arxiv: 1208.2533 1211.5284 1405.3813 1409.5975 1412.2971

Phase Space Restricted Hadronization

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Talk by T. S. Biró, Zimanyi School 2015, Dec 7-11.

Outline

1. Hadron Spectra: pictures, formulas, data.
2. Fluctuation of Reservoir Parameters.
3. Simple Phase Space Filling Patterns.
4. Single particle energy distribution.

My personal view on non-extensive:

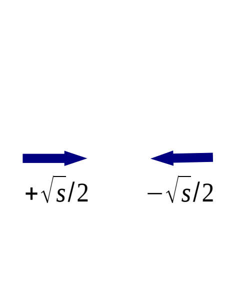
large systems behaving as $\rightarrow$ *small* ones.

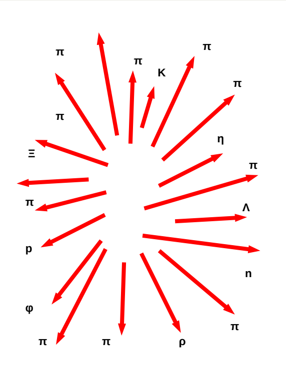
Content

1. Relativistic collisions
2. Ideal Gas with Fluctuating Finite Reservoirs
3. Cut Power-Law as an Approximation

Collision picture:

$E = \sqrt{s}/2$ fix, N fluctuates





Hadron spectra

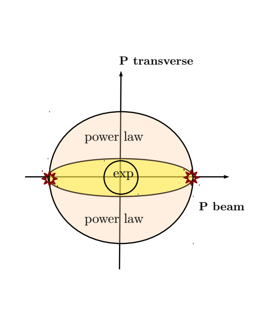
Exponential: thermal model; Power-Law: pQCD

Cut Power Law

$$\left(1+\frac{p_T}{p_0}\right)^{-\nu} \longrightarrow \begin{cases} e^{-p_T/T} \\ \\ p_T^{-\nu} \end{cases} \tag{1}$$

Deformed Exponential

$$e_q(-x) = (1+(q-1)x)^{-\frac{1}{q-1}} = \left(1+\frac{x}{n}\right)^{-n}$$



Experimental p_T spectra (G.Wilk's 2015 ERICE)

Power: pQCD = 4.5; pp data = 6-8; heavy ions = 10 - 40; thermal = ∞

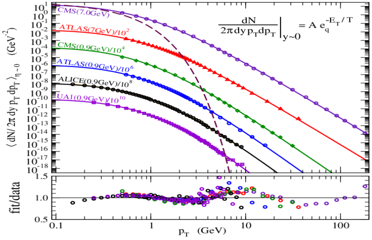


TABLE IV. Parameters used to obtain fits presented in Fig. 8. The values of A is in units of GeV$^{-2}/c^3$.

Collaboration	$\sqrt{s}$	A	q	$n=1/(q-1)$	T(GeV)
CMS [65]	pp at 7TeV	16.2	1.151	6.60	0.147
ATLAS[66]	pp at 7TeV	17.3	1.148	6.73	0.150
CMS [65]	pp at 0.9TeV	15.8	1.130	7.65	0.128
ATLAS[66]	pp at 0.9TeV	13.6	1.124	8.09	0.140
ALICE[67]	pp at 0.9TeV	9.95	1.119	8.37	0.150
UA1 [68]	$\bar{p}p$ at 0.9TeV	13.1	1.109	9.21	0.154

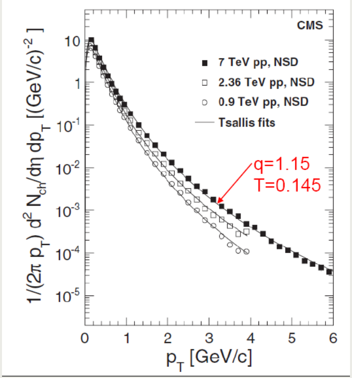
(from arXiv: 1505.02022/ Phys. Rev. D91(2015)114027)

Erice2015 96

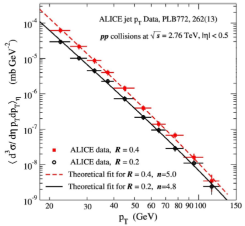
Experimental p_T spectra (G.Wilk's 2015 ERICE)

Power: pQCD = 4.5; pp data = 6-8; heavy ions = 10 - 40; thermal = ∞

CMS pp $n = 6.6$



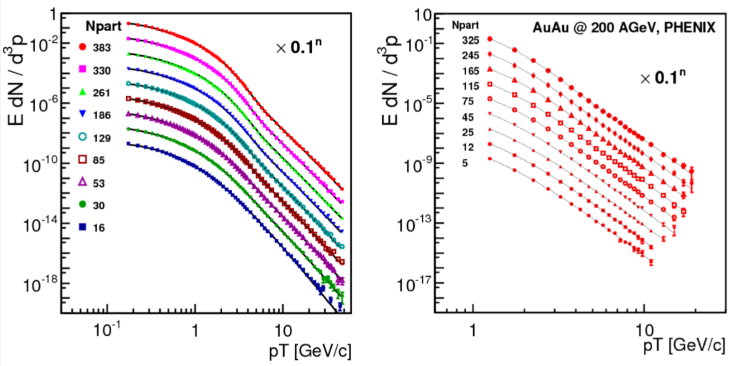
ALICE pp jet $n = 4.8$



n predicted from pQCD is 4.5 for jet production

Soft and Hard Tsallis fits:

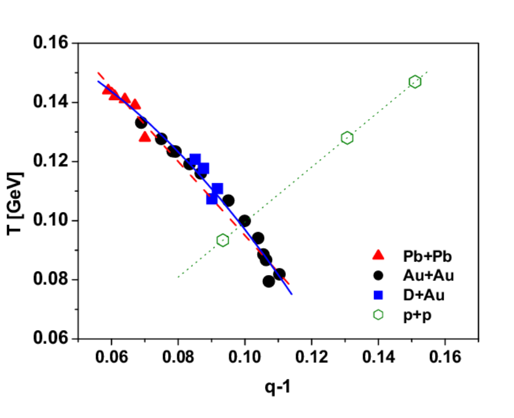
ALICE PLB 720 (2013) 52; PHENIX PRL 101 (2008) 232301



the knick is around $p_T \approx 4 - 5$ GeV.

Relation between T and q (G.Wilk's 2015 ERICE)

pp differs from AA collision!



$(q-1)$ must be a result of competing size-dependent and size-independent effects!

Content

1 Relativistic collisions

2 Ideal Gas with Fluctuating Finite Reservoirs
- The textbook limit
- Conspiracy for finite N

3 Cut Power-Law as an Approximation

Finite Reservoirs

- Atoms in classical matter $\sim 10^{24}$
- Neural synapses in human brain $\sim 10^{12}$
- Hubs in the internet $\sim 10^{7}$
- New particles from heavy ion collisons $\sim 10^{3}$
- From elementary high energy collisions (pp) ~ 10

General expectation:

smaller size $\rightarrow$ larger *relative* fluctuations.

Ideal Gas: microcanonical statistical weight

The one-particle energy, ω, out of total energy, E, is distributed in a one-dimensional relativistic jet according to a statistical weight factor which depends on the number of particles in the reservoir, N:

$$P_1(\omega) = \frac{\Omega_1(\omega)\,\Omega_N(E-\omega)}{\Omega_{N+1}(E)} = \rho(\omega) \cdot \frac{(E-\omega)^N}{E^N} \qquad (2)$$

Statistical weight factor:

$$w_{N,E}(\omega) = \left(1 - \frac{\omega}{E}\right)^N . \qquad (3)$$

The Tsallis – Pareto formula

The ideal phase space volume ratio is already a

"Tsallis distribution"

$$\left(1-\frac{\omega}{E}\right)^N = \left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} \quad (4)$$

with $T = E/N$ and $q = 1 - 1/N$.

The question is the $N \to \infty$ limit!

The Euler number

Definition

$$\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = e^x. \tag{5}$$

Composition rule:

$$\left(1+\frac{x_{12}}{N}\right)^N = \left(1+\frac{x_1}{N}\right)^N \cdot \left(1+\frac{x_2}{N}\right)^N \tag{6}$$

it follows

$$x_{12} = x_1 + x_2 + \frac{1}{N}\, x_1\, x_2. \tag{7}$$

super-additivity

Formal logarithm

we wish an additive description

Composition rule (6) in exponential form:

$$e^{K(x_{12})} = e^{K(x_1)} \cdot e^{K(x_2)}. \tag{8}$$

it follows

$$K(x) = N \ln\left(1 + \frac{x}{N}\right). \tag{9}$$

$K(x_{12}) = K(x_1) + K(x_2)$ **K-additivity**

"Thermodynamical" limit

$$w(\omega) := \lim_{\substack{E\to\infty \\ N\to\infty \\ E/N=T}} \left(1 - \frac{\omega}{E}\right)^N = e^{-\omega/T}. \tag{10}$$

$q = 1 - 1/N \to 1$ (entropy is extensive in this limit),

$T = E/N \to$ finite (energy is extensive in this limit).

Do we gain this Boltzmann–Gibbs exponential *only* in this limit?

Poisson distribution

Reservoir model: E fixed, N fluctuates.

$$w_E(\omega) := \sum_{N=0}^{\infty} P_N \, w_{N,E}(\omega). \tag{11}$$

$$P_N = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle}, \tag{12}$$

results in

$$w_E^{\mathrm{POI}} = \sum_{N=0}^{\infty} \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle} \left(1 - \frac{\omega}{E}\right)^N = e^{-\langle N \rangle \omega / E}. \tag{13}$$

This is exponential even for $\langle N \rangle < 1$, with $T = E / \langle N \rangle$.

An Ideal Reservoir: binomial n-distribution

n particles among k phase space cells: $\binom{k}{n}$ combinations

Consider a subspace (n, k) out of (N, K)!

Limit: $K \to \infty$, $N \to \infty$; while average occupancy $f = N/K$ is given.

$$F_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{k}{n}\binom{K-k}{N-n}}{\binom{K}{N}} = \binom{k}{n} f^n (1-f)^{k-n}. \qquad (14)$$

"Fermionic" reservoir

E is fixed, n is distributed according to BD:

$$\sum_{n=0}^{\infty} \left(1 - \frac{\omega}{E}\right)^n F_{n,k}(f) = \left[(1-f) + f\left(1 - \frac{\omega}{E}\right)\right]^k = \left(1 - f\frac{\omega}{E}\right)^k \quad (15)$$

Note that $\langle\, n \,\rangle = kf$ for BD. Then with $T = E/\langle\, n \,\rangle$ and $q - 1 = -\frac{1}{k}$ we get

$$\left(1 + (q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}$$

This is **exactly** a $q < 1$ Tsallis-Pareto distribution. BG limit: $q \to 1$, Poisson limit: $k \to \infty$.

Ideal Reservoir: Negative binomial n-distribution

n particles multiply among k cells: $\binom{n+k}{n}$ repeated combinations

Consider a subspace (n,k) out of (N,K)

Limit: $K \to \infty$, $N \to \infty$; average occupancy $f = N\,/\,K$ is fixed.

$$B_{n,k}(f) := \lim_{K\to\infty} \frac{\binom{n+k}{n}\binom{N-n+K-k}{N-n}}{\binom{N+K+1}{N}} = \binom{n+k}{n} f^n \,(1+f)^{-n-k-1}. \tag{16}$$

"Bosonic" reservoir

Reservoir in hep: E is fixed, n fluctuates according to NBD.

$$\sum_{n=0}^{\infty}\left(1-\frac{\omega}{E}\right)^{n} B_{n,k}(f) = \left[(1+f) - f\left(1-\frac{\omega}{E}\right)\right]^{-k-1} = \left(1+f\frac{\omega}{E}\right)^{-k-1} \tag{17}$$

Note that $\langle\, n\,\rangle = (k+1)\, f$ for NBD. Then with $T = E\,/\,\langle\, n\,\rangle$ and $q-1 = \frac{1}{k+1}$ we get

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}}$$

This is **exactly** a $q > 1$ Tsallis-Pareto distribution. BG limit: $q \to 1$, Poisson limit: $k \to \infty$.

Pascal Triangle

$$F_{n,k} = f\, F_{n-1,k-1} + (1-f)\, F_{n,k-1}$$

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

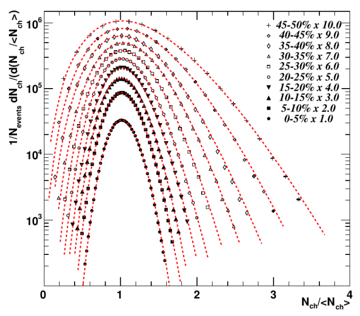
Tilted Version

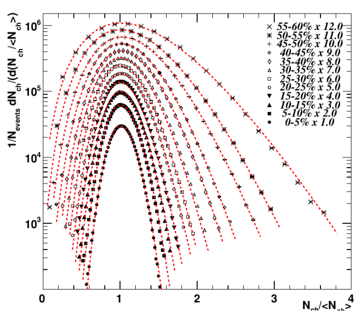
$$B_{n,k} = \frac{f}{1+f}\, B_{n-1,k} + \frac{1}{1+f}\, B_{n,k-1}$$

1	1	1	1	1
1	2	3	4	
1	3	6		
1	4			
1				

Experimental NBD distributions PHENIX PRC 78 (2008) 044902

Au + Au collisons at $\sqrt{s_{NN}} = 62$ (left) and 200 GeV (right). Total charged multiplicities.

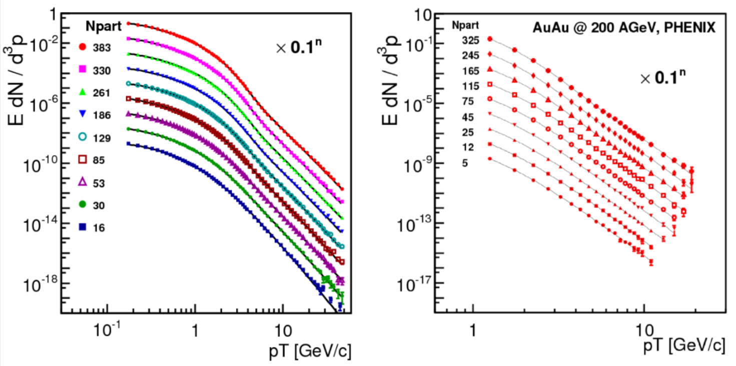




$$k \approx 10 - 20 \quad \rightarrow \quad q \approx 1.05 - 1.10.$$

Soft and Hard Tsallis fits:

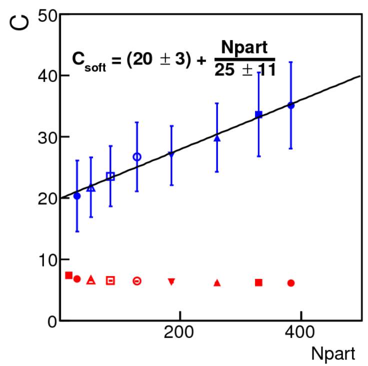
ALICE PLB 720 (2013) 52; PHENIX PRL 101 (2008) 232301



the knick is around $p_T \approx 4 - 5$ GeV.

Hard and Soft Trends with N_{part} arxiv: 1405.3963

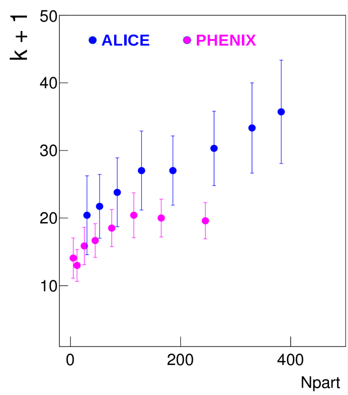
$C = K + 1 = 1/(q-1)$ **powers fitted for ALICE.**



Soft Powers vs N_{part}

arxiv: 1405.3963

Only the soft (*"statistical"*) branches for **PHENIX** and **ALICE:**



$$K \approx 10 \ldots 20 \rightarrow$$
$$q \approx 1.05 \ldots 1.10$$

Statistical vs QCD power-law

- QCD power-law: constant power $(k+1) > 4$ (conformal limit)
- statistical power: $(k+1) = \langle N \rangle / f \propto$ reservoir size
- data fits: ALICE LHC $k+1$ powers vs $N_{\rm part}$
- soft and hard power-laws differ for large $N_{\rm part}$
- power n for pQCD (jet) 4.5; pp data $6-8$; heavy ion $10-50$; thermal ∞.

Summary of ideal reservoir fluctuations

In all the three above cases

$$T = \frac{E}{\langle n \rangle}, \quad \text{and} \quad q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2} \tag{18}$$

Content

1 Relativistic collisions

2 Ideal Gas with Fluctuating Finite Reservoirs

3 Cut Power-Law as an Approximation
- Fix E, general P_n
- General $S(E)$
- Superstatistics

Ideal gas with general *n*-fluctuations

Canonical approach: expansion for small $\omega \ll E$.
Tsallis-Pareto distribution as an approximation:

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \dots \tag{19}$$

Ideal reservoir phase space up to the subleading canonical limit:

$$\left\langle \left(1-\frac{\omega}{E}\right)^n \right\rangle = 1 - \langle\, n \,\rangle \frac{\omega}{E} + \langle\, n(n-1) \,\rangle \frac{\omega^2}{2E^2} - \dots \tag{20}$$

To subleading in $\omega \ll E$

$$\mathbf{T} = \frac{\mathbf{E}}{\langle\, \mathbf{n} \,\rangle}, \qquad \mathbf{q} = \frac{\langle\, \mathbf{n(n-1)} \,\rangle}{\langle\, \mathbf{n} \,\rangle^2} = 1 - \frac{1}{\langle\, n \,\rangle} + \frac{\Delta n^2}{\langle\, n \,\rangle^2}. \tag{21}$$

General equation of state: $S(E)$

Canonical approach: expansion for small $\omega \ll E$.

$$\frac{\Omega_n(E-\omega)}{\Omega_n(E)} = e^{S(E-\omega)-S(E)} = e^{-\omega S'(E)+\omega^2 S''(E)/2-\dots}$$

$$= 1 - \omega S'(E) + \frac{\omega^2}{2}\left[S'(E)^2 + S''(E)\right] - \dots \qquad (22)$$

Compare with expansion of Tsallis

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \dots \qquad (23)$$

Interpret the parameters

$$\frac{1}{T} = S'(E), \qquad q = 1 - \frac{1}{C} \qquad (24)$$

$S''(E) = -1/CT^2$ expressed via the heat capacity of the reservoir, $1/C = dT/dE$

General system with reservoir fluctuations

Canonical approach: expansion for small $\omega \ll E$.

$$\left\langle \frac{\Omega_n(E-\omega)}{\Omega_n(E)} \right\rangle = \left\langle e^{S(E-\omega)-S(E)} \right\rangle = \left\langle e^{-\omega S'(E)+\omega^2 S''(E)/2-\dots} \right\rangle$$

$$= 1 - \omega \langle S'(E) \rangle + \frac{\omega^2}{2} \left\langle S'(E)^2 + S''(E) \right\rangle - \dots \qquad (25)$$

Compare with expansion of Tsallis

$$\left(1+(q-1)\frac{\omega}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \dots \qquad (26)$$

Interpret the parameters

$$\frac{1}{T} = \langle \beta \rangle = \langle S'(E) \rangle, \qquad q = 1 - \frac{1}{C} + \frac{\Delta\beta^2}{\langle \beta \rangle^2} \qquad (27)$$

$\langle S''(E) \rangle = \frac{d}{dE} \langle \beta \rangle = -1 / CT^2$ with heat capacity $C{=}dE/dT$

Understanding the parameter q in terms fluctuations

Opposite sign contributions from $\langle S'^2 \rangle - \langle S' \rangle^2$ and from $\langle S'' \rangle$.

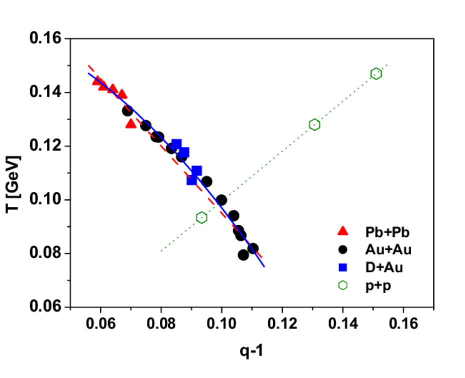
In all cases approximately

$$q = 1 - \frac{1}{C} + \frac{\Delta\beta^2}{\langle \beta \rangle^2}.$$

- $q > 1$ and $q < 1$ are both possible
- for any relative variance $\Delta\beta / \langle \beta \rangle = 1/\sqrt{C}$ $\quad q = 1$ exact
- for $N / \beta \propto E = \text{const}$ $\quad$ it is $\Delta\beta / \langle \beta \rangle = \Delta N / \langle N \rangle$.
- for ideal reservoir and $C \propto N$ distributed as NBD or BD, the Tsallis form is exact

Trends in Tsallis fits to data

G. Wilk's collection, Erice School on Complexity, 2015



Fits: $T_{AA} = 0.22 - 1.25(q-1)$ GeV;
$T_{pp} = (q-1)$ GeV.

AA: assume constant variance,
$\Delta T^2/T^2 = \sigma^2$ and ideal $C = E/T$.
Then we obtain

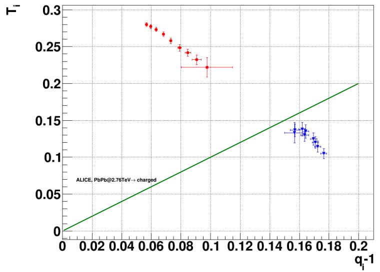
$$T = E\left(\sigma^2 - (q-1)\right)$$

pp: assume fix parameter NBD,
$(q-1) = 1/(k+1)$, $\langle n \rangle = f(k+1)$,
$T = E/\langle n \rangle$, and obtain

$$T = \frac{E}{f}(q-1)$$

Trends in Tsallis fits to data

G. Biro's collection, master thesis, 2015



soft **hard**

N-distribution from superstatistics

We demand

$$\int e^{-\beta\omega}\,\gamma(\beta)\,d\beta \;=\; \sum_N P_N(E)\,\left(1-\frac{\omega}{E}\right)^N$$

Note that

$$e^{-\beta\omega} = e^{\left(1-\frac{\omega}{E}\right)\beta E}\,e^{-\beta E}$$

Using the Taylor series of the first exponential one concludes

$$P_N(E) \;=\; \int \frac{(\beta E)^N}{N!}\,e^{-\beta E}\,\gamma(\beta)\,d\beta$$

Converter factor: Poissonian with parameter $\overline{N} = \beta E$.

Superstatistics from N-distribution

Apply the correspondence for $\omega = E$:

$$\int e^{-\beta E} \gamma(\beta) \, d\beta \; = \; P_0(E).$$

Inverse Laplace transformation delivers the superstatistical factor

$$\gamma(\beta) = \mathcal{L}^{-1} \left[P_0(E) \right]$$

Expanding for small ω one gets

$$\langle \beta \rangle = \frac{\langle N \rangle}{E} \qquad \text{and} \qquad \left\langle \beta^2 \right\rangle = \frac{\langle N(N-1) \rangle}{E^2}$$

leading to

$$1 + \frac{\Delta\beta^2}{\langle \beta \rangle^2} = 1 + \frac{\Delta N^2}{\langle N \rangle^2} - \frac{1}{\langle N \rangle}$$

so for some "super-distributions" $\Delta\beta^2$ would have to be negative...

Summary

- There are $S'(E)$-temperature fluctuations due to finite reservoirs; they cannot be Gaussian.
- Ideal gas reservoirs with NBD or BD number fluctuations lead to exact Tsallis distributions: $q = 1 + \frac{1}{k+1}$ and $q = 1 - \frac{1}{k}$.
- Tsallis distribution is the approximate canonical weight with fluctuating reservoirs: $q = 1 - 1/C + \Delta T^2/T^2$.

BACKUP SLIDES

Some literature

1. ***Ideal gas provides q-entropy,*** T.S. Biró, Physica A392 (2013) 3132-3139; arxiv:1211.5284
2. ***Quark-gluon plasma connected to finite heat bath,*** T.S. Biró, G.G. Barnaföldi, P. Ván, Eur. Phys. J. A49 (2013) 110; arxiv: 1208.2533
3. ***New Entropy Formula with Fluctuating Reservoir,*** T.S. Biró, G.G. Barnaföldi, P. Ván, Physica A 417 (2014) 215-220; arxiv: 1405.3813
4. ***Statistical Power Law due to Reservoir Fluctuations and the Universal Thermostat Independence Principle,*** T.S. Biró, P. Ván, G.G. Barnaföldi, K. Urmossy, Entropy 16 (2014) 6497-6514; arxiv: 1409.5975
5. ***Non-Extensive Quantum Statistics with Particle–Hole Symmetry,*** T.S. Biró, K.M. Shen, B.W. Zhang, Physica A (2015) in press, doi: 10.1016/j.physa.2015.01.072; arxiv: 1412.2971

Variances of functions of distributed quantities

Let x be distributed with small variance and $\langle x \rangle = a$. Consider a Taylor expandable function

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \ldots$$

Up to second order the square of it is given by

$$f^2(x) = f^2 + 2(x-a)ff' + (x-a)^2 \left[f'f' + ff''\right] + \ldots$$

denoting $f(a)$ shortly by f. Expectation values as integrals deliver

$$\langle f \rangle = f + \frac{1}{2}\Delta x^2 f'' \qquad \langle f \rangle^2 = f^2 + \Delta x^2 ff'' \qquad \langle f^2 \rangle = f^2 + \Delta x^2 (f'f' + ff'')$$

Finally we obtain $\boxed{\Delta f = |f'|\,\Delta x}$

One Variable EoS: $S(E)$

Product of variances

$$\Delta E \cdot \Delta\beta = 1 \tag{28}$$

Connection to the (absolute) temperature:

$$|C|\Delta T \cdot \frac{\Delta T}{T^2} = 1 \tag{29}$$

Relative variance scales like 1/SQRT of heat capacity!

$$\frac{\Delta T}{T} = \frac{\Delta\beta}{\beta} = \frac{1}{\sqrt{|C|}} \tag{30}$$

C is proportional to the heat bath size (volume, number of degrees of freedom) in the thermodynamical limit.

Deficiences of the Gauss picture

1. $w(\beta) > 0$ for $\beta < 0$ (finite probability for negative temperature)
2. $\langle e^{-\beta\omega} \rangle$ is not integrable in ω (it cannot be a canonical one-particle spectrum)

J.Uffink, J.van Lith: Thermodynamic Uncertainty Relations; Found.Phys.29(1999)655

”Bohr and Heisenberg suggested that the thermodynamical fluctuation of temperature and energy are complementary in the same way as position and momenta in quantum mechanics.”

B.H.Lavenda: Comments on "Thermodynamic Uncertainty Relations" by J.Uffink and J.van Lith; Found.Phys.Lett.13(2000)487

”Finally, the question about whether or not the temperature really fluctuates should be addressed. ... If the energy fluctuates so too will any function of the energy, and that includes any estimate of the temperature.”

J.Uffink, J.van Lith: Thermodynamic Uncertainty Relations Again: A Reply to Lavenda; Found.Phys.Lett.14(2001)187

”In this interpretation, the uncertainty $\Delta\beta$ merely reflects one's lack of knowledge about the fixed temperature parameter β. Thus β does not fluctuate.”

”Lavenda's book uses these ingredients to derive the uncertainty relation $\Delta\beta \cdot \Delta U \geq 1$. Our paper observes that, on the same basis, one actually obtains a result even stronger than this, namely $\Delta\beta \cdot \Delta U = 1$.”