Objectivity and material frame indifference of dissipative fluids

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Outline

1. Relativity, objectivity and material frame indifference
2. Galilean space-time and Galilean transformations
3. Galilean transformations of 1+3 tensors

Relativity, objectivity and material frame indifference

1. Philosophy? In physics tradition galore. Various interpretations. E.g. general relativity or theory of gravitation.
2. Principle of relativity (special). Galilean transformations: form invariance of basic principles (second derivative). For continua it is not trivial: Landau-Lifsic.
3. Principle of objectivity. Inertial reference frames are not enough. Rigid observers and reference frames should be considered, too. Normal tensors and objective tensors.
4. Principle of material frame indifference. Material properties are fixed to the material: use objectivity and local rest frames.
    - What is moving? Inertial contributions. Symmetry.
    - What is a material? Flow-frame.
    - Confusion: heat conduction (Müller, 1972; Müller-Weiss, 2012), kinetic theory.

Objectivity and relativity

Transformation rules

- Galilei invariance
- Rigid body motion

Transformation rule of Noll (1958):

$$x'^a = \begin{pmatrix} t' \\ x'^i \end{pmatrix} = \begin{pmatrix} t \\ h^i(t) + Q^{ij}(t)x^j \end{pmatrix},$$

where $Q^{-1} = Q^T$ is an orthogonal tensor, $a \in \{0,1,2,3\}$.
Jakobian:

$$J'^{ab} = \frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} 1 & 0^j \\ \dot{h}^i + \dot{Q}^{ij}x^j & Q^{ij} \end{pmatrix}$$

Transformation rule:

$$C'^a = J'^{ab}C^b$$

The need of four dimensions

$V^i := \dot{h}^i$

Objectivity of spatial vectors

$$\begin{pmatrix} 1 & 0 \\ V^i + \dot{Q}^{ij}x^j & Q^{ij} \end{pmatrix} \begin{pmatrix} 0 \\ C^j \end{pmatrix} = \begin{pmatrix} 0 \\ Q^{ij}C^j \end{pmatrix} \quad \rightarrow \quad C'^i = Q^{ij}C^j.$$

Galilean transformations ($Q^{ij} = \delta^{ij}$) and four-vectors?

$$\begin{pmatrix} \hat{\rho} \\ \hat{j}^i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ V^i & \delta^{ik} \end{pmatrix} \begin{pmatrix} \rho \\ j^k \end{pmatrix} = \begin{pmatrix} \rho \\ j^i + \rho V^i \end{pmatrix} \quad \rightarrow \quad \begin{matrix} \rho' = \rho \\ j'^i = j^i + \rho V^i \end{matrix}$$

Velocity $v^i := \dot{x}^i(t)$. By definition: $v'^i = \frac{d}{dt}x'^i = V^i + \dot{Q}^{ij}x^j + Q^{ij}v^j$
This is not a transformation of three-vectors.

Velocity as four-vector: $\dot{x}^a = (1, v^i)$

$$\begin{pmatrix} 1' \\ v'^i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ V^i + \dot{Q}^{ij}x^j & Q^{ij} \end{pmatrix} \begin{pmatrix} 1 \\ v^j \end{pmatrix} = \begin{pmatrix} 1 \\ V^i + \dot{Q}^{ij}x^j + Q^{ij}v^j \end{pmatrix}$$

Rational Extended Thermodynamics I

Moment series expansion of kinetic theory.
Monatomic gases:

$$
\begin{aligned}
\partial_t F + \partial_i F_i &= 0, \\
\partial_t F_i + \partial_j F_{ij} &= 0, \\
\partial_t F_{ij} + \partial_k F_{ijk} &= g_{ij}, \\
\partial_t F_{kki} + \partial_j F_{kkji} &= g_{kki}.
\end{aligned}
$$

$$
F = \rho, \quad F_i = \rho v_i, \quad F_{ij} = P_{ij}, \quad F_{kki} = q_i, \quad \ldots
$$

$P_{kk} = e$: ideal gas equation of state. Dense gases?? Solution: doubled hierarchy? (Ruggeri et. al. 2011)

Galilean transformations in RET I

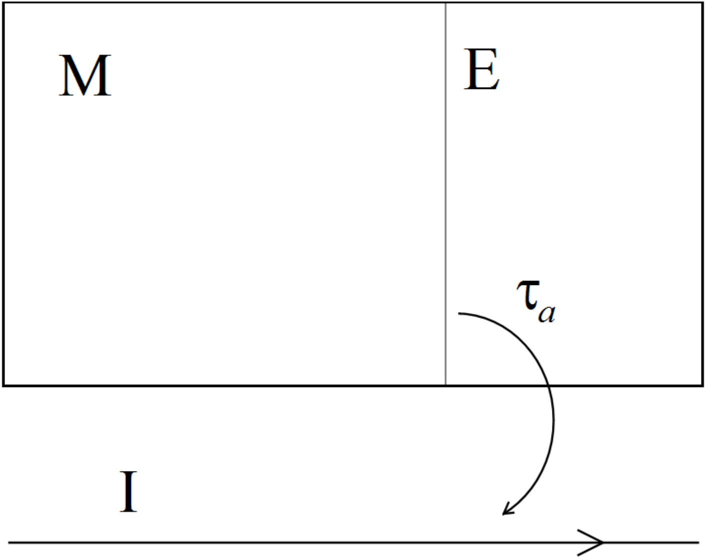
$$\partial_\alpha \mathbf{F}^\alpha = 0, \qquad \alpha = \{0,1,2,3\}$$

Objectivity: invariant balance system. Comoving derivatives, convective and conductive fluxes are distinguished. Transformation rules follow.

$$\begin{aligned}
\rho' &= \rho, \\
e' &= e + \rho\frac{v^2}{2}, \\
P'^{ij} &= P^{ij} + \rho v^i v^j \\
q'^i &= q^i + e v^i + P^{ij} v_j + \rho v^i \frac{v^2}{2}.
\end{aligned}$$

Usual, but internal and kinetic. No dissipation.

The four dimensions of Galilean relativistic space-time

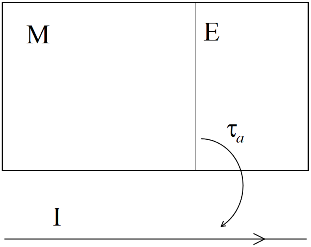


Mathematical structure of Galilean relativistic space-time

1. The *space-time* $\mathbb{M}$ is an oriented four dimensional vector space of the $x^a \in \mathbb{M}$ *world points or events*. There are no Euclidean or pseudoeuclidean structures on $\mathbb{M}$: the length of a space-time vector does not exist.
2. The *time* $\mathbb{I}$ is a one dimensional oriented vector space of $t \in \mathbb{I}$ *instants*.
3. $\tau_a : \mathbb{M} \to \mathbb{I}$ is the *timing* or *time evaluation*, a linear surjection.
4. $\delta_{\bar{a}\bar{b}} : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \otimes \mathbb{R}$ Euclidean structure is a symmetric bilinear mapping, where $\mathbb{E} := Ker(\tau) \subset \mathbb{M}$ is the three dimensional vector space of *space vectors*.

- Simplification: space-time and time are affine spaces
- Simplification: measure lines.
- Abstract indexes: $a, b, c, \ldots$ for $\mathbb{M}$, $\bar{a}, \bar{b}, \bar{c}, \ldots$ for S

Vectors an covectors are different



$$A'^{\alpha} B'_{\beta} = A^{\alpha} B_{\beta} = AB + A^i B_i$$

$$\begin{pmatrix} t' \\ x'^i \end{pmatrix} = \begin{pmatrix} t \\ x^i + v^i t \end{pmatrix}$$

Vector transformations (extensives):

$$\begin{pmatrix} A' \\ A'^i \end{pmatrix} = \begin{pmatrix} A \\ A^i + v^i A \end{pmatrix}$$

Covector transformations (derivatives):

$$\begin{pmatrix} B' & B'_i \end{pmatrix} = \begin{pmatrix} B - B_k v^k & B_i \end{pmatrix}$$

Balances: absolute, local and substantial

$$\boxed{\partial_a A^a = 0} \qquad \longrightarrow \qquad \begin{aligned} u^a: \quad & D_u A + \partial_i A^i &= \quad & d_t A + \partial_i A^i = 0, \\ u'^a: \quad & D_{u'} A + \partial_i A'^i &= \quad & \partial_t A + \partial_i A'^i = 0. \end{aligned}$$

$(a,b,c \in \{0,1,2,3\})$

Transformed: $(d_t - v^i \partial_i) A + \partial_i (A^i + A v^i) = d_t A + A \partial_i v^i + \partial_i A^i = 0$

Mass, energy and momentum

What kind of quantity is the energy?

- Square of the relative velocity: 2nd order tensor
- Kinetic theory: trace of a contravariant second order tensor.
- Energy density and flux: additional order

Basic field:

$$\boxed{Z^{abc} = z^{bc}u^a + z^{\bar{a}bc} : \qquad \text{mass-energy-momentum density-flux tensor}}$$

$a, b, c \in \{0,1,2,3\}, \quad \bar{a}, \bar{b}, \bar{c} \in \{1,2,3\}$

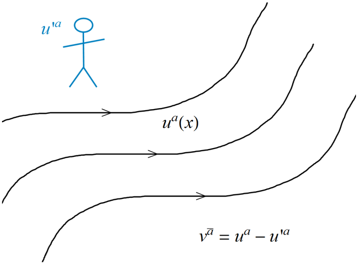
$$z^{bc} \to \begin{pmatrix} \rho & p^{\bar{b}} \\ p^{\bar{c}} & e^{\bar{b}\bar{c}} \end{pmatrix}, \qquad z^{\bar{a}bc} \to \begin{pmatrix} j^{\bar{a}} & P^{\bar{a}\bar{b}} \\ P^{\bar{a}\bar{c}} & q^{\bar{a}\bar{b}\bar{c}} \end{pmatrix}, \qquad e = \frac{e^{\bar{b}}{}_{\bar{b}}}{2}$$

$$\boxed{Z'^{\alpha\beta\gamma} = G^{\alpha}_{\mu} G^{\beta}_{\nu} G^{\gamma}_{\kappa} Z^{\mu\nu\kappa}}$$

$$Z^{\alpha\beta\gamma} = \left(\begin{pmatrix} \rho & p^i \\ p^j & e^{ji} \end{pmatrix} \quad \begin{pmatrix} j^k & P^{ki} \\ P^{kj} & q^{kij} \end{pmatrix} \right), \quad G^{\alpha}_{\nu} = \begin{pmatrix} 1 & 0^i \\ v^j & \delta^{ji} \end{pmatrix}, \quad e = \frac{e^i{}_i}{2}$$

Transformation rules follow:

$$\begin{aligned} \rho' &= \rho, \\ p'^i &= p^i + \rho v^i, \\ e' &= e + p^i v_i + \rho \frac{v^2}{2}, \\ j'^i &= j^i + \rho v^i, \end{aligned}$$



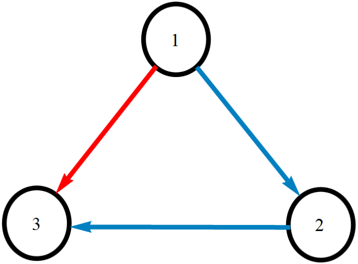
$$\begin{aligned} P'^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\ q'^i &= q^i + e v^i + P^{ij} v_j + p^j v_j v^i + (j^i + \rho v^i) \frac{v^2}{2}. \end{aligned}$$

Galiean transformation of energy

Transitivity:

$$\left.\begin{aligned} e_2 &= e_1 + p_1 v_{12} + \rho\frac{v_{12}^2}{2} \\ e_3 &= e_2 + p_2 v_{23} + \rho\frac{v_{23}^2}{2} \end{aligned}\right\} \rightarrow e_3 = e_1 + p_1 v_{13} + \rho\frac{v_{13}^2}{2}$$

$$p_2 = p_1 + \rho v_{12}, \qquad v_{13} = v_{12} + v_{23}$$



Balance transformations

Absolute

$$\partial_a Z^{abc} = \dot{z}^{bc} + z^{bc}\partial_a u^a + \partial_a z^{\bar{a}bc} = 0$$

Rest frame

$$\begin{aligned} \dot{\rho} + \partial_i j^i &= 0, \\ \dot{p}^i + \partial_k P^{ik} &= 0^i, \\ \dot{e} + \partial_i q^i &= 0. \end{aligned}$$

Inertial reference frame

$$\begin{aligned} \dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\ \dot{p}^i + p^i\partial_k v^k + \partial_k P^{ik} + \rho\dot{v}^i + j^k\partial_k v^i &= 0^i, \\ \dot{e} + e\partial_i v^i + \partial_i q^i + p^i\dot{v}_i + P^{ij}\partial_i v_j &= 0. \end{aligned}$$

Conclusions

- Fluid mechanics, thermodynamics and entropy production are absolute: independent of reference and flow-frames.
- Four-tensors are useful. Transformation rules can be calculated easily. For inertial frames those are the same as in RET.
- Thermodynamics: four-cotensor of intensive quantities. Thermovelocity equation of state ($p^i = \rho w^i$). (Self)diffusion cannot be eliminated. Galilean invariant thermodynamic fluxes and forces.
- Linear asymptotic stability of homogeneous equilibrium.
- Kinetic theory? (Ruggeri-Sugiyama)
- Special relativistic hydro? Fluxes and forces?
- Hyperbolicity, symmetry? (Godunov, Lax, Boillat, Ruggeri, ...)
- Material frame indifference. Rigid reference frames.

Thank you for the attention !

More details are in :

http://arxiv.org/pdf/1508.00121v2

Balances of simple fluids

Local

$$\begin{aligned}\partial_t\rho + \partial_k(\rho v^k) &= 0,\\ \partial_t(\rho v^i) + \partial_k(P^{ik} + \rho v^i v^k) &= 0^i,\\ \partial_t e_{tot} + \partial_k(q^k_{tot} + e_{tot} v^k) &= 0.\end{aligned}$$

Substantial

$$\begin{aligned}\dot\rho + \rho\partial_k v^k &= 0,\\ (\rho v^i)^{\dot{}} + \rho v^i\partial_k v^k + \partial_k P^{ik} &= 0^i,\\ \dot e_{tot} + e_{tot}\partial_k v^k + \partial_k q^k_{tot} &= 0.\end{aligned}$$

Notation:

- $\partial_t = \frac{\partial}{\partial_t}$, $\partial_i = \nabla$, $v^i = \mathbf{v}$, indices are not coordinates. $i, j, k \in \{1,2,3\}$
- e_{tot} is the total energy density.

Transformations

v^i relative velocity,
$\partial_t + v^i\partial_i = \frac{d}{dt}$, comoving derivative,
$\hat q^i = q^i + e_{tot} v^i$, conductive and convective

Fluid thermodynamics

total - kinetic = internal , $e = e_{tot} - \rho v^2/2$

$$\frac{d}{dt}\left(\rho\frac{v^2}{2}\right) + \rho\frac{v^2}{2}\partial_i v^i + \partial_i(P^{ik}v_k) - P^{ik}\partial_i v_k = 0.$$

$$\dot{e} + e\partial_k v^k + \partial_k(\underbrace{q^k_{tot} - P^{ik}v_i}_{q^k}) + P^{ik}\partial_i v_k = 0.$$

Thermodynamics:

$s(e,\rho), \quad de = Tds + \mu d\rho; \quad e + p = Ts + \mu\rho, \quad s^i = \frac{q^i}{T}$

$$\dot{s} + s\partial_i v^i + \partial_i s^i = \frac{1}{T}\dot{e} - \frac{\mu}{T}\dot{\rho} + s\partial_i v^i + \partial_i\frac{q^i}{T} =$$

$$-\frac{1}{T}\left(e\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j\right) + \frac{\mu}{T}\left(\rho\partial_i v^i\right) + s\partial_i v^i + \frac{\mu}{T}\partial_i q^i + q^i\partial_i\frac{1}{T} =$$

$$\boxed{q^i\partial_i\frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i v_j \geq 0.}$$

Basic fields: ρ, e, v^i; Constitutive functions: q^i, P^{ij}

Absolute and relative fields

$$Z^{abc} = z^{bc}u^a + z^{\bar{a}bc} : \qquad \text{mass-energy-momentum density-flux tensor}$$

u-form:

$$\begin{aligned} Z^{abc} \;&=\; \left(\rho u^b u^c + p^{\bar{b}} u^c + u^b p^{\bar{c}} + e^{\bar{b}\bar{c}}\right) u^a + \\ &\quad \left(j^{\bar{a}} u^b u^c + P^{\bar{a}\bar{b}} u^c + P^{\bar{a}\bar{c}} u^b + q^{\bar{a}\bar{b}\bar{c}}\right) \end{aligned}$$

ρ	$= \tau_b \tau_c z^{bc} = \tau_a \tau_b \tau_c Z^{abc},$	density
$p^{\bar{b}}$	$= \pi^{\bar{b}}{}_d \tau_c z^{dc} = \tau_a \pi^{\bar{b}}{}_d \tau_c Z^{adc},$	momentum density
$e^{\bar{b}\bar{c}}$	$= \pi^{\bar{b}}{}_d \pi^{\bar{c}}{}_e z^{de} = \tau_a \pi^{\bar{b}}{}_d \pi^{\bar{c}}{}_e Z^{ade},$	energy density tensor
$j^{\bar{a}}$	$= \pi^{\bar{a}}{}_d \tau_b \tau_c Z^{dbc},$	(self)diffusion flux
$P^{\bar{a}\bar{b}}$	$= \pi^{\bar{a}}{}_d \pi^{\bar{b}}{}_e \tau_c Z^{dec},$	pressure
$q^{\bar{a}\bar{b}\bar{c}}$	$= \pi^{\bar{a}}{}_d \pi^{\bar{b}}{}_e \pi^{\bar{c}}{}_f Z^{def}.$	heat flux tensor

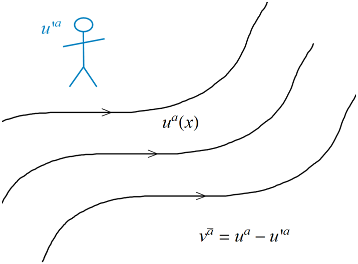
$e = \frac{1}{2} e^{\bar{a}}{}_{\bar{a}}$ energy density $\qquad$ $q^{\bar{a}} = \frac{1}{2} q^{\bar{a}\bar{b}}{}_{\bar{b}}$ heat flux

Galilean transformation

$$\boxed{Z^{abc} = z^{bc} u^a + z^{\bar{a}bc}}$$

$$z^{bc} \overset{u}{\prec} \begin{pmatrix} \rho & p^{\bar{b}} \\ p^{\bar{c}} & e^{\bar{b}\bar{c}} \end{pmatrix}, \qquad z^{\bar{a}bc} \overset{u}{\prec} \begin{pmatrix} j^{\bar{a}} & P^{\bar{a}\bar{b}} \\ P^{\bar{a}\bar{c}} & q^{\bar{a}\bar{b}\bar{c}} \end{pmatrix}, \qquad e = \frac{e^{\bar{b}}{}_{\bar{b}}}{2}$$

Transformation rules (with relative indexes):

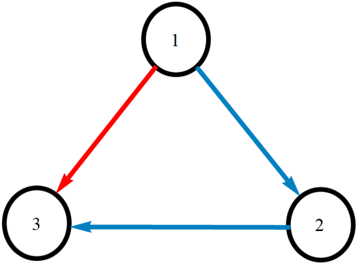


$$\begin{aligned}
\tau_a \tau_b \tau_c Z^{abc} = \hat{\rho} &= \rho, \\
\ldots = p'^i &= p^i + \rho v^i, \\
\ldots = e' &= e + p^i v_i + \rho \frac{v^2}{2}, \\
\ldots = j'^i &= j^i + \rho v^i, \\
\ldots = P'^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\
\ldots = q'^i &= q^i + e v^i + P^{ij} v_j + p^j v_j v^i + (j^i + \rho v^i) \frac{v^2}{2}.
\end{aligned}$$

Kinetic energy and Galiean transformation?

Total energy, kinetic energy and internal energy:

$$e_{tot} = e + \rho\frac{v^2}{2}, \quad \text{therefore} \quad e_2 = e_1 + \rho\frac{v_{12}^2}{2}$$



Transitivity?

$$e_2 = e_1 + \rho\frac{v_{12}^2}{2}, \quad e_3 = e_2 + \rho\frac{v_{23}^2}{2} \quad \rightarrow \quad e_1 = e_3 + \rho\frac{v_{31}^2}{2}$$

Thermodynamics. Gibbs relation I.

$\boxed{ds = Y_{bc} dz^{bc}}$ Y_{bc} chemical potential-thermovelocity-temperature cotensor

Physical definitions

$$Y_{bc} \overset{u}{\prec} \begin{pmatrix} y & y_{\overline{b}} \\ y_{\overline{c}} & y_{\overline{c}\overline{b}} \end{pmatrix} = \frac{\beta}{2} \begin{pmatrix} -2\mu & -w_{\overline{b}} \\ -w_{\overline{c}} & \delta_{\overline{b}\overline{c}} \end{pmatrix},$$

Transformation rules

$$\begin{aligned} \beta' &= \beta, \\ w_i' &= w_i + v_i, \qquad \text{like a vector!} \\ \mu' &= \mu - w_i v^i - \frac{v^2}{2}. \end{aligned}$$

Calculation with classical transformation matrix.

Thermodynamics. Gibbs relation II.

Absolute Gibbs relation: $\boxed{ds = Y_{bc}dz^{bc}}$

Absolute extensivity condition: $\boxed{S^a = Y_{bc}Z^{abc} + p^a}$

Absolute and relative

Pressure decomposition: $p^a = \beta p(u^a + w^{\bar{a}})$

$$S^a = Y_{bc}Z^{abc} + p^a \quad \rightarrow \quad Ts = e + p - \mu\rho - w_i p^i,$$
$$\rightarrow \quad Ts^i = q^i - \mu j^i - P^{ij}w_j + pw^i,$$

$$ds = Y_{bc}dz^{bc} \quad \rightarrow \quad de = Tds + \mu d\rho + w_i dp^i + (\rho w_i - p_i)dv^i.$$

Relative Gibbs relation is Galilean invariant if the inertial reference frame changes.

Thermostat(odynam)ics.

Gibbs relation: $de = Tds + \mu d\rho + w_i dp^i + (\rho w_i - p_i)dv^i$

Maxwell relations

$s(e, \rho, p^i, v^i)$

$$\frac{\partial s}{\partial p^i} = \frac{w_i}{T}, \qquad \frac{\partial s}{\partial v^i} = \frac{\rho w_i - p_i}{T}$$

$$\frac{\partial^2 s}{\partial v^i p^j} = \frac{\partial^2 s}{\partial p^i v^j} = \boxed{\frac{\partial w_i}{\partial v^j} = \delta_{ij} - \rho\frac{\partial w_i}{\partial p^j}}$$

Solution:

$$w_i = \frac{p_i}{\rho} + A_{ij}\left(v^j + \frac{p^j}{\rho}\right) + \overline{w}_i$$

Galilean invariant(!) part:

$$\boxed{p_i = \rho w_i}$$

Termodynamics III. Entropy balance.

$$\partial_a S^a = \partial_a(su^a + s^{\overline{a}}) = \sigma \geq 0, \text{ condition: } \partial_a Z^{abc} = 0$$

Entropy production

$$\begin{aligned}
\partial_a S^a &= \dot{s} + s\partial_a u^a + \partial_a s^{\bar{a}} \\
&= \dots \\
&= -(j^{\overline{a}} - \rho w^{\overline{a}})\partial_a \left(\beta\mu + \beta\frac{w^2}{2}\right) + \\
&\quad \left(q^{\overline{a}} - w^{\overline{a}}(e - p^{\overline{b}} w_{\overline{b}}) + (j^{\overline{a}} - \rho w^{\overline{a}})\frac{w^2}{2} - P^{\overline{a}\overline{b}} w_{\overline{b}}\right)\partial_a \beta - \\
&\quad \beta\left(P^{\overline{a}}_{\overline{b}} + w^{\overline{a}}(\rho w_{\overline{b}} - p_{\overline{b}}) - j^{\overline{a}} w_{\overline{b}} - p\delta^{\overline{a}}_{\overline{b}}\right)\partial_a(u^b + w^{\overline{b}}) \geq 0
\end{aligned}$$

Entropy production II.

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0,\\
\dot{p}^i + p^i\partial_k v^k + \partial_k P^{ik} + \rho\dot{v}^i + j^k\partial_k v^i &= 0^i,\\
\dot{e} + e\partial_i v^i + \partial_i q^i + p^i\dot{v}_i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

$$\begin{aligned}
\Sigma = & -(j^i - \rho w^i)\partial_i\left(\beta\mu + \beta\frac{w^2}{2}\right) + \\
& \left(q^i - w^i(e - p^j w_j) + (j^i - \rho w^i)\frac{w^2}{2} - P^{ij}w_j\right)\partial_i\beta - \\
& \beta\left(P^i_j + w^i(\rho w_j - p_j) - j^i w_j - p\delta^i_j\right)\partial_i(v^j + w^j) \geq 0
\end{aligned}$$

Variables: ρ, p^i, e

Constitutive functions: j^i, P^{ij}, q^i,

Equation of state: μ, T, w^i

$\boxed{v^i?}$ flow-frame

Classical theory

Eos: $w^i = \frac{p^i}{\rho}$
Flow frame: $A^{\bar{a}} = 0$ if $u^a = \frac{A^a}{\tau_a A^a}$

Thermo-frame

$$w^{\bar{a}} = 0 \quad \rightarrow \quad p^{\bar{a}} = 0$$

$$\begin{aligned} \dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\ \rho\dot{v}^i + \partial_k P^{ik} + j^k \partial_k v^i &= 0^i, \\ \dot{e} + e\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j &= 0. \end{aligned}$$

$$-j^i \partial_i \frac{\mu}{T} + q^i \partial_i \frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i v_j \geq 0$$

Flow-frame: hidden Galilean invariance

Constitutive theory

$$-j^i\partial_i\frac{\mu}{T} + q^i\partial_i\frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i v_j \geq 0$$

	Diffusion	Thermal	Mechanical
Force	$-\partial_i\frac{\mu}{T}$	$\partial_i\frac{1}{T}$	$\partial_i v_j$
Flux	j^i	q^i	$-\frac{1}{T}\left(P^{ij} - p\delta^{ij}\right)$

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\
\rho\dot{v}^i + \partial_k P^{ik} + j^k\partial_k v^i &= 0^i, \\
\dot{e} + e\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

(Self)-diffusion : not Brenner like