An extension of the Poincaré group and its relation to SUSY

based on `arXiv:1507.08039`

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Outline

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- Structure of Poincaré group and SUSY group
- Lorentz group extension at a spacetime point
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Introduction

- Larger symmetry group means simpler model:
  - Less possibilities for invariant Lagrangians
  - Relation between otherwise unrelated coupling factors.
  - etc etc.

- This motivates unification attempts of symmetries.

- An evident candidate:
  - Unification of internal (gauge) symmetry groups.
  - Relating gauge symmetry to spacetime symmetries.
- But to actually do this is not that simple...

- Coleman-Mandula no-go theorem [Phys.Rev.**159**(1967)1251]:
  unification of a compact gauge group with spacetime symmetries not possible in QFT.
  Only possibility is the trivial one: gauge group $\times$ Poincaré group.
- Similar, simpler theorem is of McGlinn [PRL**12**(1964)467].
- O'Raifeartaigh classification theorem of Poincaré gr. extensions [PRB**139**(1965)1052]:
  traditionally, this is also interpreted supporting the above.

- **Important:** gauge group assumed to have positive definite invariant scalar product
  $\Leftrightarrow$ gauge group is assumed to be $\mathrm{U}(1) \times \ldots \times \mathrm{U}(1) \times$ compact semisimple Lie group.
  - Group theoretical convenience: classification of semisimple groups well understood.
  - Experimental justification: Standard Model has $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$, which is such.
  - Field theory reason: necessary for unitarity, or equivalently, for positivity of energy.
    (Because in energy density of Yang-Mills field one has this scalar product.)

- OK, it seems Poincaré group cannot be extended by a gauge group.
- But is there extension at all of the Poincaré Lie algebra in generalized manner?
- Haag-Lopuszanski-Sohnius theorem [Nucl.Phys.**B88**(1975)257]:
  it is possible to extend if we take a "graded" Lie algebra instead of Lie algebra.
  - This extension is the SUSY algebra.
  - The extension part is not the gauge group, but something else. Never mind.
  - At first glance, group theoretical meaning of "graded" Lie algebra is not clear.

- We show an extension, which is not the SUSY.
- It is extension in the sense of ordinary Lie group, not "graded" etc generalization.
- The extended part is really the gauge group, not something else.
- Gauge group will be not compact semisimple:
  only the "important" part of it (Levi factor) will be compact semisimple.
- No problem with unitarity or positivity of energy density.

Structure of Lie groups

Example: Poincaré group.

$$\text{Poincaré group} \quad = \quad \mathcal{P} \rtimes \mathcal{L}$$

- $\mathcal{L}$: homogeneous Lorentz group ($\longleftarrow$ “important part, represented with nice matrices”).
- $\mathcal{P}$: group of spacetime translations ($\longleftarrow$ “trivial part, a bit pathological”).

Levi decomposition theorem states that this is generic: every Lie group has such structure.

Let G be a group.

- If N is a subgroup of G such that it is G-invariant, then it is called **normal subgroup**. (For all $g \in G$: $g\,N\,g^{-1} \subset N$.)
- We say that G is **semi-direct product** of subgroups N and H ($G = N \rtimes H$) whenever: N is normal subgroup and any element $g \in G$ is split uniquely $g = n\,h$ ($n \in N$, $h \in H$).
- We say that G is **direct product** of subgroups N and H ($G = N \times H$) whenever: $G = N \rtimes H$ and H is also normal subgroup.

Levi decomposition theorem:

- If G is a finite dimensional connected Lie group, then one has

$$G = \mathcal{R} \rtimes \mathcal{L},$$

  where $\mathcal{R}$ is solvable (called the **radical**) and $\mathcal{L}$ is semisimple (called the **Levi factor**).
  - Levi factor: “important part”, nondegenerate invariant scalar product on Lie algebra.
  - Radical: “pathological part”, degenerate directions of the invariant scalar product.

- A look at the Levi factor $\mathcal{L}$:

  $\mathcal{L}$ is **semisimple**: Killing form $(x, y) \mapsto \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y)$ is non-degenerate on its Lie algebra. Here, $\mathrm{ad}_x(\cdot) := [x, \cdot]$ for a Lie algebra element x. Killing form: is invariant scalar product. Remember from a Yang-Mills theory Lagrangian?

  $$\mathrm{Tr}\left(F_{ab}F^{ab}\right) \quad \longleftarrow \quad \text{(this is actually the Killing form evaluated on } F_{ab}, F^{ab}\text{)}.$$

- A look at the radical $\mathcal{R}$:

  $\mathcal{R}$ is **solvable**: its Lie algebra is the degenerate directions of the Killing form.
  If r is the Lie algebra of $\mathcal{R}$, then solvability $\Leftrightarrow$
  $r^0 := r, \quad r^1 := [r^0, r^0], \quad r^2 := [r^1, r^1], \quad \ldots, \quad r^k := [r^{k-1}, r^{k-1}] = \{0\}$ for finite k.

  Special case: $\mathcal{R}$ is **nilpotent**, i.e. there is finite k such that
  $\mathrm{ad}_{x_1} \ldots \mathrm{ad}_{x_k} = 0$ for all $x_1, \ldots, x_k \in r$.

  Special case: $\mathcal{R}$ is **abelian**, i.e. for all $x \in r$ one has
  $\mathrm{ad}_x = 0$.

Structure of Poincaré group and SUSY group

Example: (proper) Poincaré group.

$$\text{Poincaré group} \quad = \quad \mathcal{P} \rtimes \mathcal{L}$$

- $\mathcal{P}$: group of spacetime translations (radical, abelian normal subgroup).
  Acts on the flat spacetime and fields over flat spacetime as

  $$x^a \quad \mapsto \quad x^a + d^a$$

  in terms of affine spacetime coordinates.
- $\mathcal{L}$: homogeneous Lorentz group (Levi factor).

Example: (proper) SUSY group [Nucl.Phys.**B76**(1974)477, Phys.Lett.**B51**(1974)239].

$$\text{SUSY group} \quad = \quad \mathcal{S} \rtimes \mathcal{L}$$

- $\mathcal{S}$: group of supertranslations (radical, nilpotent normal subgroup). Acts on the superfields over the flat spacetime as

$$\begin{aligned} \theta^A &\quad \mapsto \quad \theta^A + \epsilon^A, \\ x^a &\quad \mapsto \quad x^a + d^a + \sigma^a_{AA'}\,\mathrm{i}\big(\theta^A \bar{\epsilon}^{A'} - \epsilon^A \bar{\theta}^{A'}\big) \end{aligned}$$

  in terms of "supercoordinates" (Grassmann valued two-spinor coordinates) and affine spacetime coordinates.

- $\mathcal{L}$: homogeneous Lorentz group (Levi factor).

Lie algebra of SUSY is often presented as "graded" Lie algebra, but this is not necessary. Can also be viewed as ordinary Lie algebra of the above group (less confusing?).

Spacetime translations $\mathcal{P}$ is normal subgroup within supertranslations $\mathcal{S}$.

But $\mathcal{S} \neq \mathcal{P} \rtimes \{\text{some other subgroup}\}$.
(Because a supertranslation generates also spacetime translation.)

Therefore:

$$\text{SUSY group} \neq \mathcal{P} \rtimes \{\text{some group acting at points of spacetime}\}$$

Our Poincaré group extension will be, however, of the form:

$$\mathcal{P} \rtimes \{\text{some group acting at points of spacetime}\}$$

The coming part of the talk expands this

$$\{\text{some group acting at points of spacetime}\}$$

Lorentz group extension at a spacetime point

Structure of our extended Lorentz group will be:

$$\underbrace{\underbrace{\underbrace{N}_{\text{nilpotent normal subgroup}} \rtimes \Big(\underbrace{\mathrm{U}(1)}_{\text{internal symmetries}}}_{\text{full gauge group}} \times \underbrace{\mathcal{L}}_{\text{spacetime symmetries}} \Big)}_{\text{full symmetry group of fields at a point of spacetime or momentum space}}$$

$\mathcal{L}$: (covering group of the) homogeneous conformal Lorentz group.

$\mathrm{U}(1)$: can be regarded as a usual $\mathrm{U}(1)$ internal (gauge) symmetry.

N: nilpotent normal subgroup belonging to the gauge group.

N glues together the otherwise independent internal and spacetime symmetries.
(Coleman-Mandula theorem circuimvented, but no problem with unitarity / energy positivity.)

Let A be a finite dimensional complex associative algebra with unit, $\mathbb{1}$.

Let us have a conjugate-linear involution $(\cdot)^+ : A \to A$ on it such that for all $x, y \in A$ one has $(x\,y)^+ = x^+\,y^+$. We call then A as a $^+$**-algebra**.

(It is *not* a *-algebra, because $(x\,y)^+ \neq y^+\,x^+$.)

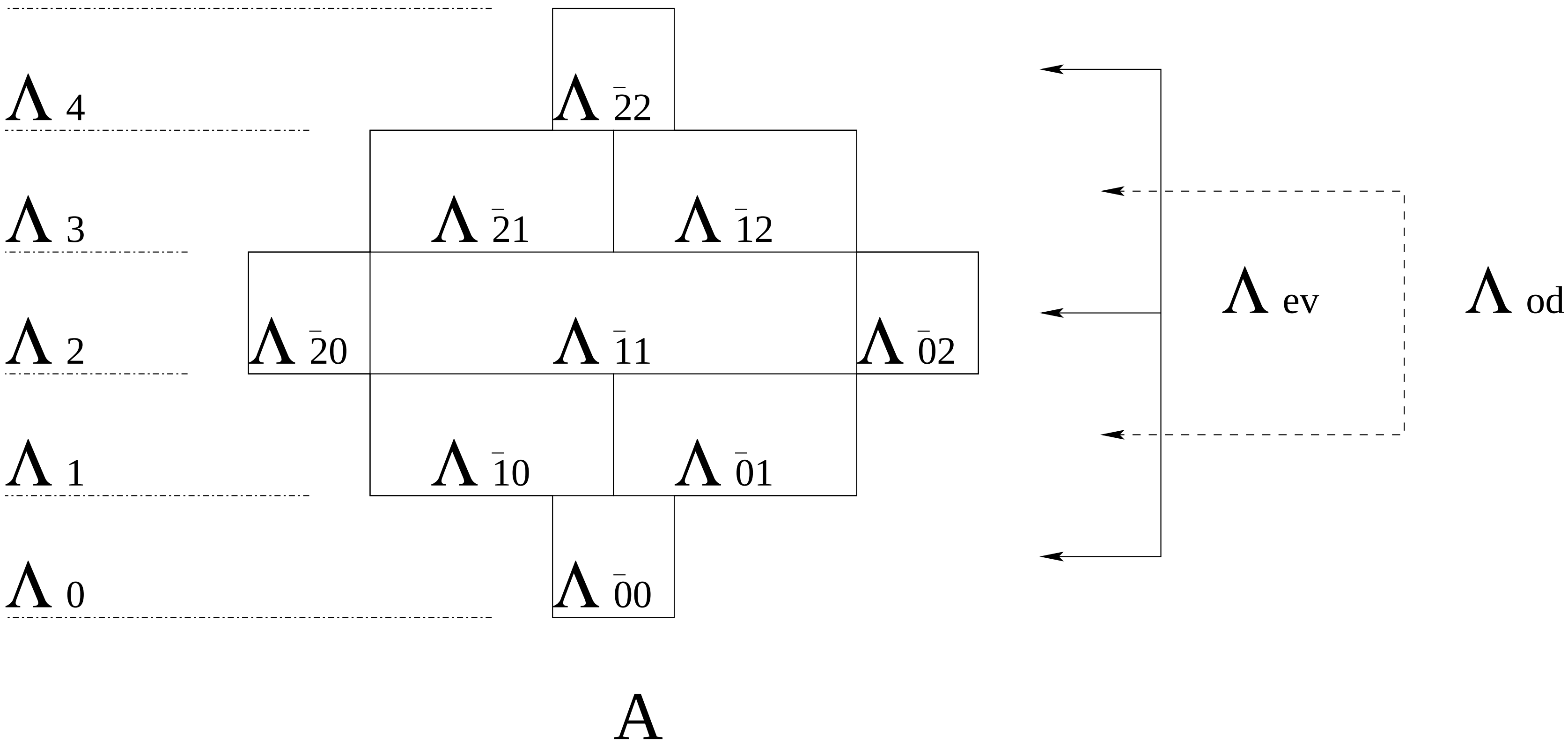
If A has a minimal generator system (e_1, e_2, e_3, e_4) such that

$$\begin{aligned} e_ie_j + e_je_i &= 0 \quad (i,j \in \{1,2\} \text{ or } i,j \in \{3,4\}), \\ e_ie_j - e_je_i &= 0 \quad (i \in \{1,2\} \text{ and } j \in \{3,4\}), \\ e_3 &= e_1^+, \\ e_4 &= e_2^+, \\ e_{i_1}e_{i_2}\dots e_{i_k} & \quad (1 \leq i_1 < i_2 < \dots < i_k \leq 4,\ 0 \leq k \leq 4) \\ & \quad \text{are linearly independent.} \end{aligned}$$

then we call it **spin algebra**,
the $^+$-operation as **charge conjugation**,
and a system of generators as above a **canonical generator system**.

Spin algebra $\sim$ algebra of creation operators of particle and antiparticle if we only had spin.

Illustration of spin algebra
(encoding 2 fundamental degrees of freedom, Pauli principle, and charge conjugation):



$\Lambda_{\bar{p}q}$:	p, q-forms, i.e. p-th polynomials of $\{e_1, e_2\}$ and q-th of $\{e_1^+, e_2^+\}$	($\mathbb{Z} \times \mathbb{Z}$-grading).
Λ_k:	k-forms, i.e. k-th polynomials of $\{e_1, e_2, e_1^+, e_2^+\}$	($\mathbb{Z}$-grading).
$\Lambda_{\rm ev}, \Lambda_{\rm od}$:	even/odd-forms, i.e. even/odd polynomials of $\{e_1, e_2, e_1^+, e_2^+\}$	($\mathbb{Z}_2$-grading).

Representation via two-spinor calculus:

Let S^* be a 2 dimensional complex vector space (“cospinor space”).

Then $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ is spin algebra.

Here $\Lambda(some\ vector\ space)$ means exterior algebra of $some\ vector\ space$:
algebra of k-fold fully antisymmetric tensors of the vector space.

(Exterior algebra $\sim$ Grassmann algebra.)

So, a spin algebra is isomorphic to (“has the structure of”) $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$.
But there is a freedom in matching the canonical generators.

An element of $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ consists of 9 spinorial sectors:

$$\left(\varphi \quad \bar{\xi}_{(+)\,A'} \quad \xi_{(-)\,A} \quad \bar{\epsilon}_{(+)\,[A'B']} \quad v_{A'B} \quad \epsilon_{(-)\,[AB]} \quad \bar{\chi}_{(+)\,[C'D']A} \quad \chi_{(-)\,A'[CD]} \quad \omega_{[A'B'][CD]}\right)$$

Our studied group:

$\mathrm{Aut}(A)$, the automorphism (“symmetry”) group of spin algebra A.

These are the invertible $A \to A$ maps, which preserve the algebraic structure.

$\alpha \in \mathrm{Aut}(A) \Leftrightarrow$

- $\alpha : A \to A$ invertible transformation,
- α is complex-linear,
- α preserves algebraic product: $\alpha(x\,y) = \alpha(x)\,\alpha(y)$ for all $x, y \in A$,
- α is $^+$-real: $\alpha(x^+) = \alpha(x)^+$ for all $x \in A$.

Structure of $\mathrm{Aut}(A)$ (arXiv:1507.08039):

$$\mathrm{Aut}(A) \quad = \underbrace{N}_{\text{grading non}-\text{preserving}} \rtimes \underbrace{\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A)}_{\mathbb{Z}\times\mathbb{Z}-\text{grading preserving}} \rtimes \underbrace{\mathcal{J}}_{\text{label exchanging}}$$

$\mathcal{J}$: particle-antiparticle label exchanging, i.e. $e_1 \mapsto e_3$, $e_2 \mapsto e_4$ and vice-versa.

$\Lambda_{\bar{p}q} \leftrightarrow \Lambda_{\bar{q}p}$.

$\equiv \mathbb{Z}_2$ discrete group, trivial part.

$\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A)$: the $\mathbb{Z} \times \mathbb{Z}$-grading preserving automorphisms, i.e. preserving each $\Lambda_{\bar{p}q}$.

Just mixing e_1 and e_2 within each-other.

$\equiv \mathrm{GL}(2,\mathbb{C}) \equiv \mathrm{U}(1) \times \mathrm{D}(1) \times \mathrm{SL}(2,\mathbb{C})$ (here $\mathrm{D}(1)$: dilatation group).

N: nilpotent normal subgroup of **dressing transformations**.

Mixes higher form contribution to lower forms. Nontrivial part!

$e_i \mapsto e_i + \text{higher forms}.$

Structure of unit connected component of $\mathrm{Aut}(A)$ (omitting the trivial discrete part):

$$\underbrace{\underbrace{\underbrace{N}_{\text{dressing transformations}} \rtimes (\underbrace{\mathrm{U}(1)}_{\text{internal symmetries}}}_{\text{full gauge group}} \times \underbrace{\mathrm{D}(1)\times \mathrm{SL}(2,\mathbb{C})}_{\text{spacetime symmetries}})}_{\text{full symmetry group of } A\text{—valued fields at a point of spacetime or mom. space}}$$

$\mathrm{D}(1)\times \mathrm{SL}(2,\mathbb{C})$: covering group of the homogeneous conformal Lorentz group.

$\mathrm{U}(1)$: can be regarded as a usual $\mathrm{U}(1)$ internal (gauge) symmetry.

N: nilpotent normal subgroup of dressing transformations, belonging to the gauge group.

N glues together the otherwise independent internal and spacetime symmetries.
(Coleman-Mandula theorem circuimvented.)

For Standard Model, one could search for something like:

$$\underbrace{\underbrace{\underbrace{N}_{\text{dressing transformations}} \rtimes \Big(\underbrace{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)}_{\text{SM}-\text{internal symmetries}}}_{\text{full gauge group}} \times \underbrace{\mathrm{D}(1) \times \mathrm{SL}(2, \mathbb{C})}_{\text{spacetime symmetries}} \Big)}_{\text{full symmetry group of fields at a point of spacetime or momentum space}}$$

Full gauge group: $N \rtimes (\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))$.

Allowed properties of the full gauge group:

- Has positive **semi**definite invariant scalar product.
- Equivalently: only its **Levi factor** needs to be compact semisimple.

$\Rightarrow$ No problem with unitarity / positivity of energy density, possible unification mechanism.

Summary

- A non-SUSY Poincaré group extension was presented.
- It is of the form $\{\text{translations}\} \rtimes \{\text{Lorentz group extension}\}$.
- The Lorentz group extension is an automorphism group.
- It has the structure
  $N \rtimes (\{\text{internal symmetries}\} \times \{\text{spacetime symmetries}\})$
  with nilpotent normal subgroup N.
- The internal and spacetime symmetries are glued by N.
- Coleman-Mandula theorem circuimvented because of N.
- Full gauge group $N \rtimes \{\text{internal symmetries}\}$
  has compact semisimple Levi factor
  $\Rightarrow$ unitarity, positivity of energy is OK.
- Unification mechanism may be used for SM group?

Backup slides

O'Raifeartaigh classification theorem on Poincaré group extensions [PRB**139**(1965)1052]:

Let G be connected and simply connected Lie group containing the (covering of the proper) Poincaré group, being of the form $\mathcal{P} \rtimes \mathcal{L}$, where $\mathcal{P}$ denotes translations and $\mathcal{L}$ denotes the (covering of the proper) Lorentz group. Let the Levi decomposition of G be $G = \mathcal{R} \rtimes \mathcal{M}$. Then exactly one of the following cases hold.

(i) $\mathcal{P} = \mathcal{R}$ and $\mathcal{M} = \{\text{semisimple Lie group}\} \times \mathcal{L}$. (⟵ trivial extension, CM)

(ii) $\mathcal{R}$ is abelian, $\mathcal{P} \subset \mathcal{R}$ but $\mathcal{P} \neq \mathcal{R}$. (⟵ possible extra translations)

(iii) $\mathcal{R}$ is not abelian but solvable, $\mathcal{P} \subset \mathcal{R}$ but $\mathcal{P} \neq \mathcal{R}$. (⟵ SUSY, and our example)

(iv) $\mathcal{P} \rtimes \mathcal{L} \subset \mathcal{M}$ and $\mathcal{M}$ is simple. (⟵ rather artificial, basically impossible)