Is the η–η' complex an ordinary two-state system?*

D. Klabučar[a]

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[a] Physics Department, University of Zagreb, Croatia

1. Introduction = QUESTION:

Do η and η' always obey von Neumann–Wigner anticrossing theorem? It states: *If a Hermitian matrix represents an observable for a system and depends on continuous real parameters, its eigenvalues cannot cross as any of the parameters vary.*

In a two-state (two-level) system, it is easy to see the avoided crossing of the levels, as the eigenvalues of the 2x2 Hamiltonian matrix $\hat{H}$ are

$$E_\pm = \frac{1}{2}(H_{11}+H_{22}) \pm \frac{1}{2}\sqrt{(H_{11}-H_{22})^2 + 4|H_{12}|^2}\,.$$

Thus $E_\pm$ obey $E_+ > E_-$ for all parameter values if the transition matrix element $H_{12} \neq 0$.

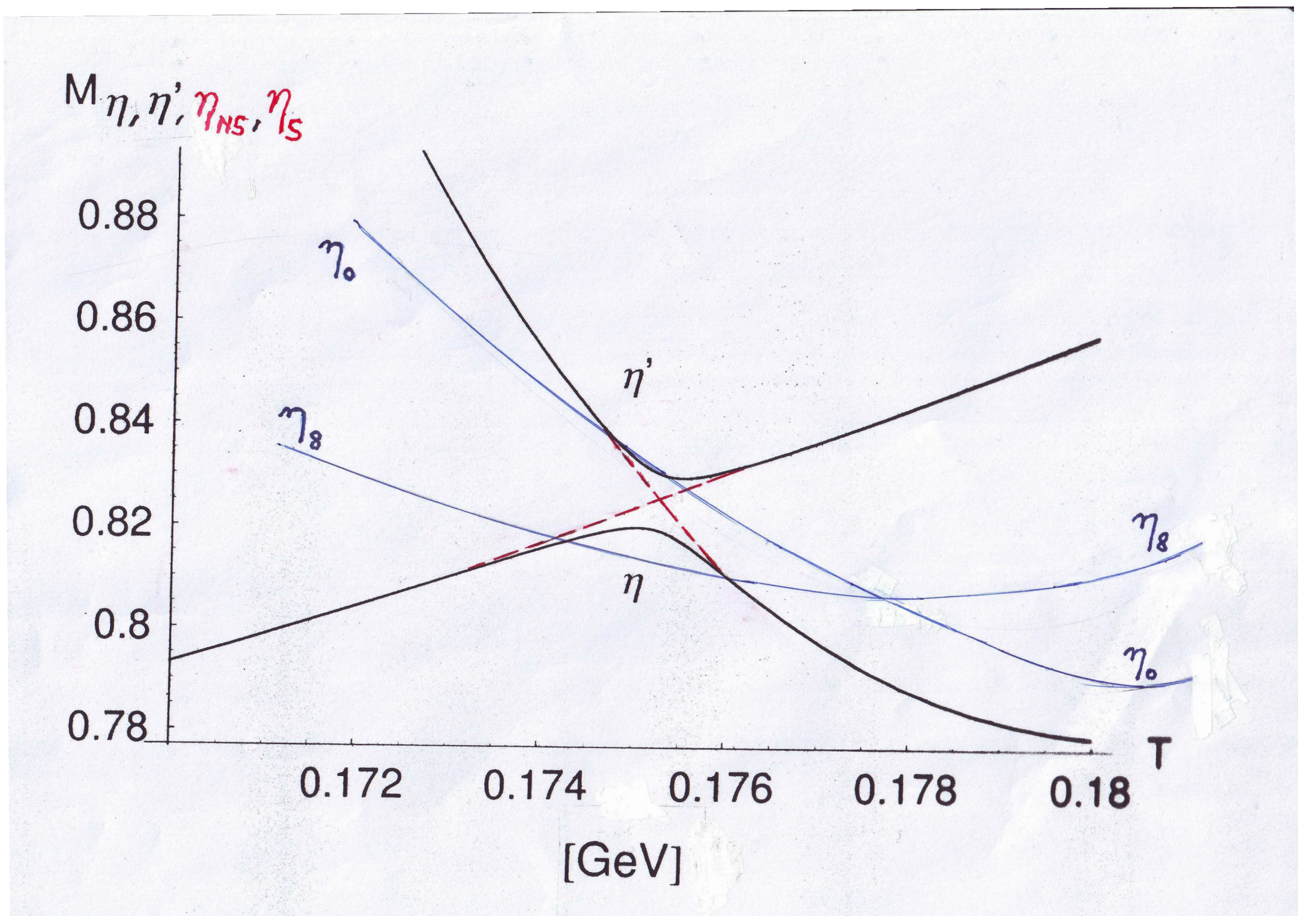
$\Rightarrow$ If they form a two-level system, any description of η and η' through their 2x2 mass matrix should also exhibit this property regarding their masses.

However, functional renormalization group (FRG) approach in a quark-meson truncation indicates that the assignment $M_{\eta'} > M_\eta$ changes as the $U_A(1)$ breaking is turned off.

$\Rightarrow$ **Re-examine η & η' being coupled in a standard two-level system**

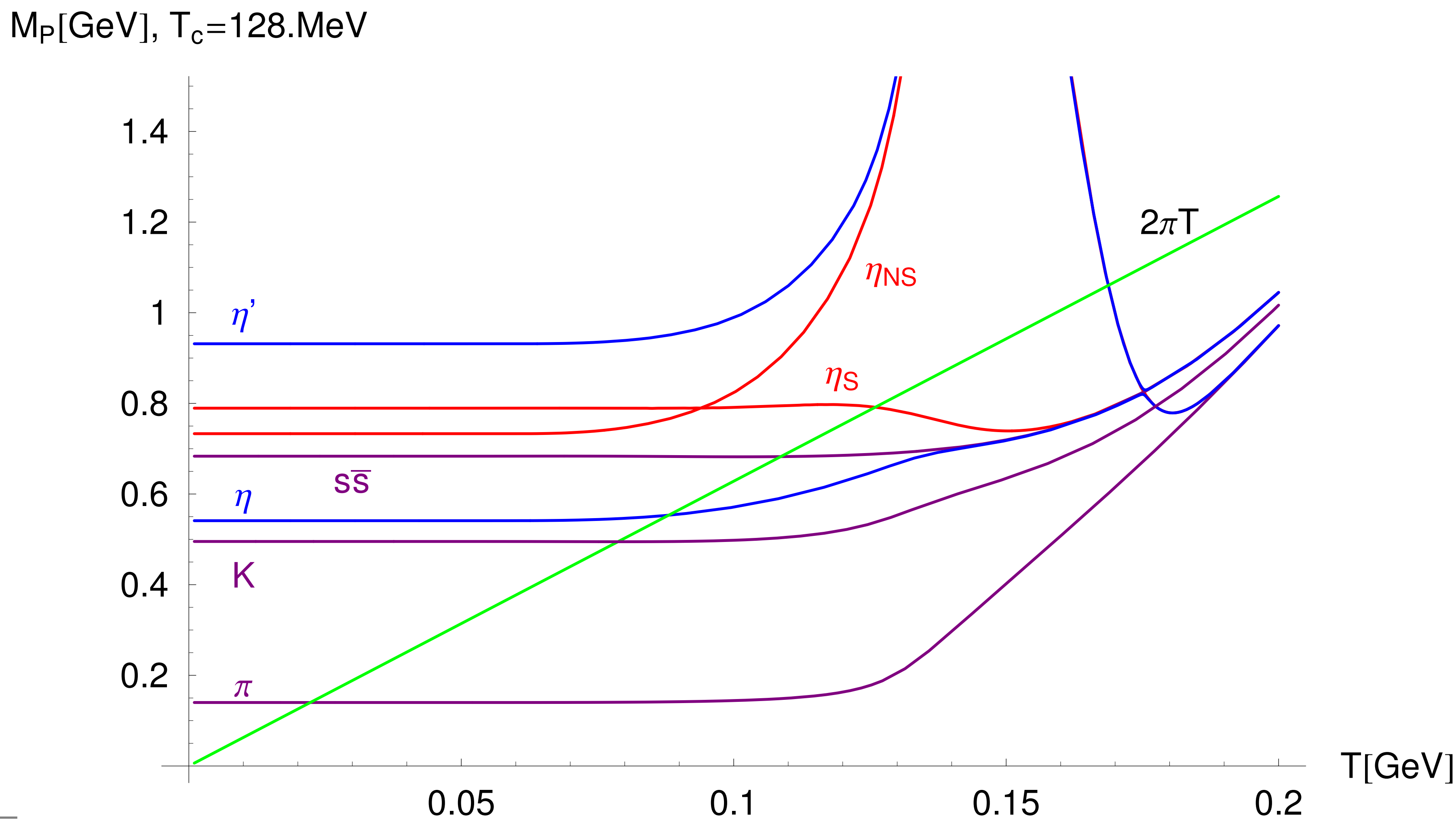
Mass matrices in η_{NS}-η_S or η_8-η_0 basis - diagonalize to get M_η, $M_{\eta'}$

$$\begin{pmatrix} M^2_{\eta_{NS}} & M^2_{\eta_S\eta_{NS}} \\ M^2_{\eta_{NS}\eta_S} & M^2_{\eta_S} \end{pmatrix} \text{ or } \begin{pmatrix} M^2_{\eta_8} & M^2_{\eta_8\eta_0} \\ M^2_{\eta_0\eta_8} & M^2_{\eta_0} \end{pmatrix} \xrightarrow{\textit{diagonalize}} \begin{pmatrix} M^2_{\eta} & 0 \\ 0 & M^2_{\eta'} \end{pmatrix}$$



$\leftarrow$ **This was enlarged anticrossing when $M_{\eta'}(T)$ is dominated by $\chi_{\rm YM}(T)/f_\pi^2(T)$**

but, **just to show anticrossing better!** Experiment EXCLUDED this scenario for $M_{\eta'}(T)$ (for $T \to T_{Chiral}$) [Horvatić&al, PRD**76** (2007) 0960]:

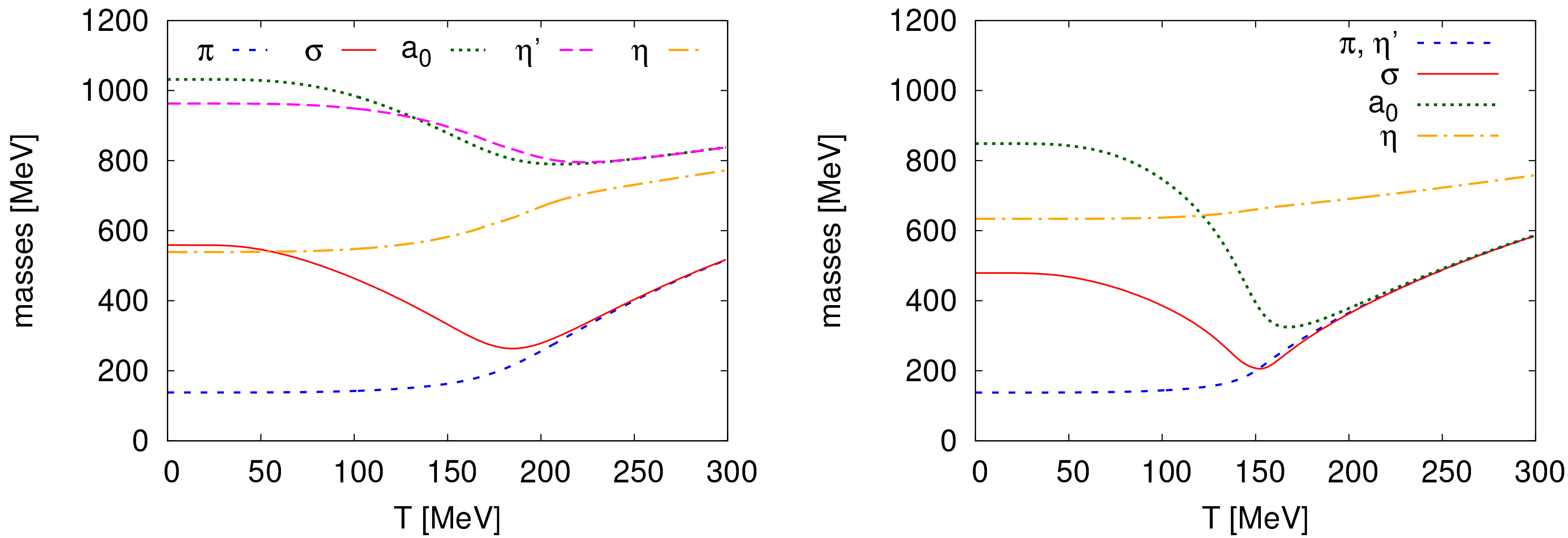


In the FRG approach to the quark-meson truncation [Mitter & Schaefer, PRD89 (2014) 054027]:

the $U_A(1)$ breaking implemented in the effective Lagrangian through the 't Hooft determinantal interaction ξ with the coupling strength c:

$$c\,\xi = c\,\left(\det\left[\Sigma\right] + \det\left[\Sigma^\dagger\right]\right)\ , \text{ where } \ \Sigma = T^a\left(\sigma_a + i\pi_a\right)$$

First panel: $c \neq 0$ necessary for the empirical, very large mass of η':



Second panel: no $U_A(1)$ breaking, due to $c = 0$. η changed little, but η' is now degenerate with π, i.e., lighter than η [see Weinberg PRD11 (1975) 3583].
Thus, varying of the coupling parameter c between 0 and its phenomenological value would lead to level crossing for any T.

2. Why $\eta_0 \approx \eta'$ has an anomalous piece of mass:

$U_A(1)$ symmetry is broken by nonabelian ("gluon") axial anomaly: even in the chiral limit (ChLim, where $m_q \to 0$),

$$\partial_\alpha \bar{\psi}(x)\gamma^\alpha \gamma_5 \frac{\lambda^0}{2}\psi(x) \propto F^a(x)\cdot \widetilde{F}^a(x) \equiv \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) \neq 0\,.$$

This breaks the $U_A(1)$ symmetry of QCD and precludes the 9^{th} Goldstone pseudoscalar meson $\Rightarrow$ very massive η': even in ChLim, where $m_\pi, m_K, m_\eta \to 0$, still ('ChLim WVR')

$$0 \neq \Delta M^2_{\eta_0} = \Delta M^2_{\eta'} = \frac{(A = \text{qty.dim.mass})^4}{(\text{“}f_{\eta'}\text{”})^2} = \frac{6\,\chi_{\text{YM}}}{f_\pi^2} + O(\frac{1}{N_c})$$

$$\text{Out of ChLim}: \quad {M_{\eta'}}^2 + {M_\eta}^2 - 2\,{M_K}^2 = \frac{2N_f}{f_\pi^2}\,\chi_{\text{YM}} \quad \left(+O(\frac{1}{N_c})\right)$$

Anomalous part of η_0 mass: $\Delta M_{\eta_0}^2 = \chi_{\rm YM}\,\frac{2N_f}{f_\pi^2} + O(\frac{1}{N_c})$

- Except η-η', pseudoscalars are qualitatively understood at $T = 0$ and $T > 0$, *e.g.*, in the $q\bar{q}$ bound-state Dyson-Schwinger (DS) approach

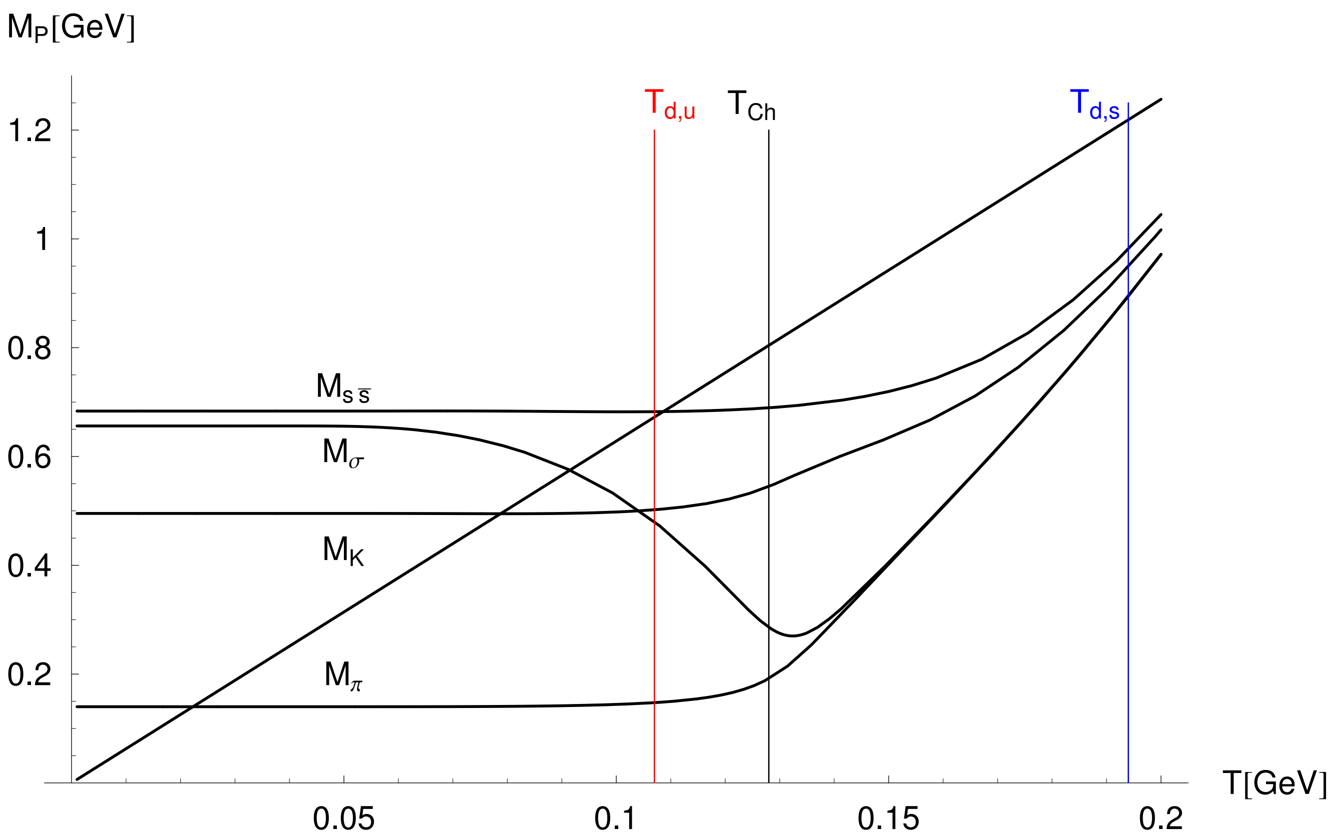
QCD chiral behavior (reproduced by DS approach) of the non-anomalous parts of masses of light $q\bar{q}'$ pseudoscalars (i.e., all parts except ΔM_{η_0}) : $M_{q\bar{q}'}^2 = \mathrm{const}\,(m_q + m_{q'}),\;\; (q, q' = u, d, s)$.

$\Rightarrow$ non-anomalous parts of the masses in WVR cancel:
$M_{\eta'}{}^2 + M_\eta{}^2 - 2\,M_K{}^2 \approx \Delta M_{\eta_0}{}^2\,,$ approx. as in ChLim WVR

$$\chi = \int d^4x\;\langle 0|Q(x)Q(0)|0\rangle\;, \qquad Q(x) = \frac{g^2}{64\pi^2}\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x)$$

- $Q(x)$ = topological charge density operator
- In WV rel., χ is not the one of the full QCD with quarks, but the pure-glue, YM one, $\chi_{\rm YM} \leftrightarrow \chi_{quenched}$.

An illustration of 'non-anomalous' meson M's(T) by simple 'separable' DS model:



- 'Deconfinement' $T_{d,q}$ from S_q pole - very different $T_{d,u}$, $T_{d,s}$ ... can be cured/synchronized with $T_{\rm Ch}(= T_{cri})$ by Polyakov loop
- But interplay of $U_A(1)$ anomaly and DChSB makes η and η' harder to understand both at $T = 0$ and $T > 0$

3. Should η and η' really form a two-state system?

- Pseudoscalar mesons of light quarks $q = u, d, s$ are (almost) Goldstone bosons of DChSB, so for $m_{u,d,s} \to 0$ also vanishing meson masses^2 $M^2_{u\bar{d}} = M^2_{\pi^+}, M^2_{u\bar{s}} = M^2_K, ..., \hat{M}^2_{NA} = \mathrm{diag}(M^2_{u\bar{u}}, M^2_{d\bar{d}}, M^2_{s\bar{s}})$ QCD chiral behavior reproduced correctly by Dyson-Schwinger-Bethe-Salpeter approach (DS) – except anomalously heavy η' !

- $|u\bar{d}\rangle = |\pi^+\rangle, |u\bar{s}\rangle = |K^+\rangle$, ... but $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ do not correspond to any physical particles (at $T = 0$ at least!), although in the isospin limit (adopted from now on) $M_{u\bar{u}} = M_{d\bar{d}} = M_{u\bar{d}} = M_\pi$. $I =$ good Q.no. $\Rightarrow$ recouple into "more physical" $I_3 = 0$ octet-singlet basis

$$
\begin{aligned}
I = 1 \qquad |\pi^0\rangle &= \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)\ , \\
\text{but}\ \ I = 0 \qquad |\eta_8\rangle &= \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle) \approx |\eta\rangle \quad \text{mixes with} \\
I = 0 \qquad |\eta_0\rangle &= \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) \approx |\eta'\rangle \quad \text{...too heavy for GB}\ .
\end{aligned}
$$

Physical η and η' must have a diagonal mass matrix

- the "non-anomalous" (chiral-limit-vanishing!) part of the mass-squared matrix of π^0 and η's is (in π^0-η_8-η_0 basis)

$$\hat{M}^2_{NA} = \begin{pmatrix} M_\pi^2 & 0 & 0 \\ 0 & M_{88}^2 & M_{80}^2 \\ 0 & M_{08}^2 & M_{00}^2 \end{pmatrix} \overset{\text{diagonalization}}{\underset{\text{U}_\text{A}(1)\text{ problem}}{\Longrightarrow}} \begin{pmatrix} M_\pi^2 & 0 & 0 \\ 0 & M_\pi^2 & 0 \\ 0 & 0 & M_{s\bar{s}}^2 \end{pmatrix}$$

$$M_{88}^2 \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_8 \rangle = \frac{2}{3}(M_{s\bar{s}}^2 + \frac{1}{2} M_\pi^2), \qquad M_{00}^2 \equiv \langle \eta_0 | \hat{M}^2_{NA} | \eta_0 \rangle = \frac{2}{3}(\frac{1}{2} M_{s\bar{s}}^2 + M_\pi^2),$$

$$M_{80}^2 \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_0 \rangle = M_{08}^2 = \frac{\sqrt{2}}{3}(M_\pi^2 - M_{s\bar{s}}^2)$$

- What reproduces M_π & M_K cannot also $M_\eta = 548$ & $M_{\eta'} = 958$ MeV!
- $\hat{M}^2_{NA}$ not enough! To avoid the U$_A$(1) problem, one must break the U$_A$(1) symmetry (as it is destroyed by the gluon anomaly) at least at the level of the masses.

Gluon anomaly is not accessible to ladder approximation

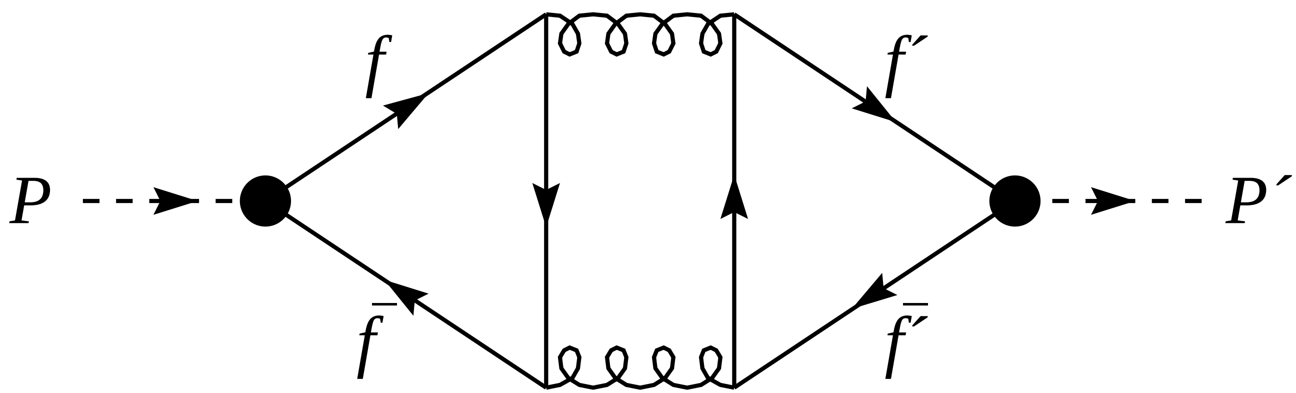
- All masses in $\hat{M}^2_{NA}$ are calculated in the ladder approx., which cannot include the gluon anomaly contributions.
- Large N_c: the gluon anomaly suppressed as $1/N_c$! $\rightarrow$ Include its effect just at the level of masses: break the $U_A(1)$ symmetry and avoid the $U_A(1)$ problem by shifting the η_0 (squared) mass by anomalous contribution 3β.
- complete mass matrix is then $\hat{M}^2 = \hat{M}^2_{NA} + \hat{M}^2_A$ where

$$\hat{M}^2_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\beta \end{pmatrix} \quad \text{does not vanish in the chiral limit.}$$

$3\beta = \Delta M^2_{\eta_0}$ = the anomalous mass^2 of η_0 [in SU(3) limit incl. ChLim] is related to the YM topological susceptibility. Fixed by phenomenology or (here) taken from the lattice.

Transitions related to the $U_A(1)$ anomaly

- Transitions between hidden flavors $|q\bar{q}\rangle \to |q'\bar{q}'\rangle$ $(q, q' = u, d, s)$



- Diamond graph: **just the simplest example** of a transition $|q\bar{q}\rangle \to |q'\bar{q}'\rangle$ $(q, q' = u, d, s)$, contributing to the anomalous masses in the η-η' complex, but not included in the interaction kernel in the ladder approximation.

Anomaly & flavor breaking conspire to make $\eta \approx \eta_8$ and $\eta' \approx \eta_0$:

- rewrite the anomalous part $\hat{M}_A^2$ in the $q\bar{q}$ basis $|u\bar{u}\rangle$, $|d\bar{d}\rangle$, $|s\bar{s}\rangle$:

$$\hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{breaking}]{\text{flavor}} \hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & X \\ 1 & 1 & X \\ X & X & X^2 \end{pmatrix}$$

- We introduced effects of the flavor breaking on the anomaly-induced transitions $|q\bar{q}\rangle \to |q'\bar{q}'\rangle$ $(q, q' = u, d, s)$: $s\bar{s}$ transition suppression is estimated by $X \approx f_\pi / f_{s\bar{s}} < 1$. Then, $\hat{M}_A^2$ in the octet-singlet basis is

$$\hat{M}_A^2 = \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3}(1-X)^2 & \frac{\sqrt{2}}{3}(2 - X - X^2) \\ 0 & \frac{\sqrt{2}}{3}(2 - X - X^2) & \frac{1}{3}(2+X)^2 \end{pmatrix}$$

- π^0 remains decoupled $\Rightarrow$ can restrict to 2×2 submatrix of the etas.
- Off-diagonal elements reduced by $\frac{\sqrt{2}}{3}(2 - X - X^2)$ in the complete mass matrix $\hat{M}^2 = \hat{M}_{NA}^2 + \hat{M}_A^2 \approx$ roughly diagonal in the η_8-η_0 basis!

Mass matrix and mixing in *NS*–*S* basis

- nonstrange (*NS*) – strange (*S*) basis:

$$|\eta_{NS}\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}}|\eta_8\rangle + \sqrt{\frac{2}{3}}|\eta_0\rangle \ ,$$
$$|\eta_S\rangle = |s\bar{s}\rangle = -\sqrt{\frac{2}{3}}|\eta_8\rangle + \frac{1}{\sqrt{3}}|\eta_0\rangle \ .$$

- the η–η' mass matrix in this basis:

$$\hat{M}^2 = \begin{pmatrix} M^2_{\eta_{NS}} & M^2_{\eta_S \eta_{NS}} \\ M^2_{\eta_{NS}\eta_S} & M^2_{\eta_S} \end{pmatrix} = \begin{pmatrix} M^2_{u\bar{u}} + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} M^2_\eta & 0 \\ 0 & M^2_{\eta'} \end{pmatrix}$$

- *NS*–*S* mixing relations – states rotation diagonalizing $\hat{M}^2$:

$$|\eta\rangle = \cos\phi|\eta_{NS}\rangle - \sin\phi|\eta_S\rangle \ , \quad |\eta'\rangle = \sin\phi|\eta_{NS}\rangle + \cos\phi|\eta_S\rangle \ .$$

$$\theta = \phi - \arctan\sqrt{2}$$

Finally, fix anomalous contribution to η-η':

- Equal traces of diagonalized & non-diagnlz. $\hat{M}^2$ demand 1^{st} eqality in

$$\beta(2{+}X^2) = M_\eta^2{+}M_{\eta'}^2{-}2M_K^2 = \frac{2N_f}{f_\pi^2}\chi_{\rm YM} \quad (2^{\rm nd}\text{equality} = \text{WV rel.})$$

- requiring that the experimental trace $(M_\eta^2 + M_{\eta'}^2)_{exp}$ ≈ 1.22 GeV^2 be reproduced by the theoretical $\hat{M}^2$, yields $\beta_{\rm fit} = \frac{1}{2+X^2}[(M_\eta^2 + M_{\eta'}^2)_{exp} - (M_{u\bar{u}}^2 + M_{s\bar{s}}^2)]$
- Or, get β from lattice $\chi_{\rm YM}$! Then no free parameters!
- then, can calculate the NS-S mixing angle ϕ

$$\tan 2\phi = \frac{2\, M_{\eta_S\eta_{NS}}^2}{M_{\eta_S}^2 - M_{\eta_{NS}}^2} = \frac{2\sqrt{2}\beta X}{M_{\eta_S}^2 - M_{\eta_{NS}}^2} \qquad \text{and}$$

$$M_{\eta_{NS}}^2 = M_{u\bar{u}}^2{+}2\beta = M_\pi^2{+}2\beta, \quad M_{\eta_S}^2 = M_{s\bar{s}}^2{+}\beta X^2 = M_{s\bar{s}}^2{+}\beta\,\frac{f_\pi^2}{f_{s\bar{s}}^2}$$

Physical η, η' eigenmasses – of the two-level type:

- The diagonalization of the NS-S mass matrix then gives

$$
\begin{aligned}
M_\eta^2 &= \cos^2\phi\, M_{\eta_{NS}}^2 - M_{\eta_S\eta_{NS}}^2 \sin 2\phi + \sin^2\phi\, M_{\eta_S}^2 \qquad (\text{note } M_{\eta_S\eta_{NS}}^2 = \sqrt{2}\beta X) \\
M_{\eta'}^2 &= \sin^2\phi\, M_{\eta_{NS}}^2 + M_{\eta_S\eta_{NS}}^2 \sin 2\phi + \cos^2\phi\, M_{\eta_S}^2
\end{aligned}
$$

- Equivalently, secular determinant $\Rightarrow$ the eigenvalues of 2$\times$2 matrix:

$$
\begin{aligned}
M_\eta^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 - \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 4\, M_{\eta_S\eta_{NS}}^4}\,\right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) - \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2}\,\right] \\
M_{\eta'}^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 + \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 4\, M_{\eta_S\eta_{NS}}^4}\,\right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) + \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2}\,\right]
\end{aligned}
$$

This exhibits the Goldstone-b. character of η in ChLim & degeneracy with π in $SU(3)$ limit

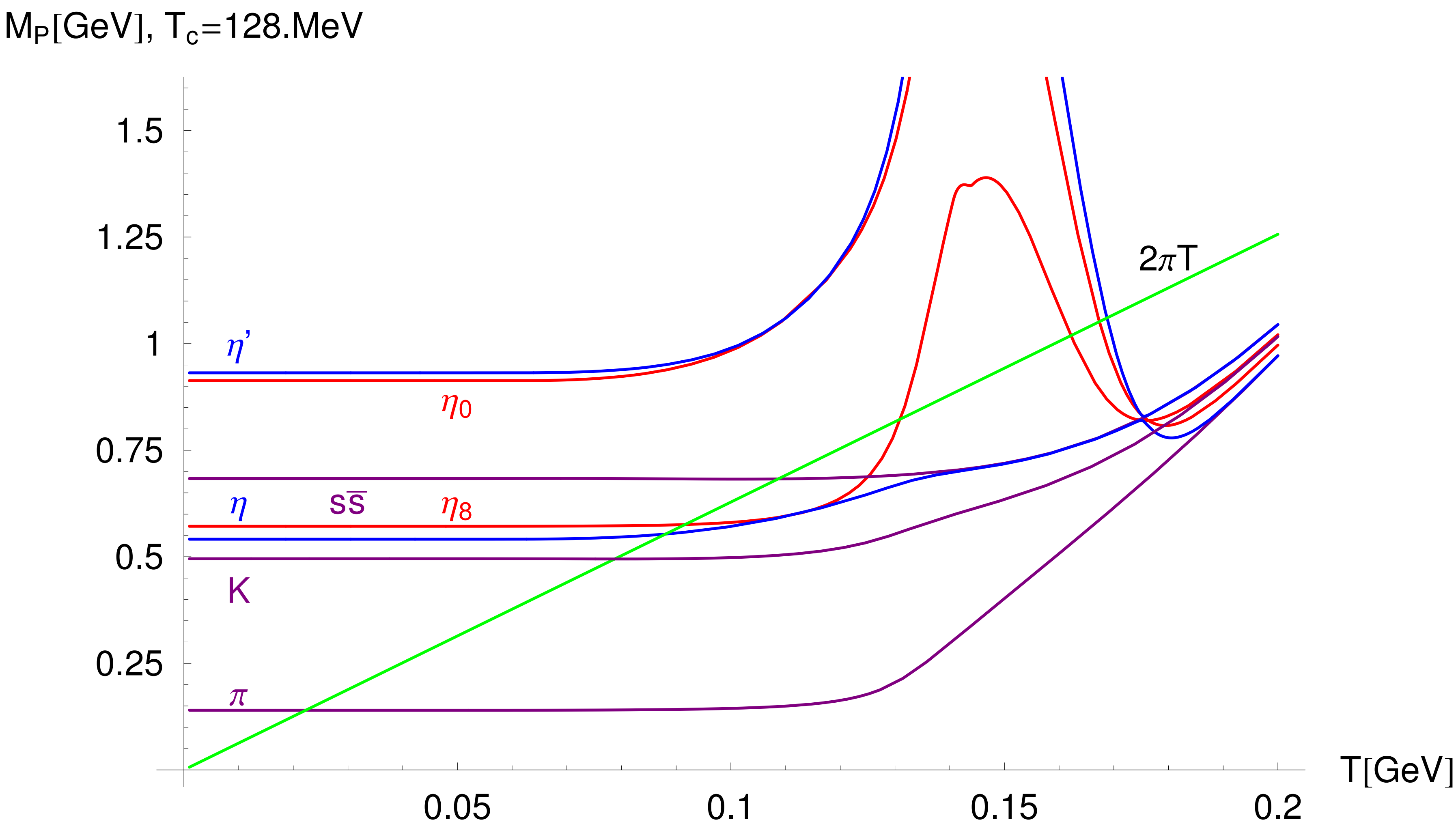
Separable model results on η and η' at $T = 0$

	$\beta_{\rm fit}$	$\beta_{\rm latt.}$	Exp.
θ	-12.22°	-13.92°	
M_η [MeV]	548.9	543.1	547.75
$M_{\eta'}$[MeV]	958.5	932.5	957.78
X	0.772	0.772	
3β [GeV^2]	0.845	0.781	

- $X = f_\pi / f_{s\bar{s}}$ as well as the whole $\hat{M}^2_{NA}$ (consisting of M_π and $M_{s\bar{s}}$) are calculated model quantities.
- $\beta_{\rm latt.}$ was obtained from $\chi_{\rm YM}(T=0) = (175.7 \text{ MeV})^4$
- But is an extension to high T possible, as there is a large mismatch of characteristic temperature scales of the pure-gauge YM ($T_c \sim 270$ MeV) vs. full QCD ($T_c \sim 160$ MeV) with quarks?
- Concretely in WVR, $\chi_{\rm YM}$ is more T-resistant than QCD quantities $M_{\eta,\eta',K}$ and f_π. Does WVR become unusable as T approaches the (pseudo-)critical temperatures of full QCD, such as $T \sim T_{\rm Ch}$?

Scenario that $2N_f\chi_{\rm YM}(T)/f_\pi^2(T)$ dominates $M_{\eta'}$ excluded, but actually only for $T \to T_{Chiral}$

[Horvatić&al, PRD**76** (2007) 0960]:



Clash with phenomenology removed by another relation connecting YM and QCD!

Shore's generalization of WV valid to all orders in $1/N_c$

- WV rel. – lowest order in $1/N_c$ – improved like this:

$$(f_{\eta'}^0)^2 M_{\eta'}^2 + (f_\eta^0)^2 M_\eta^2 \;=\; \frac{1}{3}\left(f_\pi^2 M_\pi^2 + 2 f_K^2 M_K^2\right) + 6A \qquad (1)$$

$$f_{\eta'}^0 f_{\eta'}^8 M_{\eta'}^2 + f_\eta^0 f_\eta^8 M_\eta^2 \;=\; \frac{2\sqrt{2}}{3}\left(f_\pi^2 M_\pi^2 - f_K^2 M_K^2\right) \qquad (2)$$

$$(f_{\eta'}^8)^2 M_{\eta'}^2 + (f_\eta^8)^2 M_\eta^2 \;=\; -\frac{1}{3}\left(f_\pi^2 M_\pi^2 - 4 f_K^2 M_K^2\right) \qquad (3)$$

A is the full QCD topological charge parameter (replacing $\chi_{\rm YM}$ in WV)

$$A = \frac{\chi}{1 + \chi\left(\frac{1}{\langle \bar{u}u\rangle\, m_u} + \frac{1}{\langle \bar{d}d\rangle\, m_d} + \frac{1}{\langle \bar{s}s\rangle\, m_s}\right)} \qquad (4)$$

= seemed hard to calculate on lattice (maybe easier today?) ...

However, it is known that $A = \chi_{\rm YM} + \mathcal{O}(\frac{1}{N_c})$ (at $T=0$)

Approximating the full QCD topological charge parameter A

Replacing 3 different condensates by the chiral one, $\langle\bar{q}q\rangle_0$, reduces the full QCD topological charge A (4) to the combination $\widetilde{\chi}$ on the RHS of Leutwyler-Smilga relation:

$$\chi_{\rm YM} = \frac{\chi}{1+\frac{\chi}{\langle\bar{q}q\rangle_0}\sum\limits_{q=u,d,s}\frac{1}{m_q}} \to \widetilde{\chi}(T,\mu) = \frac{\langle\bar{q}q(T,\mu)\rangle_0}{\sum\limits_{q=u,d,s}\frac{1}{m_q}} + corr's \approx A(T,\mu)$$

because of Di Vecchia-Veneziano result for small m_q:

$$\chi = -\frac{m\,\langle\bar{q}q\rangle_0}{N_f} + \mathrm{corrections}(m)\,,$$

(Previously, we only **conjectured** $\chi_{\rm YM}(T) \to \widetilde{\chi}(T)$ [Benić& al, Phys.Rev.D84 (2011)016006].)

$\Rightarrow$ The quark condensates $\langle\bar{q}q(T,\mu)\rangle$, and not the pure-gauge quantity $\chi_{\rm YM}$, determine the T (and μ) dependence of (partial) restoration of $U_A(1)$. $\Rightarrow$ Linked with the chiral restoration!

T-dependence of χ and $\widetilde{\chi}$

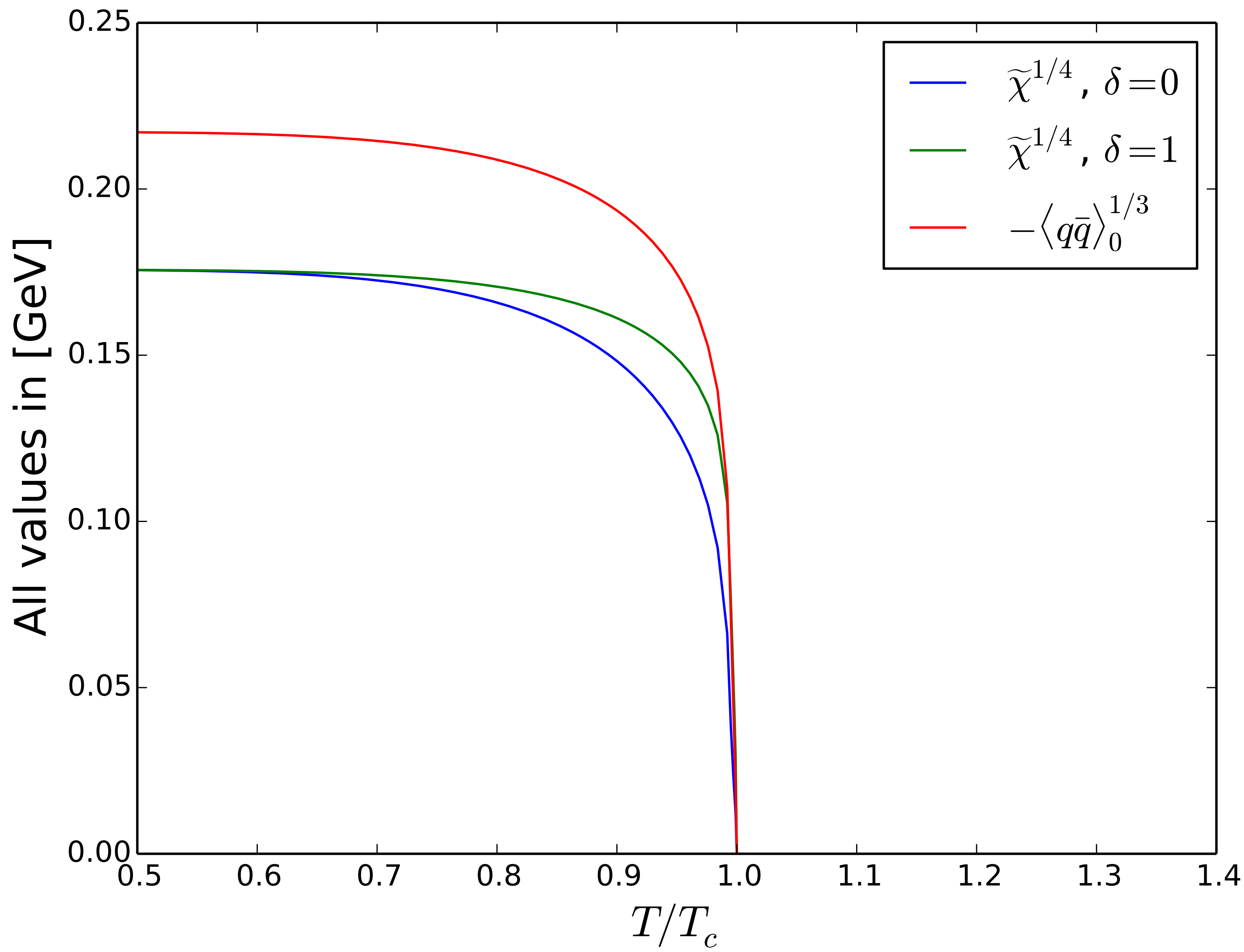
- Extending the light-quark full-QCD topol. susceptibility χ is somewhat uncertain, as there is no guidance from lattice [unlike for $\chi_{\rm YM}(T)$].
- The leading term in Di Vecchia-Veneziano relation $\propto \langle\bar{q}q\rangle_0(T)$ very plausibly, but for the correction term we have to explore a range of Ansätze, i.e.,

$$\chi(T) = -\frac{m\,\langle\bar{q}q\rangle_0(T)}{N_f} + \mathcal{C}(m)\left[\frac{\langle\bar{q}q\rangle_0(T)}{\langle\bar{q}q\rangle_0(T=0)}\right]^\delta, \quad (0 \le \delta < 2).$$

Then, $\quad \widetilde{\chi}(T) =$

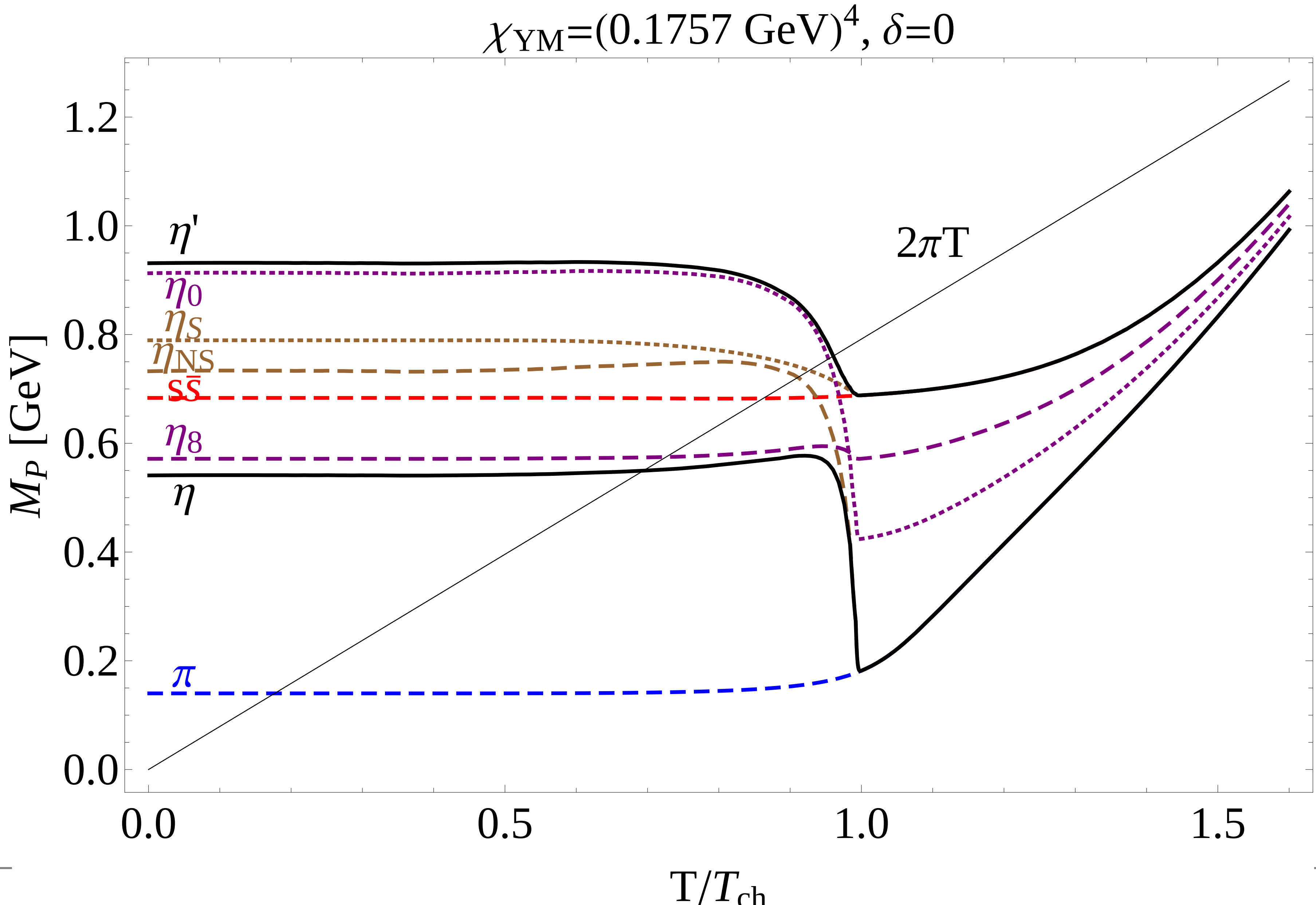
$$= \frac{\langle\bar{q}q\rangle_0(T)}{\sum_{q=u,d,s}\left(\frac{1}{m_q}\right)}\left\{1 - \frac{\langle\bar{q}q\rangle_0(T)}{\sum_{q=u,d,s}\left(\frac{1}{m_q}\right)}\frac{1}{\mathcal{C}(m)}\left[\frac{\langle\bar{q}q\rangle_0(T=0)}{\langle\bar{q}q\rangle_0(T)}\right]^\delta\right\}.$$

Chiral condensate $\langle q\bar{q}\rangle_0(T)$ and resulting $\widetilde{\chi}(T)$

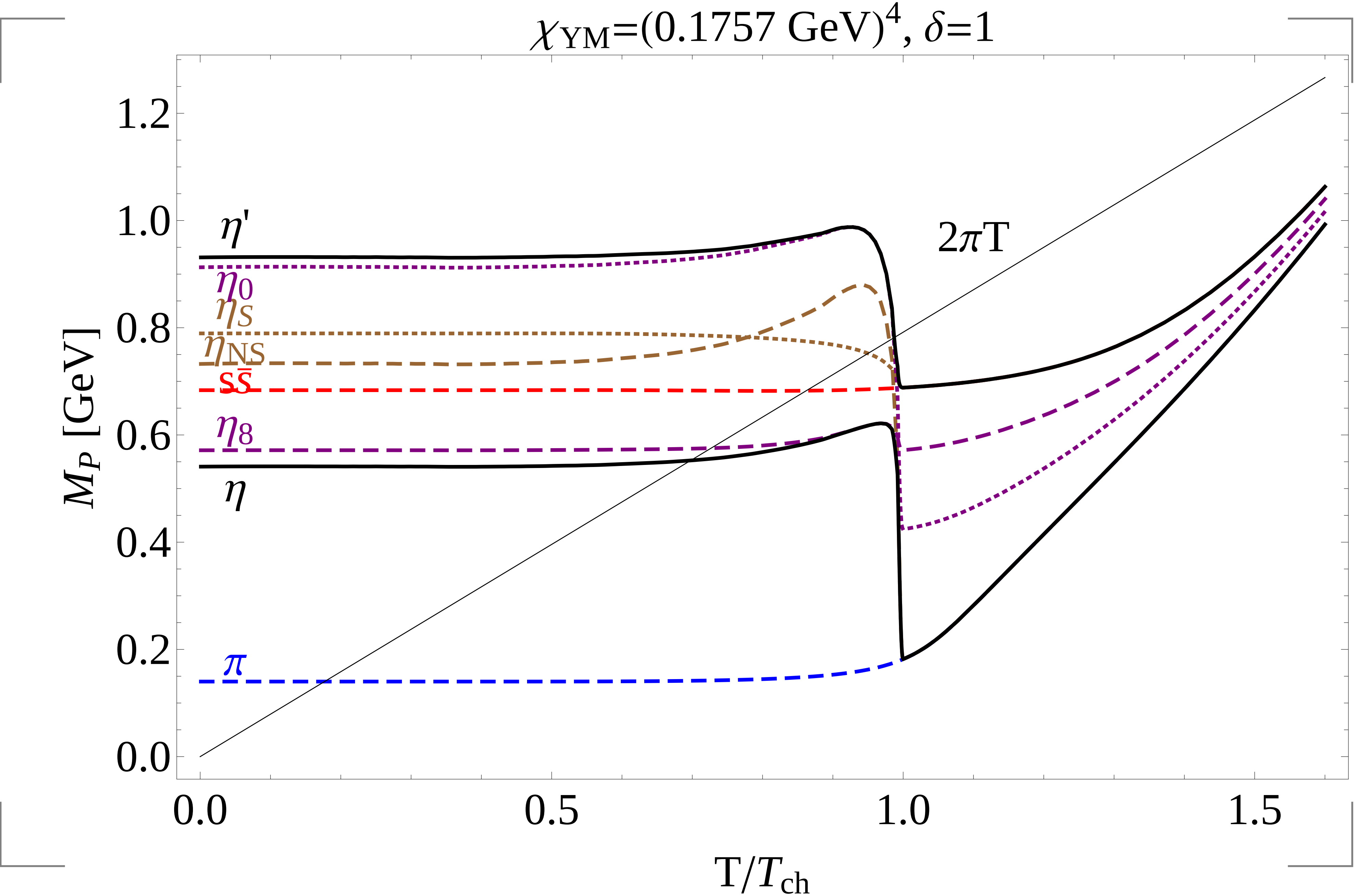


Case 1: T-independent correction term in χ

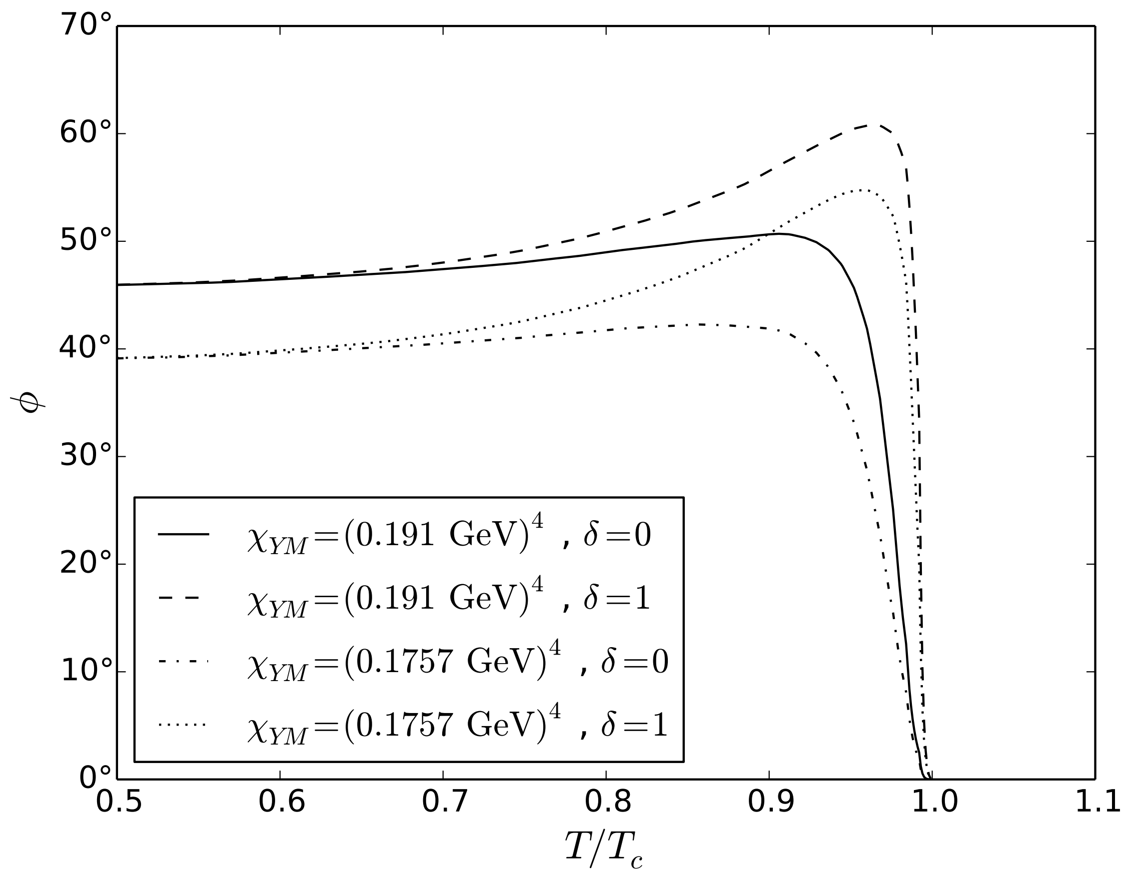
[Benić, Horvatić, Kekez and Klabučar, Phys. Rev. D **84** (2011) 016006.]



Case 2: Strongly T-dependent correction term $\propto \langle \bar{q}q \rangle_0(T)$



T-dependence of the NS-S mixing angle $\phi(T)$



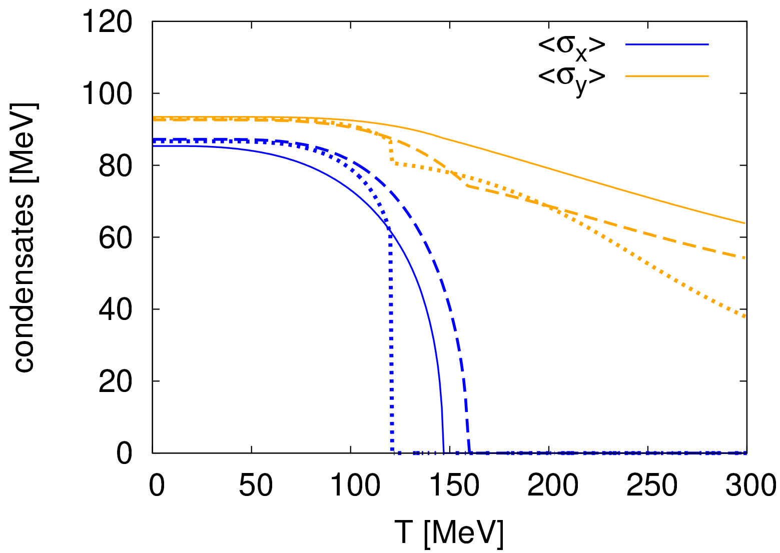
$\phi(T)$ for the cases of the T-independent correction term in $\chi(T)$ ($\delta = 0$) and the correction term in $\chi(T)$ behaving like the leading term, i.e., like the chiral condensate ($\delta = 1$), and for two values of $\widetilde{\chi}(T = 0) = \chi_{YM}$.

A functional renormalization group (FRG) approach

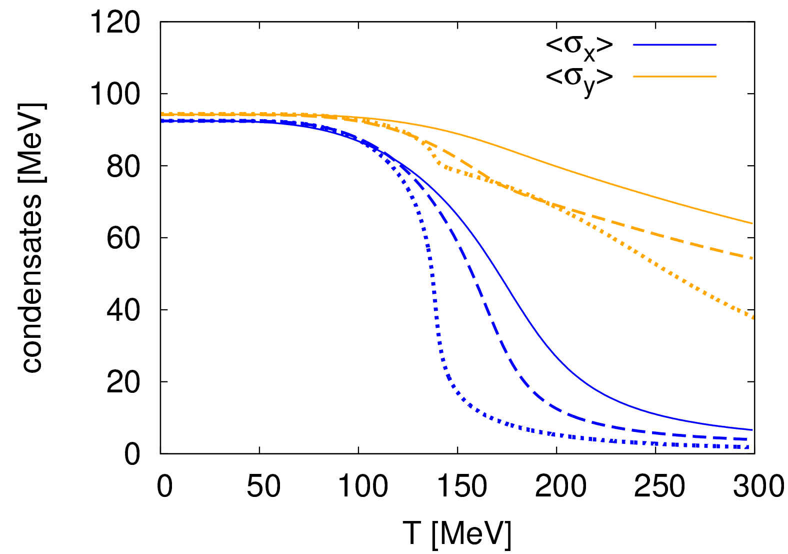
[M. Mitter & B. J. Schaefer, Phys. Rev. D **89** (2014) 5, 054027]: Axial anomaly & chiral symmetry investigated by a FRG approach in a three flavor quark-meson truncation.

- Chiral order parameters: quark condensates $\langle \bar{q}q \rangle$ related via bosonization to vacuum expectation values of the scalar-isoscalar mesonic fields $\sigma_{x,y} = \sigma_{NS,S}$:

$$\langle \Sigma \rangle = \mathrm{diag}(\langle \sigma_x \rangle/2, \langle \sigma_x \rangle/2, \langle \sigma_y \rangle/\sqrt{2})$$



NS (x) & S (y) condensates in the nonstrange chiral limit



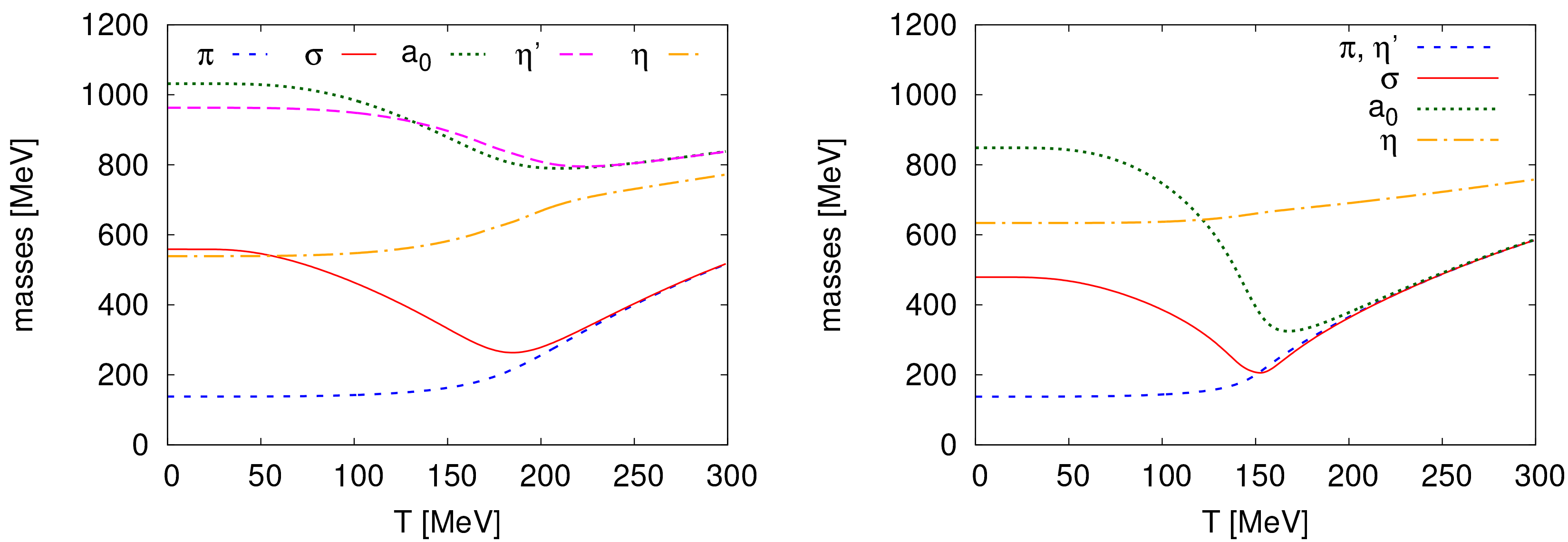
With explicit chiral breaking for all flavors

In the FRG approach to the quark-meson truncation, condensates dominate $U_A(1)$ breaking

The $U_A(1)$ breaking implemented in the effective Lagrangian through the 't Hooft determinantal interaction ξ with the coupling strength c:

$$c\,\xi = c\,\left(\det\left[\Sigma\right] + \det\left[\Sigma^\dagger\right]\right)\ , \text{ where } \ \Sigma = T^a\left(\sigma_a + i\pi_a\right)$$

First panel: $c \neq 0$ contributes to good agreement with the present data:

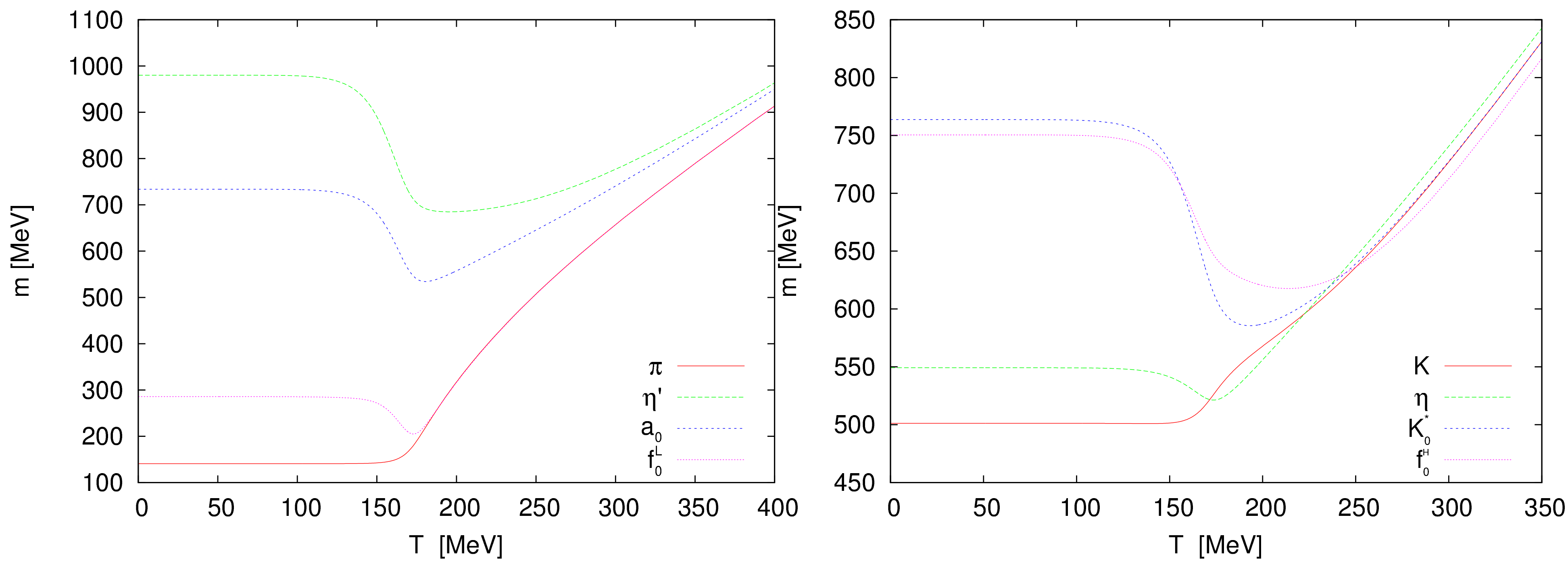


Second panel: no $U_A(1)$ breaking, due to $c = 0$. η changed little, but η' is now degenerate with π, i.e., lighter than η.

Thus, varying of the coupling parameter c between 0 and its phenomenological value would lead to level crossing for any T.

In between FRG & mass matrix: linear Σ-model with quarks, Polyakov loop & (axi-)vector mesons:

Larger drop in $M_{\eta'}$, but a smaller one also in M_η, in the preliminary results by P. Kovacs, Z. Szep and G. Wolf, J. Phys. Conf. Ser. **599** (2015) 1, 012010



... should be examined for possible crossing (?) in the $U_A(1)$ limit ...

4. Discussion and Summary

- It is clear that crossing of the M_η and M_η eigenvalues cannot happen IF the traditional description through the 2×2 η-η' mass matrix is not flawed.
- The general condition enabling crossings of eigenvalues is belonging to different irreducible representations of the pertinent symmetry group. Then, the possible objection here is that the assumption of nonet symmetry was used in forming this matrix for the two states, the $SU(3)$ octet member η_8, and the $SU(3)$ singlet η_0. However, what is then missing is a conserved quantity which would prevent the mixing of the isoscalars η_8 and η_0 – such as strangeness, charge and (approx.) isospin, preventing the mixing of etas with K's and π's.
- Besides, the usage of nonet symmetry is founded in the regime without $U_A(1)$ symmetry breaking. On the other hand, we saw that the presence of this breaking was needed for an approximate identification of η with η_8 described by GMO formula, which is needed for claiming that Weinberg's argument cannot pertain to η.
- Conclusion: the η–η' complex remains an ordinary two-level system, while the mass crossing seemingly exhibited by a FRG approach in a quark-meson truncation is probably an artifact of this level of truncation (which, however, can be systematically improved).
- Besides the main issue of (anti)crossing, I presented the results of the model where the mass contribution of the axial anomaly is expressed through $q\bar{q}$ condensates, and thus diminishes as the chiral symmetry is restored. The excessive drop of M_η already at $T \to T_{Chiral}$ is the consequence of the exclusive usage of the chiral $q\bar{q}$ condensate and will be mended by using massive $q\bar{q}$ condensates.

Additional slides

On Shore's generalization of WV relation

and
its combining with the
Feldmann–Kroll–Stech (FKS) scheme

η' and η have 4 independent decay constants

$$f^0_{\eta'}, f^8_\eta, f^0_\eta, f^8_{\eta'} : \quad \langle 0|A^{a\,\mu}(x)|P(p)\rangle = i f^a_P\, p^\mu e^{-ip\cdot x}, \;\; a=8,0; \;\; P=\eta,\eta' .$$

- Equivalently, one has 4 related but different constants $f^{NS}_{\eta'}, f^{NS}_{\eta}, f^{S}_{\eta}, f^{S}_{\eta'}$ if instead of octet and singlet axial currents ($a=8,0$) one takes this matrix element of the nonstrange-strange axial currents ($a=NS,S$)

$$A^\mu_{NS}(x) = \frac{1}{\sqrt{3}} A^{8\,\mu}(x) + \sqrt{\frac{2}{3}} A^{0\,\mu}(x) = \frac{1}{2}\left(\bar{u}(x)\gamma^\mu\gamma_5 u(x) + \bar{d}(x)\gamma^\mu\gamma_5 d(x)\right) ,$$

$$A^\mu_{S}(x) = -\sqrt{\frac{2}{3}} A^{8\,\mu}(x) + \frac{1}{\sqrt{3}} A^{0\,\mu}(x) = \frac{1}{\sqrt{2}}\bar{s}(x)\gamma^\mu\gamma_5 s(x) ,$$

$$\begin{bmatrix} f^{NS}_\eta & f^S_\eta \\ f^{NS}_{\eta'} & f^S_{\eta'} \end{bmatrix} = \begin{bmatrix} f^8_\eta & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} ,$$

$$a,P=NS,S: \qquad \langle 0|A^\mu_{NS}(x)|\eta_{NS}(p)\rangle = i f_{NS}\, p^\mu e^{-ip\cdot x} , \qquad \langle 0|A^\mu_{NS}(x)|\eta_S(p)\rangle = 0 ,$$

$$a,P=NS,S: \qquad \langle 0|A^\mu_{S}(x)|\eta_{S}(p)\rangle = i f_{S}\, p^\mu e^{-ip\cdot x} , \qquad \langle 0|A^\mu_{S}(x)|\eta_{NS}(p)\rangle = 0 ,$$

- Note: in a DS approach, $f_{NS} = f_{u\bar{u}} = f_{d\bar{d}} = f_\pi$, $f_S = f_{s\bar{s}}$ are calculated quantities

Two Mixing Angles and FKS one-angle scheme

- Any 4 η-η' decay constants conveniently parametrized in terms of two decay constants and two angles:

$$f_\eta^8 = \cos\theta_8\, f_8\ , \qquad f_\eta^0 = -\sin\theta_0\, f_0\ , \qquad f_\eta^{NS} = \cos\phi_{NS}\, f_{NS}\ , \qquad f_\eta^S = -\sin\phi_S\, f_S\ ,$$

$$f_{\eta'}^8 = \sin\theta_8\, f_8\ , \qquad f_{\eta'}^0 = \cos\theta_0\, f_0\ , \qquad f_{\eta'}^{NS} = \sin\phi_{NS} f_{NS}\ , \qquad f_{\eta'}^S = \cos\phi_S f_S$$

- Big practical difference between 0-8 and NS-S schemes:

- while θ_8 and θ_0 differ a lot from each other and from $\theta \approx (\theta_8 + \theta_0)/2$, FKS showed that $\phi_{NS} \approx \phi_S \approx \phi$.

$$\begin{bmatrix} f_\eta^{NS} & f_\eta^S \\ f_{\eta'}^{NS} & f_{\eta'}^S \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} .$$

For four decay constants, can use FKS one-angle scheme!

- ϕ relates $\{f_\eta^8, f_{\eta'}^8, f_\eta^0, f_{\eta'}^0\}$ with $\{f_{NS}, f_S\} = \{f_\pi, f_{s\bar{s}}\}$:

$$\begin{bmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

- Some other useful relations between quantities of NS-S (FKS) and 0-8 schemes:

$$f_8 = \sqrt{\frac{1}{3} f_{NS}^2 + \frac{2}{3} f_S^2}\,, \qquad \theta_8 = \phi - \arctan\left(\frac{\sqrt{2} f_S}{f_{NS}}\right)\,,$$

$$f_0 = \sqrt{\frac{2}{3} f_{NS}^2 + \frac{1}{3} f_S^2}\,, \qquad \theta_0 = \phi - \arctan\left(\frac{\sqrt{2} f_{NS}}{f_S}\right)\,.$$

Solve numerically Shore's Eqs. (1)-(3) for $M_{\eta'}$, M_η, and ϕ:

Inputs:	$M_\pi, M_K, f_\pi = f_{\rm NS}, f_{s\bar{s}} = f_{\rm S}$ and f_K, calculated in 3 different DS models					
$\chi_{\rm YM}$	191^4	175.7^4	191^4	175.7^4	191^4	175.7^4
M_η	499.8	485.7	496.7	482.8	526.2	507.0
$M_{\eta'}$	931.4	815.8	934.9	818.4	983.2	868.7
ϕ	52.01°	46.11°	51.85°	46.07°	47.23°	40.86°
θ	-2.72°	-8.62°	-2.89°	-8.67°	-7.51°	-13.87°
θ_0	7.74°	1.84°	7.17°	1.39°	-0.33°	-6.69°
θ_8	-12.00°	-17.90°	-11.85°	-17.6°	-14.12°	-20.47°
f_0	108.8	108.8	107.9	107.9	101.8	101.8
f_8	122.6	122.6	121.1	121.1	110.7	110.7
f_η^0	-14.7	-3.5	-13.5	-2.6	0.6	11.9
$f_{\eta'}^0$	107.9	108.8	107.1	107.9	101.8	101.1
f_η^8	119.9	116.7	118.5	115.4	107.4	103.7
$f_{\eta'}^8$	-25.5	-37.7	-2.49	-37.6	-27.0	-38.7

(in D. Horvatić et al., Eur. Phys. J. A **38** (2008) 257.) $M_{\eta,\eta'}$ and f's in MeV, $\chi_{\rm YM}$ is in MeV^4.

The same is now reproduced analytically:

- Eqs. (1)-(3) $\Rightarrow$ two closed-form solutions for M_η, $M_{\eta'}$ and $\tan\phi$ in terms of $f_\pi, f_{s\bar{s}}, M_\pi$, M_K and A.
  The set reproducing the previous numerical results is:

$$\tan\phi = \frac{-2Af_\pi^2 + 4Af_{s\bar{s}}^2 - 2f_K^2 f_\pi^2 M_K^2 + f_\pi^4 M_\pi^2 + f_\pi^2 f_{s\bar{s}}^2 M_\pi^2 + \Delta}{4\sqrt{2}\,A\,f_\pi\,f_{s\bar{s}}}$$

$$M^2_{\eta,\eta'} = \frac{2Af_\pi^2 + 4Af_{s\bar{s}}^2 + 2f_K^2 f_\pi^2 M_K^2 - f_\pi^4 M_\pi^2 + f_\pi^2 f_{s\bar{s}}^2 M_\pi^2 \mp \Delta}{2f_\pi^2 f_{s\bar{s}}^2}$$

where $\quad \Delta^2 =$

$$32\,A^2\,f_\pi^2 f_{s\bar{s}}^2 + \left\{2A(f_\pi^2 - 2f_{s\bar{s}}^2) + f_\pi^2\,\left[2f_K^2 M_K^2 - (f_\pi^2 + f_{s\bar{s}}^2)M_\pi^2\right]\right\}^2$$

[Benić, Horvatić, Kekez & Klabučar, Phys. Lett. B738 (2014) 113]

Find matrix elem's in NS-S basis from these $M_\eta, M_{\eta'}, \phi$:

$$\begin{aligned} M^2_{\eta_{NS}} \equiv M^2_{\mathsf{NS}} &= \cos^2\phi\, M^2_\eta + \sin^2\phi\, M^2_{\eta'} \\ M^2_{\eta_S} \equiv M^2_{\mathsf{S}} &= \sin^2\phi\, M^2_\eta + \cos^2\phi\, M^2_{\eta'} \\ M^2_{\eta_{NS}\eta_S} \equiv M^2_{\mathsf{NSS}} &= \sin\phi\, \cos\phi\, (M^2_\eta - M^2_{\eta'}) \end{aligned}$$

to use $M^2_{\eta,\eta'} = \frac{1}{2}\left[M^2_{\mathsf{NS}} + M^2_{\mathsf{S}} \mp \sqrt{(M^2_{\mathsf{NS}} - M^2_{\mathsf{S}})^2 + 4M^4_{\mathsf{NSS}}}\right]$

Mathematica leads to surprisingly simple results:

$$M^2_{\mathsf{NS}} = M^2_\pi + \frac{4A}{f^2_\pi}, \qquad M^2_{\mathsf{NSS}} = \frac{2\sqrt{2}A}{f_\pi f_{s\bar{s}}}$$

$$M^2_{\mathsf{S}} = \frac{1}{f^2_{s\bar{s}}}\left[2\, f^2_K\, M^2_K - f^2_\pi\, M^2_\pi\right] + \frac{2A}{f^2_{s\bar{s}}} = M^2_{s\bar{s}} + \frac{2A}{f^2_{s\bar{s}}}$$

$f^2_\pi\, M^2_\pi = -m_u\langle u\bar{u}\rangle - m_d\langle d\bar{d}\rangle$ and $f^2_K\, M^2_K = -m_u\langle u\bar{u}\rangle - m_s\langle s\bar{s}\rangle$

$\Rightarrow \quad 2\, f^2_K\, M^2_K - f^2_\pi\, M^2_\pi = f^2_{s\bar{s}}\, M^2_{s\bar{s}}$ "eq. (23)"

Compare $M_{\text{NS}}, M_{\text{NSS}}$ and M_{S} with NS-S mass matrix:

$$\begin{bmatrix} M^2_{\text{NS}} & M^2_{\text{NSS}} \\ M^2_{\text{NSS}} & M^2_{\text{S}} \end{bmatrix} = \begin{bmatrix} M^2_\pi + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{bmatrix}$$

$\Rightarrow$ Very similar formulas in WV case and "Shore case":

$$1.) \quad \beta_{\text{WV}} = \frac{6\chi_{YM}}{f_\pi^2(2+X^2)}\,, \qquad \beta_{\text{Shore+FKS}} = \frac{2A}{f_\pi^2} \approx \frac{2\chi_{YM}}{f_\pi^2}$$

Explains why Shore's scheme needs higher values of χ_{YM} than WV, to approach empirical masses.

$2.) \quad X = \dfrac{f_\pi}{f_{\text{ss}}}$ the SAME in the both WV and Shore cases ...

... but in the "Shore case", it follows from equations! Before, incl. WV, it was an input – estimate, educated guess.

T-dependence of pseudoscalar decay constants