^{-(\mu_w-\mu_T)w}\right)}=\sqrt{\varphi_0\, w^2 e^{-\mu_T L}\left(2-(\mu_w-\mu_T)w\right)}$$

$$\approx\sqrt{2\varphi_0\, w^2 e^{-\mu_T L}}$$

Signal to noise ratio

$$SNR_{in}^2=\left(\frac{\Delta N}{\sigma}\right)^2=\frac{1}{2}\cdot\varphi_0\cdot w^4\cdot e^{-\mu_T\cdot L}\cdot(\mu_w-\mu_T)^2$$

$$\varphi_0=\frac{2\cdot SNR_{in}^2\cdot e^{\mu_T\cdot L}}{w^4(\mu_w-\mu_T)^2}$$

Photon flux

$$D_{entrance}=\frac{\Delta E}{\Delta M}=\frac{\Delta E}{\rho\cdot\Delta V}=\frac{\Delta\left(N_0\cdot E_\gamma\cdot(1-e^{-\mu_T\cdot x})\right)}{\rho\cdot A\cdot\Delta x}\sim\frac{\varphi_0\cdot E_\gamma}{\rho}$$

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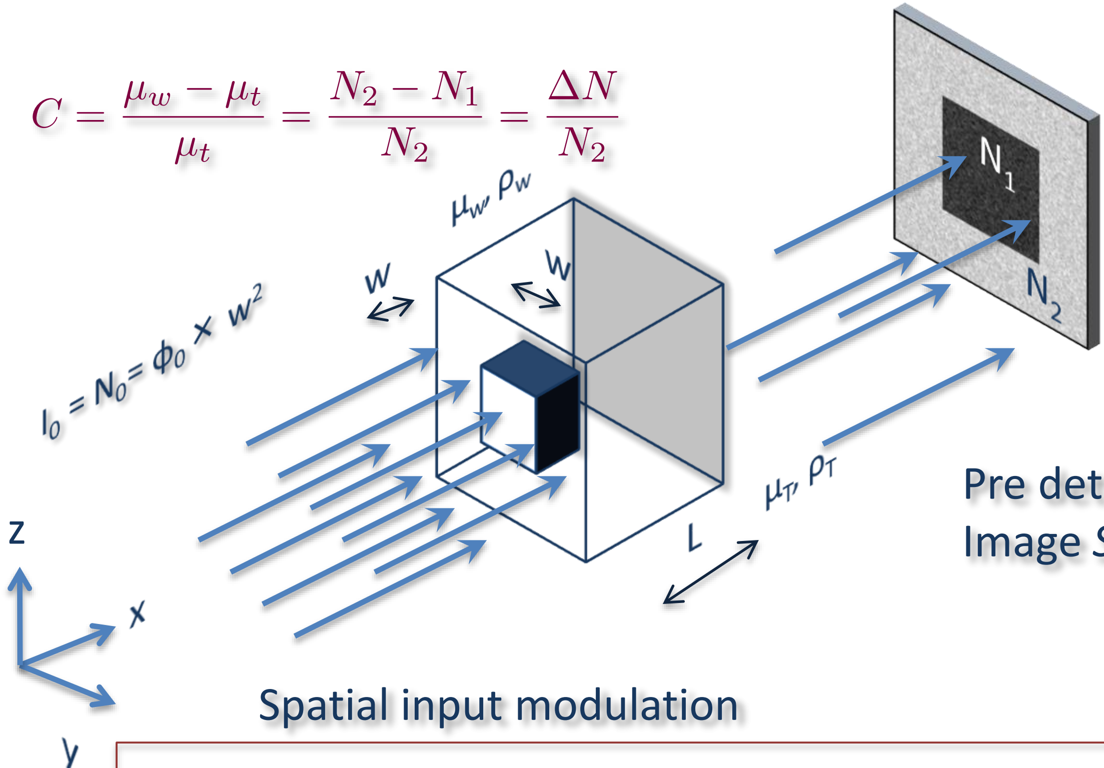
Signal to noise

$$\varphi_0 = \frac{2 \cdot SNR_{in}^2 \cdot e^{\mu_T \cdot L}}{w^4 (\mu_w - \mu_T)^2}$$

$$C = \frac{\mu_w - \mu_t}{\mu_t} = \frac{N_2 - N_1}{N_2} = \frac{\Delta N}{N_2}$$



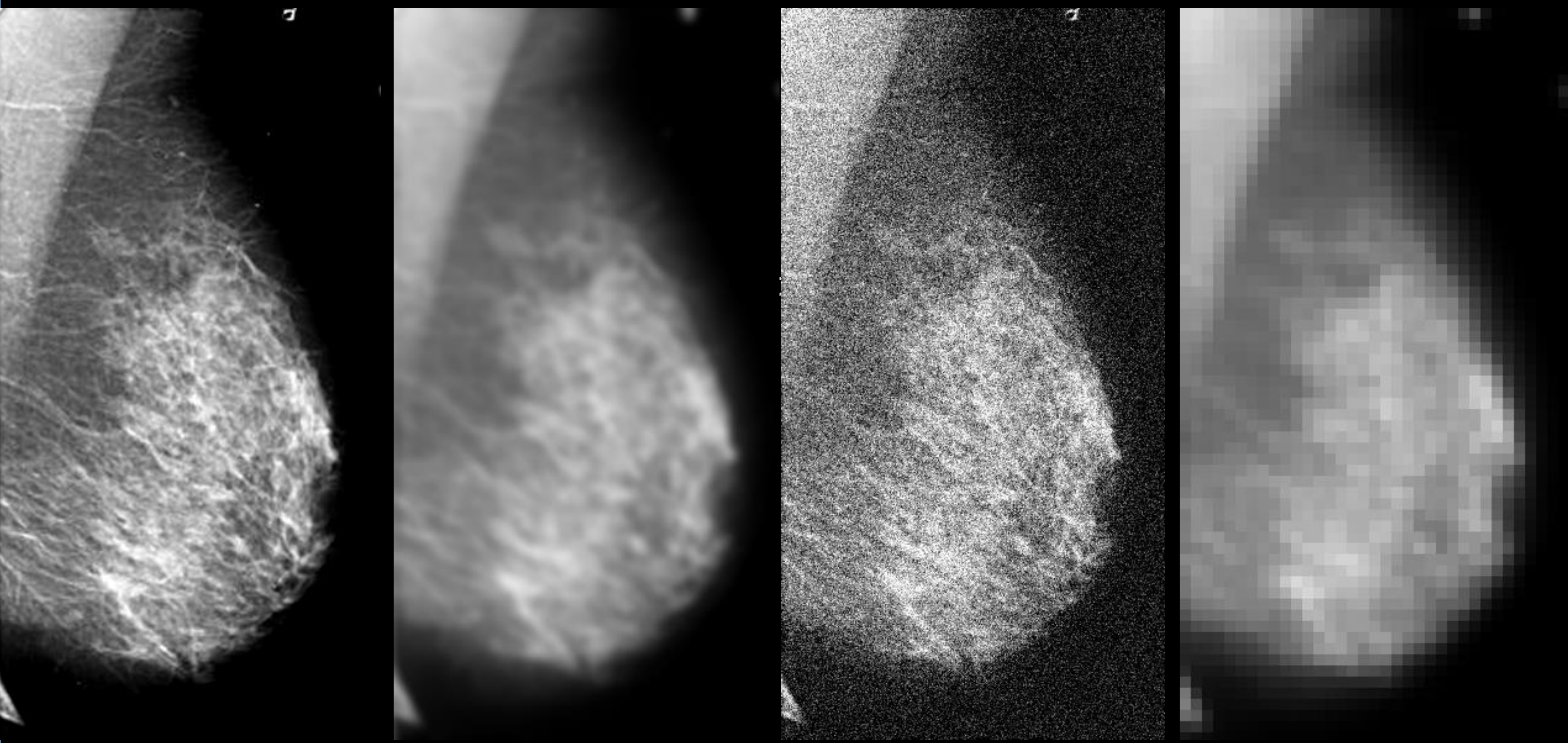
Pre detection
Image *SNR_{in}*

Spatial input modulation

$$SNR_{in}^2 = \left(\frac{\Delta N}{\sigma}\right)^2 = \frac{1}{2} \cdot \varphi_0 \cdot w^4 \cdot e^{-\mu_T \cdot L} \cdot (\mu_w - \mu_T)^2$$

Detective quantum efficiency & Detector problems

Original image | Image blurring | Detector noise | Fixed pattern



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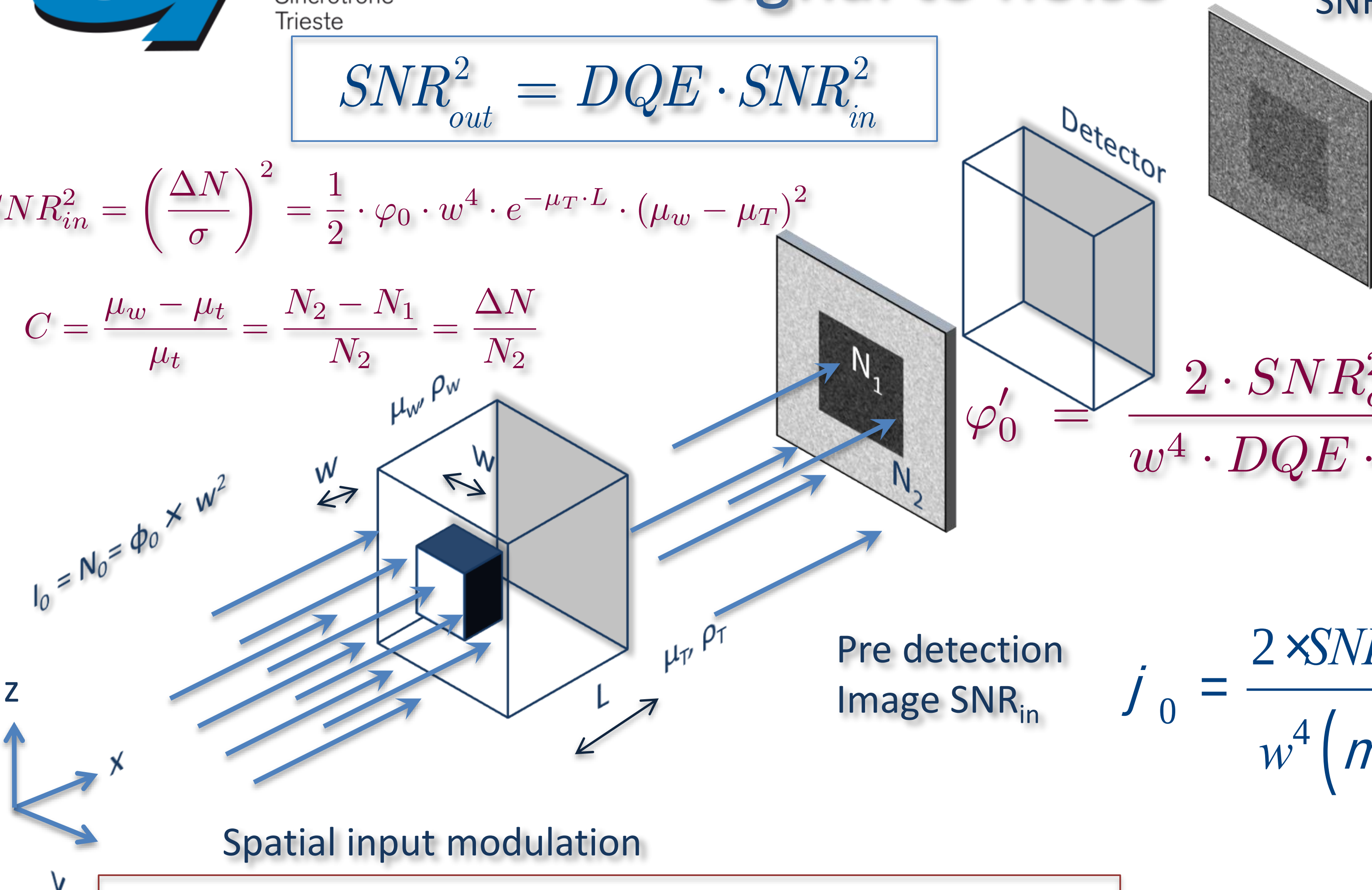
Signal to noise

$$SNR^2_{out} = DQE \cdot SNR^2_{in}$$

$$SNR^2_{in} = \left(\frac{\Delta N}{\sigma}\right)^2 = \frac{1}{2} \cdot \varphi_0 \cdot w^4 \cdot e^{-\mu_T \cdot L} \cdot (\mu_w - \mu_T)^2$$

$$C = \frac{\mu_w - \mu_t}{\mu_t} = \frac{N_2 - N_1}{N_2} = \frac{\Delta N}{N_2}$$



$$\varphi'_0 = \frac{2 \cdot SNR^2_{out} \cdot e^{\mu_T \cdot L}}{w^4 \cdot DQE \cdot (\mu_w - \mu_T)^2}$$

$$\varphi_0 = \frac{2 \times SNR^2_{in} \times e^{\mu_T \times L}}{w^4 \left(\mu_w - \mu_T\right)^2}$$

$$D_{entrance} = \frac{2 \cdot SNR^2_{out} \cdot e^{\mu_T \cdot L} \cdot E_\gamma}{w^4 \cdot C^2_\mu \cdot DQE \cdot \mu^2_T} \cdot \left(\frac{\mu_T}{\rho}\right)$$

0 < DQE < 1

What is the DQE?

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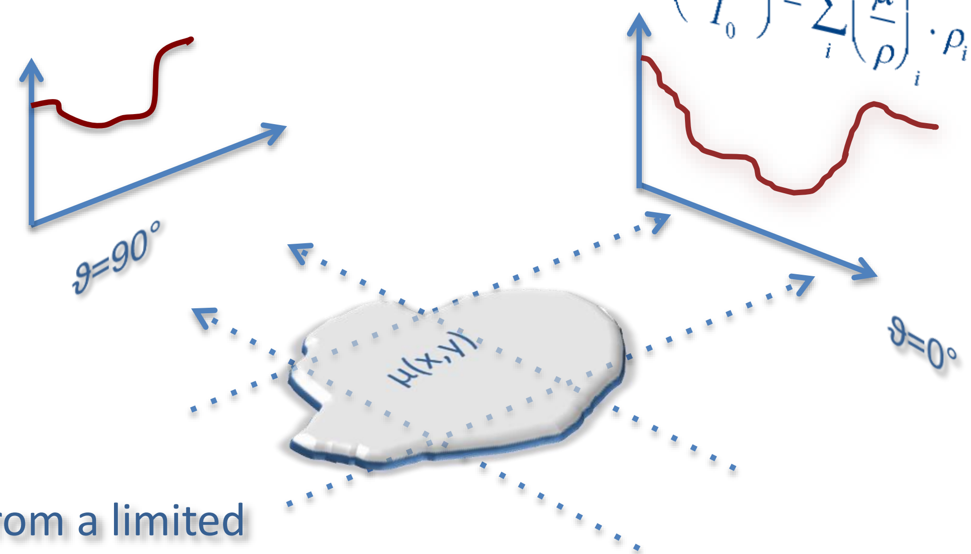
Johann Radon
(1887 - 1956)

Allan Cormack
(1924 - 1998)

Godfrey Hounsfield
(1919 - 2004)

$$P = -\ln\left(\frac{I(x)}{I_0}\right) = \sum_i \left(\frac{\mu}{\rho}\right)_i \cdot \rho_i \cdot \Delta x_i$$



Is it possible to reconstruct μ(x,y) from a limited number of projections

$$m(x,y) = \frac{1}{2 \cdot p} \int_{-\infty}^{\infty} \frac{d}{dy} \aleph\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(x,y) \cdot d\left[y - (m \cdot x + y) \cdot dx \cdot dy\right]\right] \cdot dm$$

11	15	10

3	4	3
4	3	4
4	8	3

10
11
15

3.66	5	3.33

3.49	4.16	3.33
3.66	4.16	3.49
4.33	5	4.16

3.33
3.66
5

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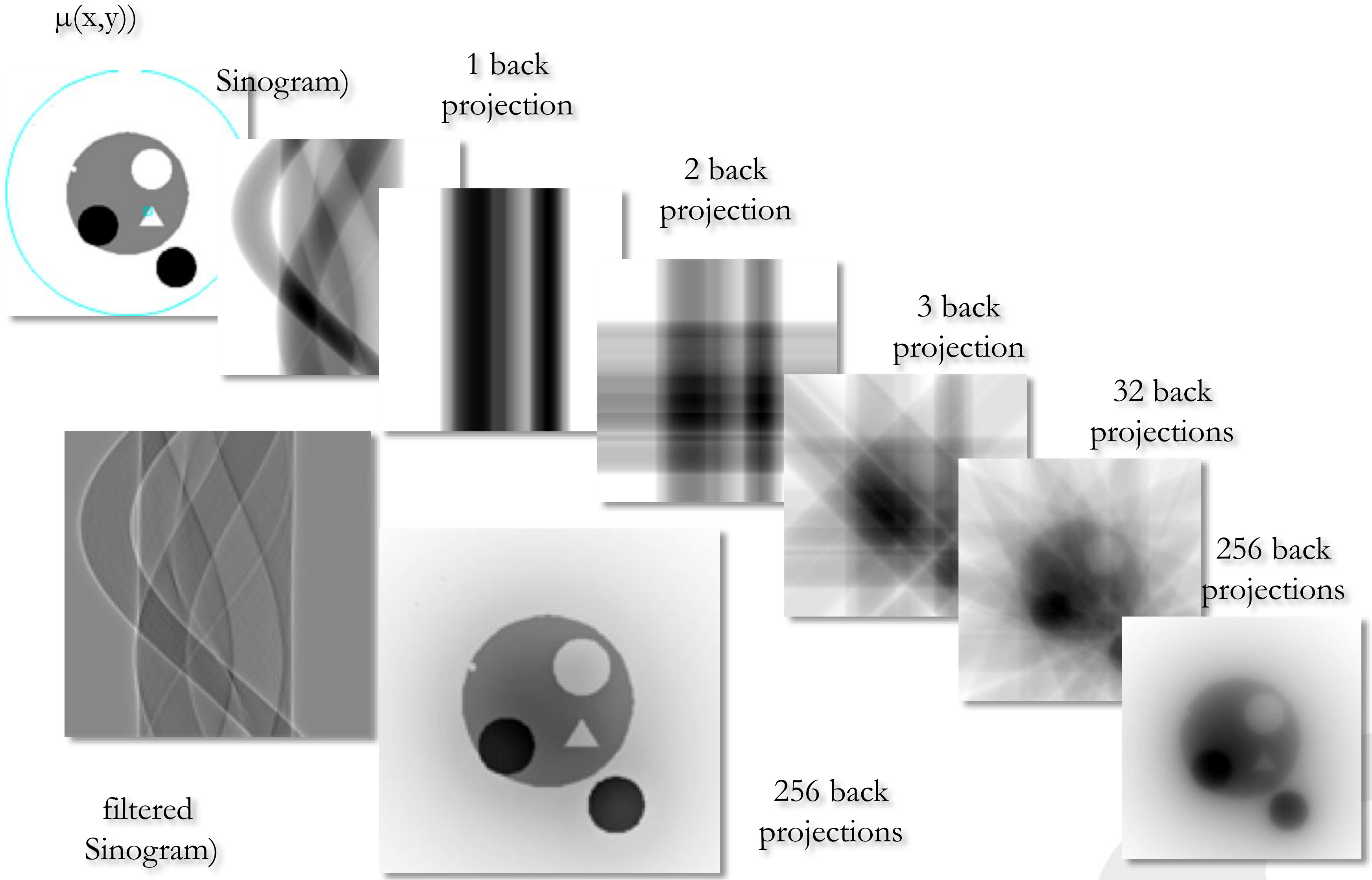

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DQE

$$D_{entrance} = \frac{2 \cdot SNR_{out}^2 \cdot e^{\mu_T \cdot L} \cdot E_\gamma}{w^4 \cdot C_\mu^2 \cdot DQE \cdot \mu_T^2} \cdot \left(\frac{\mu_T}{\rho} \right)$$

What is the DQE?

0 < DQE < 1