Non-extensive Statistical Models for Heavy-Ion Collisions

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1. Preface
   - q in Non-extensive Statistics
   - q in Heavy-Ion Collisions
2. p_T Spectra with Finite Heat Capacity
   - General Systems with Finite-Fluctuating Reservoirs
   - Superstatistics
3. p_T Spectra with $T-$independent and $\langle p_T^2 \rangle$
   - A 'Soft+Hard' Model
   - Fittings in p_T Spectra
4. Summary and Outlook

Outline

Tsallis Statistical Mechanics

In 1988 **[1]** C. Tsallis suggested to use the non-extensive entropy formula,

$$S_q = \frac{\sum_{i=1}^{W} p_i^q - 1}{1-q} := -\sum_{i=1}^{W} p_i^q \ln_q p_i \qquad (1)$$

where $q > 0$ is the non-extensive parameter. Here we introduce the deformed q-exponential function $e_q^x \equiv [1+(1-q)x]^{\frac{1}{1-q}}$ and its inverse function $\ln_q(x) \equiv \frac{x^{1-q}-1}{1-q}$ $(x > 0)$.

Tsallis-Pareto distribution

Next we use the OLM(Optimal Lagrange Multipliers)-Tsallis technique to give the generalized q-equilibrium probability distribution, namely, the Tsallis-Pareto distribution,

$$p_i = \frac{1}{Z_q}[1-(1-q)\beta\omega_i]^{\frac{1}{1-q}} \tag{2}$$

Tsallis Fits to p_T Spectra for pp Collisions at LHC

In 2012 C. Wong and G. Wilk **[2]** considered the different cross section with a transverse Tsallis distribution in the form,

$$E\frac{d^3N_{ch}}{dp^3} = C\frac{dN_{ch}}{dy}(1+\frac{E_T}{nT})^{-n} \quad (3)$$

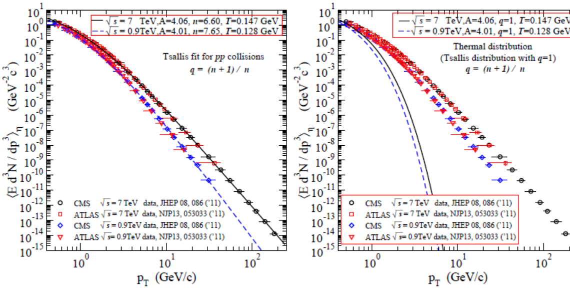

Figure : Left panel: Tsallis fits to the CMS and ATLAS Collaborations data for pp at 7 and 0.9 TeV. Right panel: the same data compared with the corresponding $q = 1$ (or $n \to \infty$) curves.

Outline

the Non-extensive Parameter q

1. Considering the general system with reservoir fluctuations, we have **[4]**

$$q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2} \tag{4}$$

where C is the heat capacity.

2. Ideal gas: finite heat capacity,

$$C = \frac{E}{T} \tag{5}$$

Constant Relative Variance due to $\beta-$Fluctuations

When the relative variance due to $\beta-$fluctuations is a constant, namely,

$$\sigma^2 = \frac{\Delta T^2}{T^2} = const. \tag{6}$$

Easily we get

$$T = E[\sigma^2 - (q-1)] \tag{7}$$

where both E and σ^2 are constant.

Constant Average Occupancy

1. Next consider that the average occupancy in Negative Binomial $n-$Distributions(NBD) is a constant, namely,

$$f = \frac{N}{K} = const. \tag{8}$$

2. Note that for NBD,

$$\langle n \rangle = f(k+1) \tag{9}$$

$$\Delta n^2 = f(k+1)(1+f) \tag{10}$$

3. With the connection for $N/\beta \propto E$ being a constant, $\Delta T^2/T^2 = \Delta n^2/\langle n \rangle^2$, we can have

$$T = \frac{E}{f}(q-1) \tag{11}$$

where E and f are constant.

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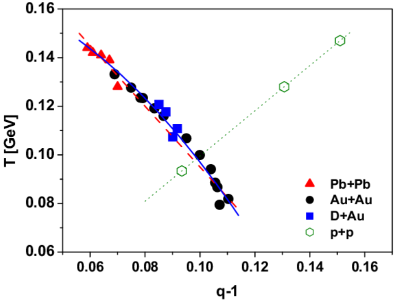
Compared with Data



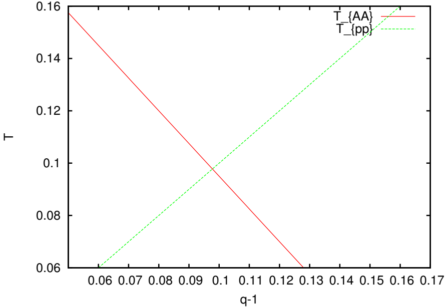


Figure : Left panel: G. Wilk's collection, Eric School on complexity, 2015. Right panel: Our fitting plot with $T_{AA} = 0.22 - 1.25(q-1)$ for the constant relative variance σ^2 case ($E = 1.25$ and $\sigma^2 = 0.176$) and $T_{pp} = q - 1$ for the constant average occupancy f case ($E/f = 1$).

Outline

A 'Soft+Hard' Model

In ultra-relative heavy-ion collisions, p_T spectra are effected by hadrons yields stemming not only from the QGP(Quark-Gluon-Plasma) ("soft"), but also from jets ("hard"). For a first approximation, therefore, we make out the distributions as two part, **[5]**

$$p^0\frac{dN}{d^3p} = (p^0\frac{dN}{d^3p})^{soft} + (p^0\frac{dN}{d^3p})^{hard} \tag{12}$$

That is to say, the transverse momentum spectrum of charged hadrons will be

$$\frac{dN}{2\pi p_T dp_T dy}|_{y=o} = \sum_i A_i\{1 + \frac{q_i - 1}{T_i}[\gamma_i(m_T - v_i p_T) - m]\}^{-\frac{1}{q_i-1}} \tag{13}$$

where $i = soft,\ hard$ and $E_i \equiv \gamma_i(m_T - v_i p_T) - m$ is the co-moving energy.

the Constant $\langle p_T^2 \rangle$ with $q-$exponential

1. With the distributions of p_T, we have

$$\langle p_T^2 \rangle = \frac{\int p_T dp_T \frac{dN}{2\pi p_T dp_T dy}|_{y=o} p_T^2}{\int p_T dp_T \frac{dN}{2\pi p_T dp_T dy}|_{y=o}} \approx \frac{\int p_T^3 dp_T (1 + \frac{q-1}{T} p_T)^{-1/(q-1)}}{\int p_T dp_T (1 + \frac{q-1}{T} p_T)^{-1/(q-1)}}$$
$$= T^2 \frac{6}{(1-3(q-1))(1-4(q-1))} \quad (14)$$

where $q = q_{soft}$ and $T := T_{soft}\sqrt{\frac{1+v_{soft}}{1-v_{soft}}}$ is the Doppler-shifted parameter.

2. Consider the p_T spectra for centrality-dependent collisions, and the same $\langle p_T^2 \rangle$, we get

$$T = \sqrt{\frac{(1-3(q-1))(1-4(q-1))}{6}\langle p_T^2 \rangle} \quad (15)$$

where $\langle p_T^2 \rangle$ is constant.

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the Constant $\langle p_T^2 \rangle$ with $\kappa-$exponential

In comparison, we study the 'soft+hard' model with non-extensive $\kappa-$exponential distributions by just replacing the $q-$exponential in Eq.(13) with $\kappa-$exponential**[6]**, $[\sqrt{1+(\kappa E_i/T_i)^2}+\kappa E_i/T_i]^{-1/\kappa}$.
Then we recalculate the $\langle p_T^2 \rangle$ as,

$$\langle p_T^2 \rangle = T^2 \frac{6}{1-16\kappa^2} \tag{16}$$

namely,

$$T = \sqrt{\frac{1-16(q-1)^2}{6}\langle p_T^2 \rangle} \tag{17}$$

with $\kappa \equiv q-1$.

Constant $\langle p_T^2 \rangle$

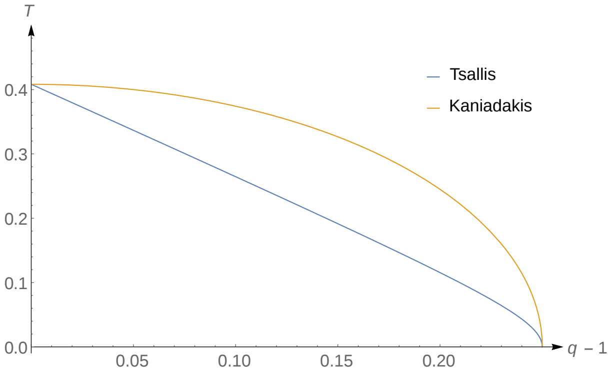

Figure : $T = \sqrt{\frac{(1-3(q-1))(1-4(q-1))}{6} \langle p_T^2 \rangle}$ for Tsallis $q-$exponential and $T = \sqrt{\frac{1-16(q-1)^2}{6} \langle p_T^2 \rangle}$ for Kaniadakis $\kappa-$exponential.

Tsallis' $q-$exponential Fittings

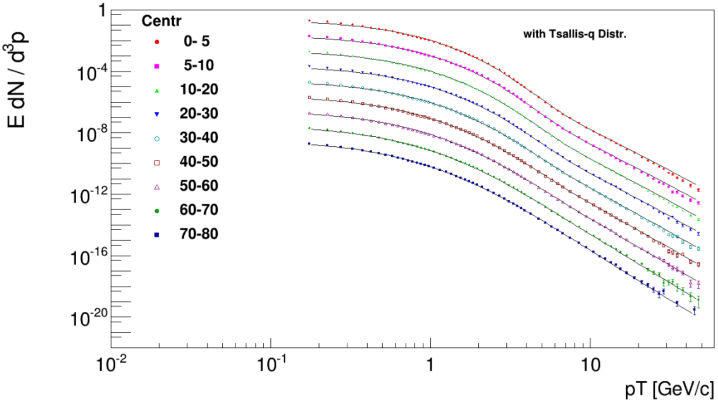

Figure : p_T spectra of charged hadrons stemming from various centrality $PbPb$ collisions at $\sqrt{s} = 2.76\ ATeV$ for 'soft+hard' model with Tsallis $q-$exponential distributions.

Kaniadakis' $\kappa-$exponential Fittings with $\kappa = q - 1$

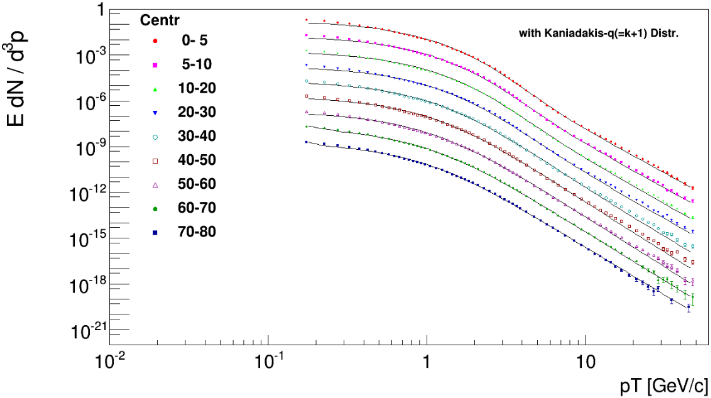

Figure : p_T spectra of charged hadrons stemming from various centrality $PbPb$ collisions at $\sqrt{s} = 2.76\ ATeV$ for 'soft+hard' model with Kaniadakis $\kappa-$exponential distributions.

Fitting Parameters with Constant $< p_T^2 >$

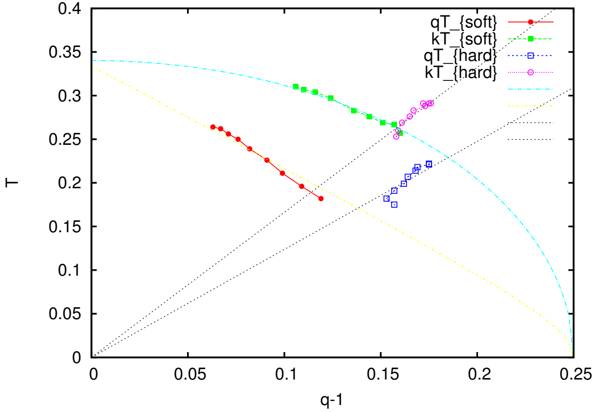

Figure : Fitting parameters T $vs.$ $\kappa = q-1$ ($T := T_s\sqrt{\frac{1+v_s}{1-v_s}}$) in p_T spectra of charged hadrons stemming from various centrality $PbPb$ collisions at $\sqrt{s} = 2.76\ ATeV$ for 'soft+hard' model with Tsallis $q-$ and Kaniadakis $\kappa-$exponential distributions, respectively.

Outline

Summary and Outlook

1. The Tsallis-Pareto distribution is introduced as well as some of its application in heavy-ion collisions.
2. For the non-extensive parameter q, two different models are studied to show its connection with T in fitting the data.
3. A "Soft+Hard" model is re-considered with two different non-extensive distrbutions. Moreover, with constant $\langle p_T^2 \rangle$ the fitting parameters are studied.
4. Next the connections and comparisons of these models will be studied further.

Thank You!!!

Backup Slides

Cited Papers

- C. Tsallis, J. Stat. Phys. 52, (1988) 479.
- C. Wong and G. Wilk, A. Phys. Pol. B. 43 2047 (2012)
- Zebo Tang, *et. al.*, Phys. Rev. C 79 051901 (R) (2009)
- T. S. Biro, G. G. Barnafoldi and P. Van, Phys. A 417 (2015) 215
- K. Urmossy, T. S. Biro, G. G. Barnafoldi and Z. Xu, arXiv: 1405.3963v2
- G. Kaniadakis, Statistical mechanics in the context of special relativity II, Phys. Rev. E 72, 036108 (2005)
- J. Adams et al. (STAR Collaboration), Phys. Rev. Lett. 92, 112301 (2004).

Useful Derivations

In non-extensive statistics, for an arbitrary physical quantity A, the following normalized q-expectation value is introduced,

$$< A >_q \equiv \frac{\sum_{i=1}^{W} p_i^q A_i}{\sum_{j=1}^{W} p_j^q} \equiv \sum_{i=1}^{W} P_i A_i \tag{18}$$

where $\{A_i\}$ are the corresponding eigenvalues in the system and $P_i \equiv \frac{p_i^q}{\sum_{j=1}^{W} p_j^q}$ are the escort probabilities, which are normalized naturally. Consider the canonical ensemble. To obtain the thermal equilibrium distribution associated with a conservative physical system in contact with the thermostat we shall extremize S_q under the constraints,

$$\sum_{i=1}^{W} p_i = 1, \qquad \sum_{i=1}^{W} P_i \epsilon_i = U_q \tag{19}$$

Useful Derivations

Next we use the OLM(Optimal Lagrange Multipliers)-Tsallis technique to give the generalized q-equilibrium probability distribution.

1. $\Phi[p_i] = S_q - \alpha \sum_i^W p_i - \beta \sum_i^W P_i \epsilon_i$
2. $\partial \Phi[p_i] / \partial p_i = 0$
3.

$$
\begin{aligned}
p_i &= \frac{1}{\bar{Z}_q}[1-(1-q)\frac{\beta^*}{\sum_{i=1}^W p_j^q}(\epsilon_i - U_q)]^{\frac{1}{1-q}} \\
&= \frac{1}{Z_q}[1-(1-q)\frac{\beta}{\sum_{i=1}^W p_j^q}\epsilon_i]^{\frac{1}{1-q}} \equiv \frac{1}{Z_q} e_q^{-\beta' \epsilon_i}
\end{aligned}
\quad (20)
$$

where $\beta' = \frac{\beta}{\sum p_j^q} = \frac{\beta^*}{\sum p_j^q + (1-q)\beta^* U_q}$, Z_q and $\bar{Z}_q$ denotes the corresponding normalized constant.

Useful Derivations

For the general system with reservoir fluctuations, we consider the canonical approach: expansion for samll $\omega \ll E$,

$$< \frac{\Omega_n(E-\omega)}{\Omega_n(E)} > =< e^{S(E-\omega)-S(E)} >$$
$$=< e^{-\omega S'(E)+\omega^2 S''(E)/2-\cdots} >$$
$$= 1 - \omega < S'(E) > + \frac{\omega^2}{2} < S'(E)^2 + S''(E) > - \cdots \quad (21)$$

Compare it with the expansion of Tsallis-Pareto distribution

$$[1 + (q-1)\frac{\omega}{T}]^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q\frac{\omega^2}{2T^2} - \cdots \quad (22)$$

with respect to the relations, $C = dE/dT$, easy to get

$$\frac{1}{T} =< \beta >=< S'(E) >, \quad q = 1 - \frac{1}{C} + \frac{\Delta\beta^2}{< \beta >^2} \quad (23)$$

Useful Derivations

Consider the function, $G(t) := \ln \sum p_n e^{nt}$, easily we have

$$G(0) = 0 \tag{24}$$

Expand it at $t = 0$,

$$G(0) + tG'(t) + \frac{t^2}{2} + \cdots = t < n > + \frac{t^2}{2}\Delta n^2 + \cdots \tag{25}$$

where $\Delta n^2 \equiv < n^2 > - < n >^2$. So we have

$$G'(0) = < n >, \quad G''(0) = \Delta n^2 \tag{26}$$

Useful Derivations

Consider NBD, $p_n = \begin{pmatrix} n+k \\ n \end{pmatrix} f^n (1+f)^{-n-k-1}$, so

$$\begin{aligned} G(t) &= \ln \sum \begin{pmatrix} n+k \\ n \end{pmatrix} f^n (1+f)^{-n-k-1} e^{nt} \\ &= \ln \sum \begin{pmatrix} n+k \\ n \end{pmatrix} (fe^t)^n (1+f)^{-n-k-1} \\ &= \ln(1+f-fe^t)^{-k-1} \\ &= -(1+k)\ln(1+f-fe^t) \end{aligned} \tag{27}$$

Then we can have

$$\begin{aligned} G'(0) &= f(k+1) = < n >, \\ G''(0) &= f(k+1)(f+1) = \Delta n^2 \end{aligned} \tag{28}$$

Note

1. For the derivation of Eq.(14), during the integrals the hard part is neglected as well as the mass of particles, with respect to the integrating range and p_T spectra.
2. For the Fig in p17, it is plotted with $\langle p_T^2 \rangle = 1$.
3. For the Fig in p20, the four fitting functions are $0.34015\sqrt{1-16\kappa^2}$, $0.333699\sqrt{1-7(q-1)+12(q-1)^2}$, $1.23754(q-1)$ and 1.66185κ respectively.

Note

1. In C. Wong and G. Wilk's fig, $E_T = m_T - m = \sqrt{m^2 + p_T^2} - m$ and the integral of it is used to fit the experimental data,

$$\langle E\frac{d^3N_{ch}}{dp^3}\rangle_\eta = \frac{C}{2\eta_0}\frac{dN_{ch}}{dy}\int_{-\eta_0}^{\eta_0} d\eta\frac{dy}{d\eta}(1+\frac{E_T}{nT})^{-n} \quad (29)$$

with assuming now a rapidity plateau structure with a constant $A = CdN_{ch}/dy$ and $m = m_\pi = 140\ MeV$. So the fitting parameters are $A = 4.06$, $n = 6.60$ and $T = 147\ MeV$ for $\sqrt{s} = 7\ TeV$ and $A = 4.01$, $n = 7.65$ and $T = 128\ MeV$ for $\sqrt{s} = 0.9\ TeV$. Moreover, the ATLAS measurement has a slightly larger η window, $|\eta| \leq 2.5$, instead of CMS's $|\eta| \leq 2.4$ but same spectra or data. And the p_T values extends from 0.5 to 36 for 7 and from 0.5 to 31 for 0.9 (GeV).

Power-Law Functions in $Au + Au$ Collisions

Z. Tang, *et.al.* **[3]** also rewrite the usual Boltzmann distribution in an m_T exponential form as a power-law distribution,

$$\frac{d^2N}{2\pi m_T dm_T dy} \propto (1 + \frac{q-1}{T} m_T)^{-1/(q-1)} \quad (30)$$

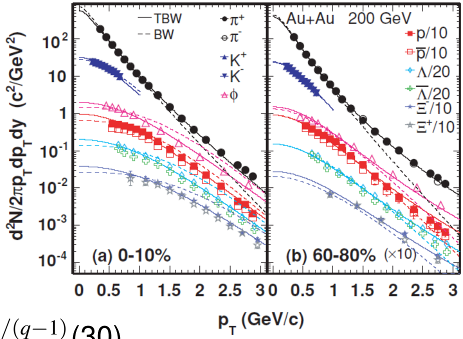

Figure : Identified particle p_T spectra in $Au + Au$ at 200 GeV in $0 \sim 10\%$ central collisions (a) and in peripheral $60 \sim 80\%$ collisions (b).

Note

1. In Z. Tang's fig, the solid curves represent the TBW(Tsallis-Blast-Wave model) fit and the dashed lines are BGBW (Boltzmann-Gibbs-Blast-Wave model) calculations with flow velocity β and temperature T values from [7]. Only fits to particles are shown because both models have the same spectral shapes for particles and anti-particles. Fitting parameters are shown as, $\beta = 0.047 \pm 0.009$, $T = 0.122 \pm 0.002$ and $q - 1 = 0.018 \pm 0.005$ for $0 \sim 10\%$ and $\beta = 0 \pm 0.05$, $T = 0.114 \pm 0.003$ and $q - 1 = 0.086 \pm 0.002$ for $60 \sim 80\%$. More are seen in [3].