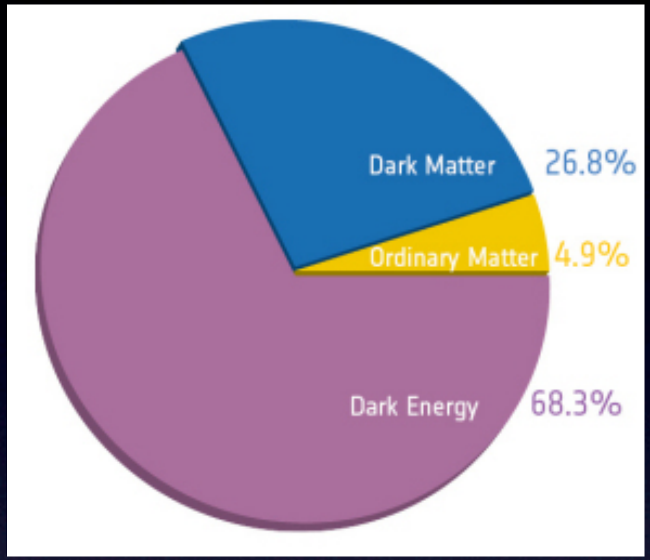
Axion dark matter in the post-inflationary Peccei-Quinn symmetry breaking scenario

Ken'ichi Saikawa (DESY)

A. Ringwald, KS, PRD93, 085031 (2016) [arXiv:1512.06436]
M. Kawasaki, KS, T. Sekiguchi, PRD91, 065014 (2015) [arXiv:1412.0789]

Dark matter

- Recent astrophysical observations
  - 27% of the total energy of the universe is occupied by unknown matter
  - "Invisible"
    (Interaction with ordinary matters is weak)



Credit: ESA and the Planck Collaboration

- Physics beyond the standard model
  - A well motivated candidate : axion

Strong CP problem

- How they are produced, and how they evolved ?
  → Key to understand the nature of dark matter
  - Prediction for axion dark matter depends largely on the early history of the universe

QCD axion as dark matter candidate

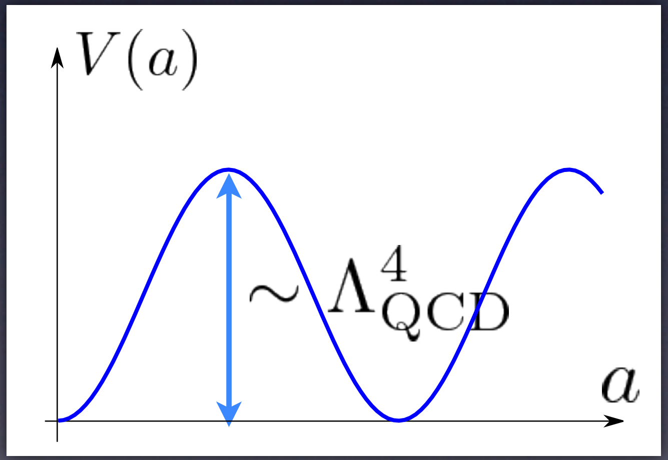
- Motivated by Pecccei-Quinn mechanism Peccei and Quinn (1977)
  as a solution of the strong CP problem

- Spontaneous breaking of global U(1) Peccei-Quinn (PQ) symmetry
  at a scale $F_a \simeq 10^{8-11}\,\mathrm{GeV}$ "axion decay constant"
  - Nambu-Goldstone theorem
    → emergence of the (massless) particle ≡ axion
    Weinberg (1978), Wilczek (1978)

- Axion has a small mass (QCD effect)
  → pseudo-Nambu-Golstone boson

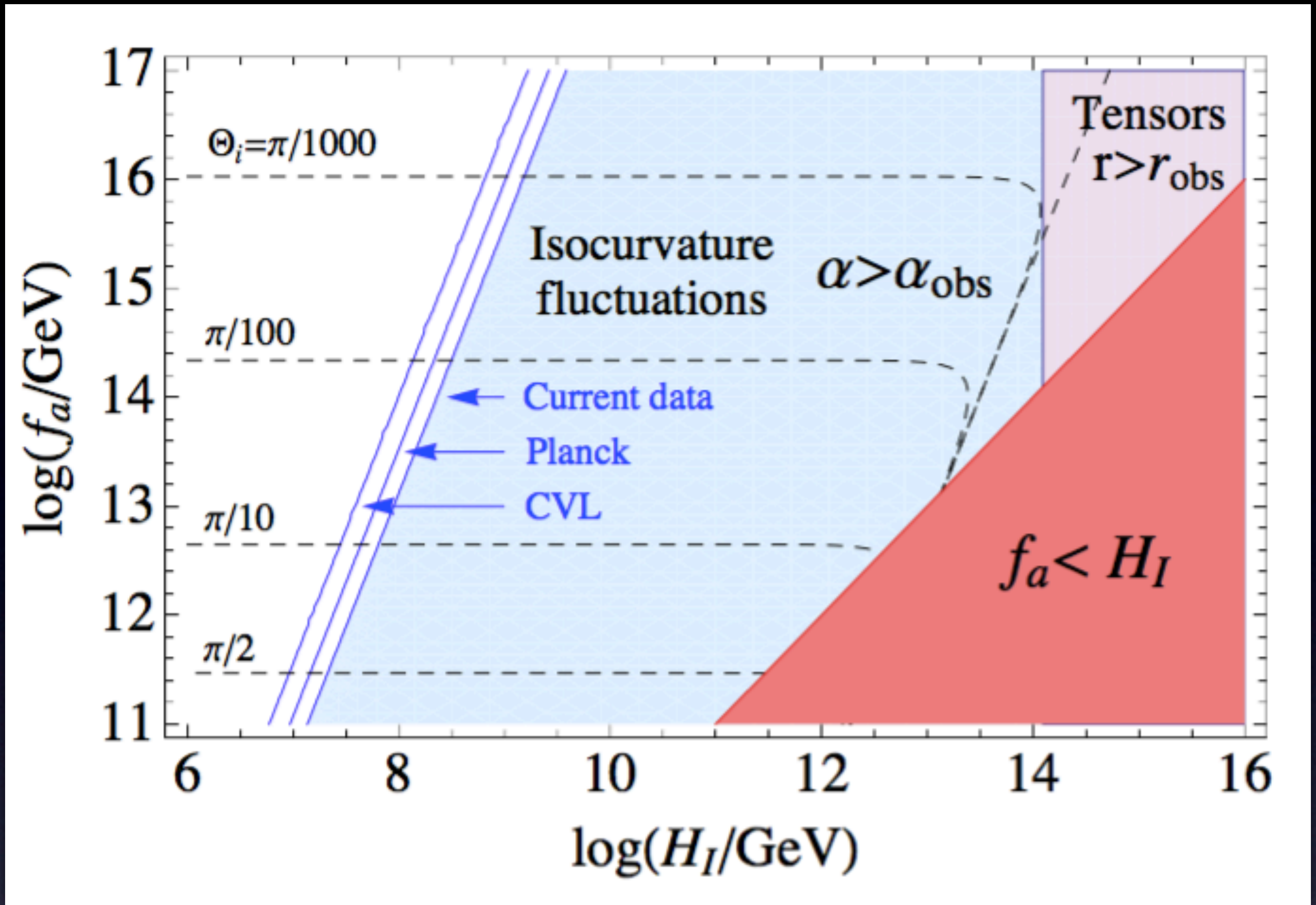
$$m_a \sim \frac{\Lambda_{\rm QCD}^2}{F_a} \sim 6\times 10^{-5}\mathrm{eV}\left(\frac{10^{11}\mathrm{GeV}}{F_a}\right)$$

$\Lambda_{\rm QCD} \simeq \mathcal{O}(100)\mathrm{MeV}$



- Tiny coupling with matter + non-thermal production
  → good candidate of cold dark matter

Axions in the inflationary universe



Hamann, Hannestad, Raffelt and Wong (2009)

- PQ symmetry is broken before inflation if $F_a > \max[H_I/2\pi, T_{\max}]$

  H_I : Hubble parameter during inflation
  $T_{\max}$: maximum temperature of the thermal bath after inflation

- In this case, axion field during inflation leads to isocurvature fluctuations that are severely constrained unless H_I is sufficiently small
- In the following, we focus on the post-inflationary PQ symmetry breaking scenario:

$$F_a < \max[H_I/2\pi, T_{\max}]$$

Axionic string and axionic domain wall

Peccei-Quinn field (complex scalar field)

$$\Phi = |\Phi| e^{ia(x)/\eta}$$

$a(x)$: axion field

$$F_a = \eta / N_{\rm DW}$$

String formation $T \lesssim F_a$

Spontaneous breaking of U(1)PQ

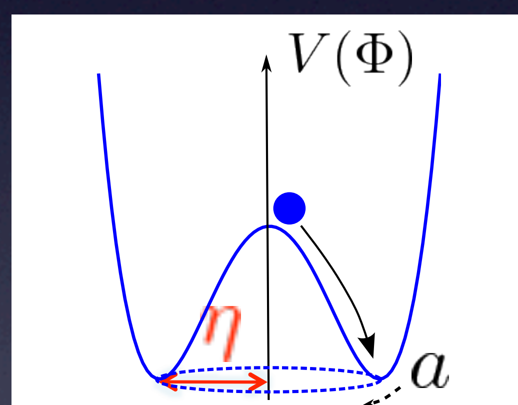
$$V(\Phi) = \frac{\lambda}{4}(|\Phi|^2 - \eta^2)^2$$

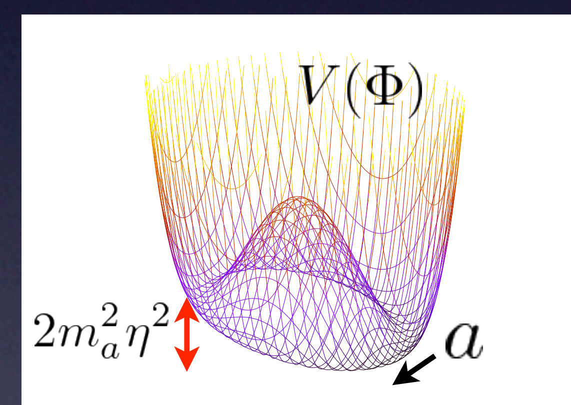
Domain wall formation $T \lesssim 1\mathrm{GeV}$

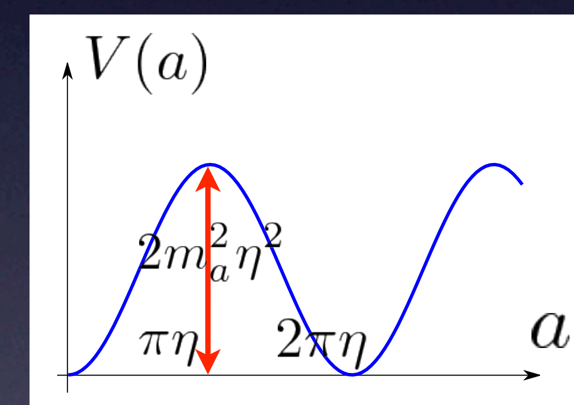
QCD effect

$$V(\Phi) = \frac{\lambda}{4}(|\Phi|^2 - \eta^2)^2 + m_a^2\eta^2(1-\cos(a/\eta))$$

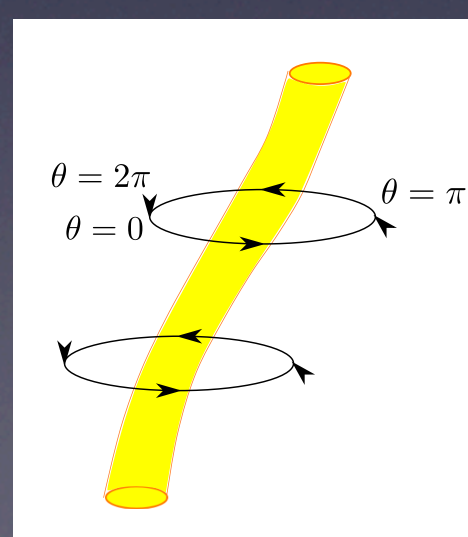
Field space

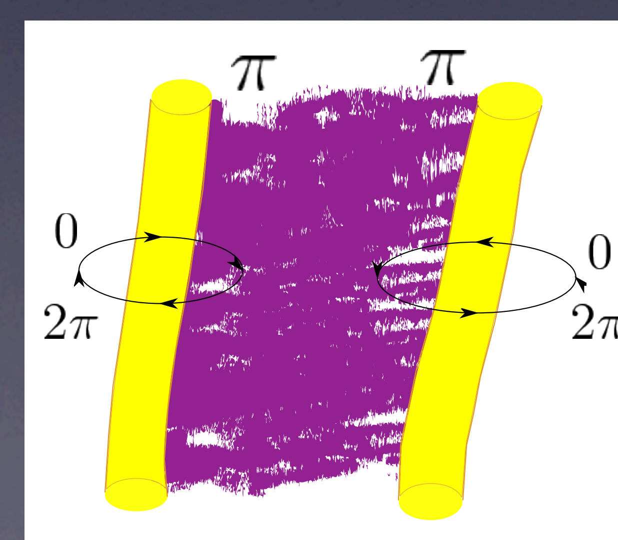






Coordinate space





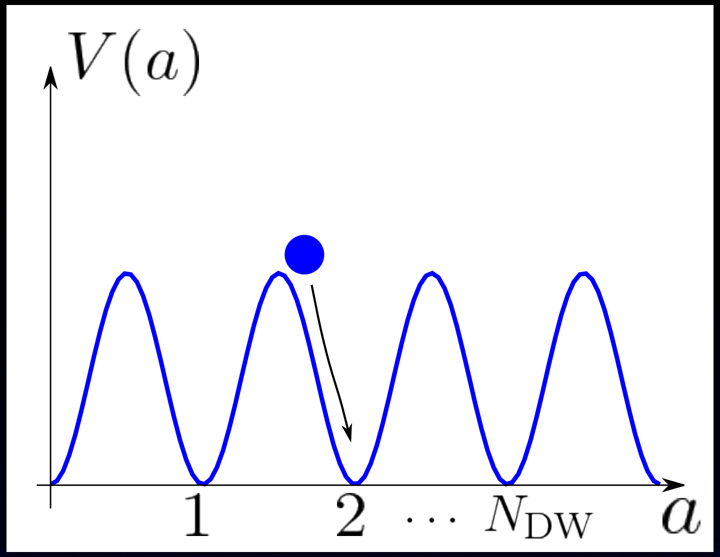
Strings attached by domain walls

Domain wall problem

Domain wall number N_{DW} (vacuum degeneracy)

$$V(a) = \frac{m_a^2 \eta^2}{N_{\rm DW}^2}(1 - \cos(N_{\rm DW} a/\eta))$$

$N_{\rm DW}$: Integer determined by QCD anomaly, which depends on particle physics model



$N_{\rm DW} = 1$ for Kim-Shifman-Vainshtein-Zakharov (KSVZ) models

Kim (1979), Shifman, Vainshtein and Zakharov (1980)

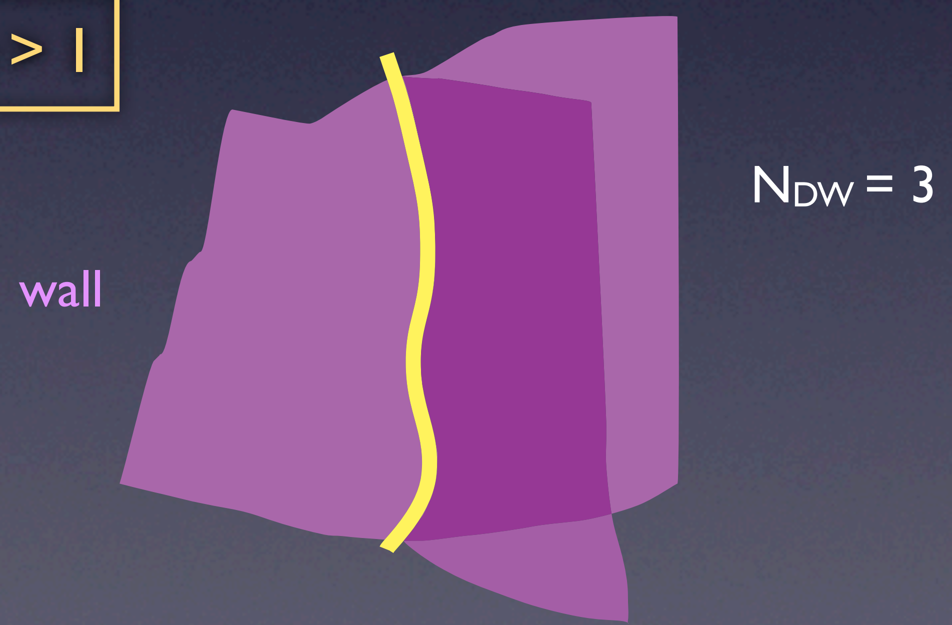
$N_{\rm DW} = 6$ for Dine-Fischler-Srednicki-Zhitnitsky (DFSZ) models

Zhitnitsky (1980), Dine, Fischler and Srednicki (1981)

$N_{DW} = 1$

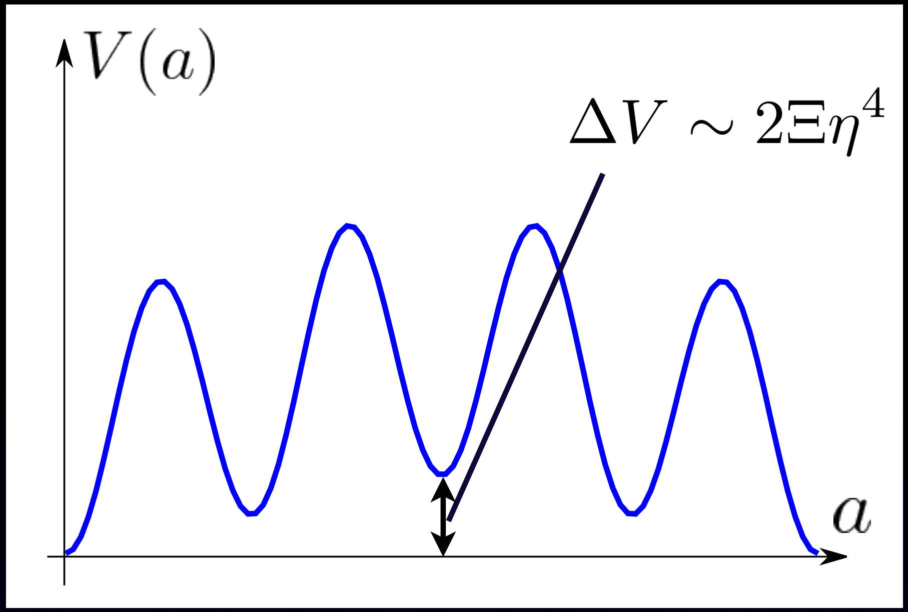
Collapse to produce additional axions

$N_{DW} > 1$



Stable, coming to overclose the universe
"Domain wall problem"

- The domain wall problem for $N_{DW}>1$ might be avoided by introducing an explicit symmetry breaking term (bias term) Sikivie (1982)



$$V(a) = \frac{m_a^2\eta^2}{N_{\rm DW}^2}\left(1-\cos\left(\frac{N_{\rm DW}a}{\eta}\right)\right) + \underline{\Delta V_{\rm bias}}$$

lifts degenerate vacua

- Origin of the bias term ?
  - $U(1)_{PQ}$ may not be an exact symmetry: any global symmetry can be spoiled by gravitational effects Holman et al. (1992), Kamionkowski and March-Russell (1992), Barr and Seckel (1992), Ghigna, Lusignoli and Roncadelli (1992), Dine (1992)
  - We can assume that the PQ symmetry is not *ad hoc* but instead an accidental symmetry of an exact discrete Z_N symmetry (with large N) Choi, Nilles, Ramos-Sanchez and Vaudrevange (2009)
  - Planck-suppressed operators allowed by the Z_N symmetry work as the bias term

$$\mathcal{L} \supset \frac{g}{M_{\rm Pl}^{N-4}}\Phi^N + {\rm h.c.}, \qquad g = |g|e^{i\Delta}$$

$$\Delta V_{\rm bias}(a) = -2\Xi\eta^4\cos\left(N\frac{a}{\eta}+\Delta_D\right)$$

$$\Xi = \frac{|g|N_{\rm DW}^{N-4}}{(\sqrt{2})^N}\left(\frac{F_a}{M_{\rm Pl}}\right)^{N-4}, \qquad \Delta_D = \Delta - N\bar{\theta}$$

$\bar{\theta}$: contribution from the QCD θ parameter and the phase of the quark masses

Annihilation mechanism of domain walls

The bias term acts as a pressure force p_V on the wall

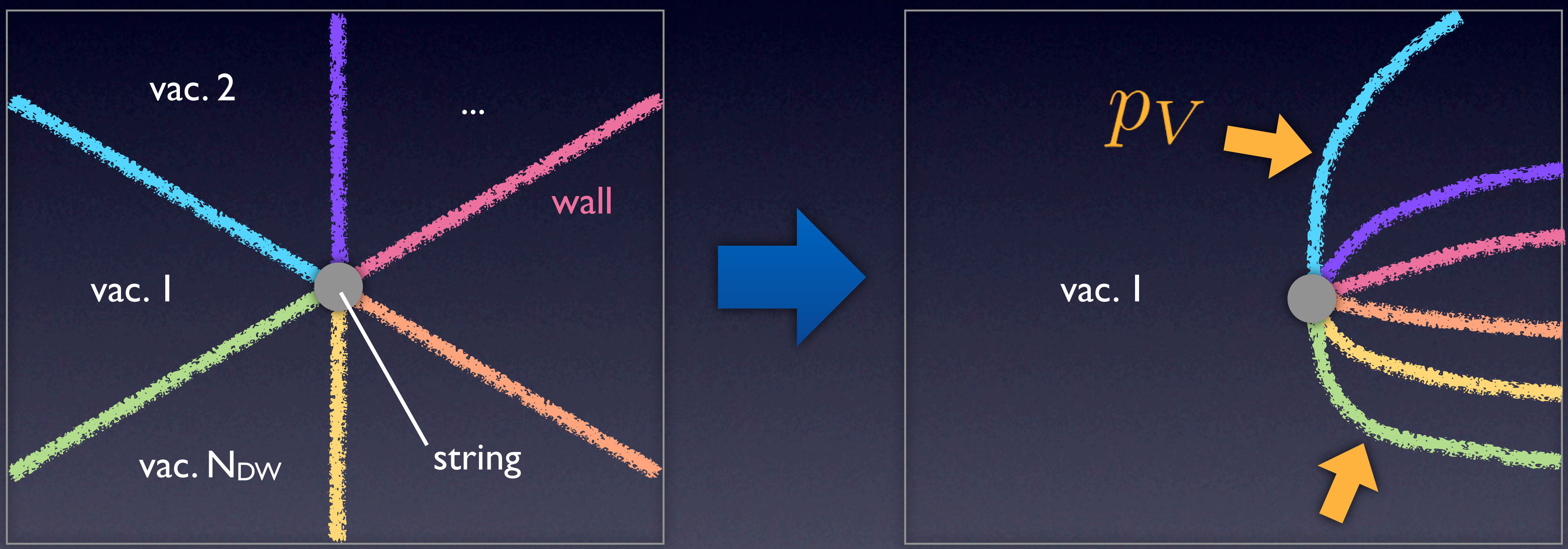
$$p_V \sim \Delta V_{\rm bias} \sim \Xi \eta^4$$

Annihilation occurs when the tension p_T becomes comparable with the pressure p_V

$$p_T \sim \sigma_{\rm wall}/R \sim m_a \eta^2 / N_{\rm DW}^2 R$$

R : curvature radius of walls

$\sigma_{\rm wall}$: surface mass density of walls



Decay time

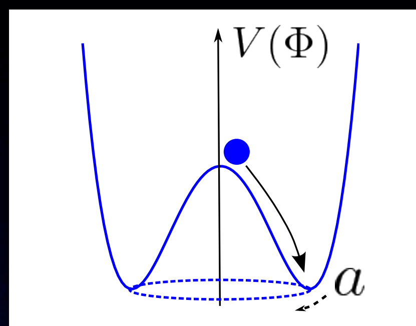
$$t_{\rm dec} \sim R|_{p_V = p_T} \sim \frac{m_a}{N_{\rm DW}^2 \Xi \eta^2}$$

$$\sim \mathcal{O}(10^{-6})\,{\rm sec} \left(\frac{6}{N_{\rm DW}}\right)^4 \left(\frac{10^{-51}}{\Xi}\right) \left(\frac{10^9\,{\rm GeV}}{F_a}\right)^3$$

Production of axions in the early universe

(post-inflationary PQ symmetry breaking scenario)

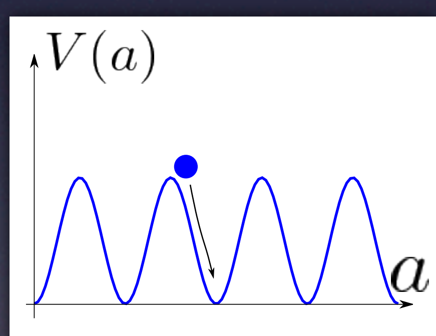
Inflation



$T \lesssim F_a \simeq 10^{8-11}\,\mathrm{GeV}$

PQ symmetry breaking
- Formation of strings

$T \lesssim 1\,\mathrm{GeV}$



QCD phase transition
- Axion acquires a mass
- Formation of domain walls

$N_{DW} = 1$

Immediately after formation

Collapse of string-wall systems

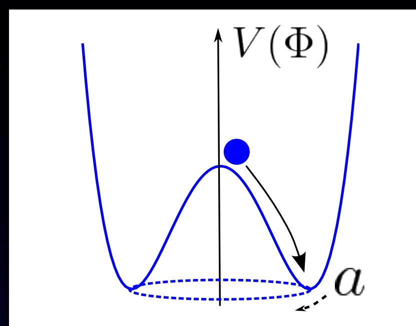
$N_{DW} > 1$

String-wall networks exist for a long time

Annihilation of domain walls before they overclose the universe

Production of axions in the early universe

(post-inflationary PQ symmetry breaking scenario)



$T \lesssim F_a \simeq 10^{8-11}\,\mathrm{GeV}$

Inflation

PQ symmetry breaking
- Formation of strings

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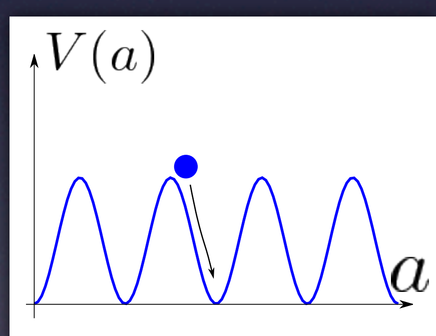


QCD phase transition
- Axion acquires a mass
- Formation of domain walls

(i) Coherent oscillation (realignment mechanism)

$\Omega_{a,\mathrm{real}}$

$N_{DW} = 1$ — Immediately after formation

Collapse of string-wall systems

$N_{DW} > 1$ — String-wall networks exist for a long time

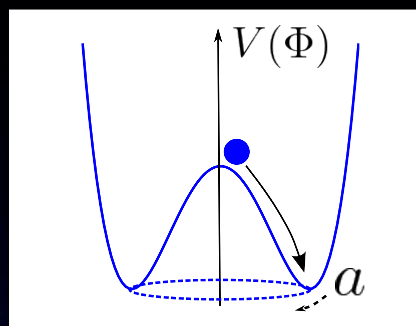
Annihilation of domain walls before they overclose the universe

Production of axions in the early universe

(post-inflationary PQ symmetry breaking scenario)

Inflation

PQ symmetry breaking
- Formation of strings



$T \lesssim F_a \simeq 10^{8-11}\,\mathrm{GeV}$

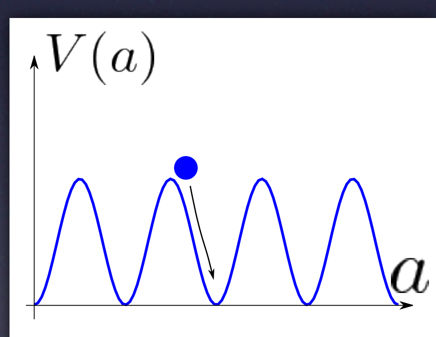
(ii) Radiation from strings

$\Omega_{a,\mathrm{string}}$

(i) Coherent oscillation (realignment mechanism)

$\Omega_{a,\mathrm{real}}$

$T \lesssim 1\,\mathrm{GeV}$

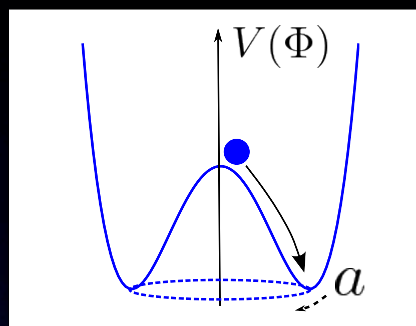


QCD phase transition
- Axion acquires a mass
- Formation of domain walls

$N_{DW} = 1$	$N_{DW} > 1$
Immediately after formation	String-wall networks exist for a long time
Collapse of string-wall systems	Annihilation of domain walls before they overclose the universe

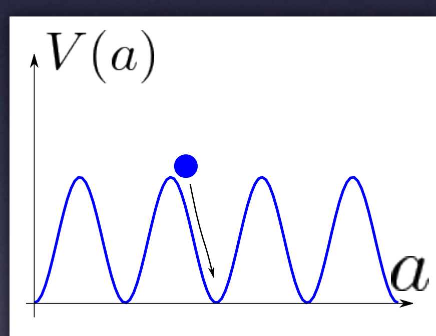
Production of axions in the early universe

(post-inflationary PQ symmetry breaking scenario)



$T \lesssim F_a \simeq 10^{8-11}\,\mathrm{GeV}$

$T \lesssim 1\,\mathrm{GeV}$



Inflation

PQ symmetry breaking
- Formation of strings

QCD phase transition
- Axion acquires a mass
- Formation of domain walls

(ii) Radiation from strings
$\Omega_{a,\mathrm{string}}$

(i) Coherent oscillation
(realignment mechanism)
$\Omega_{a,\mathrm{real}}$

(iii) Wall decay
$\Omega_{a,\mathrm{dec}}$

$N_{DW} = 1$

Immediately after formation

Collapse of string-wall systems

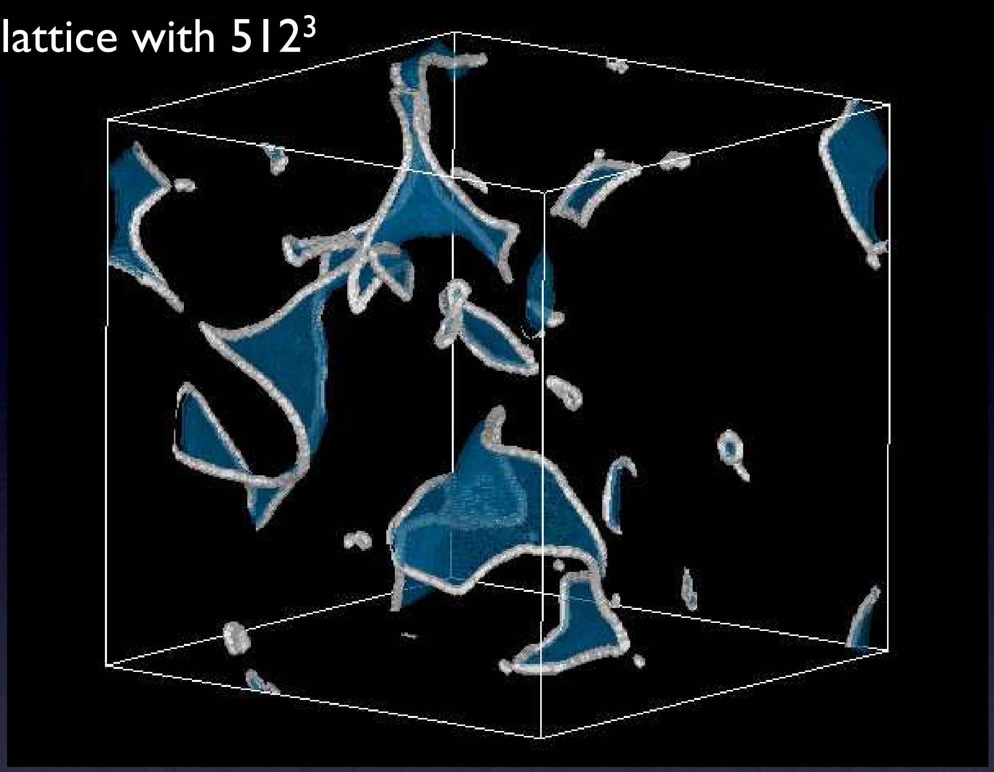
$N_{DW} > 1$

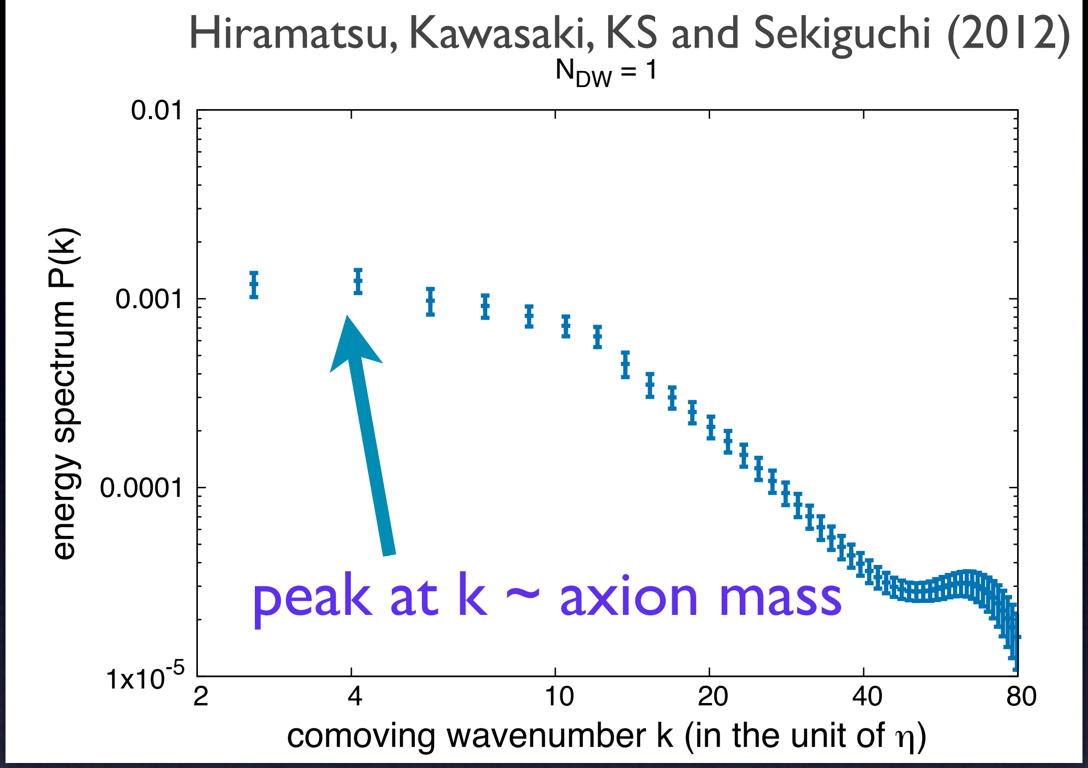
String-wall networks exist for a long time

Annihilation of domain walls before they overclose the universe

Numerical simulation of string-wall systems

- Energy spectrum of radiated axions is estimated and the total relic abundance of axions is computed by using the results of numerical simulations





- For $N_{DW} = 1$ (KSVZ-like models), axion density from the decay of string-wall systems $\Omega_{a,\mathrm{dec}}$ is comparable to axion densities from other sources

$$\Omega_{a,\mathrm{dec}} \sim \Omega_{a,\mathrm{real}} \sim \Omega_{a,\mathrm{string}}$$

- Constraint on the Peccei-Quinn scale

$$\Omega_{a,\mathrm{tot}} \leq \Omega_{\mathrm{CDM}}$$

$$\Omega_{a,\mathrm{tot}} = \Omega_{a,\mathrm{real}} + \Omega_{a,\mathrm{string}} + \Omega_{a,\mathrm{dec}}$$

$$F_a \lesssim (4.6\text{–}7.2) \times 10^{10}\,\mathrm{GeV}$$

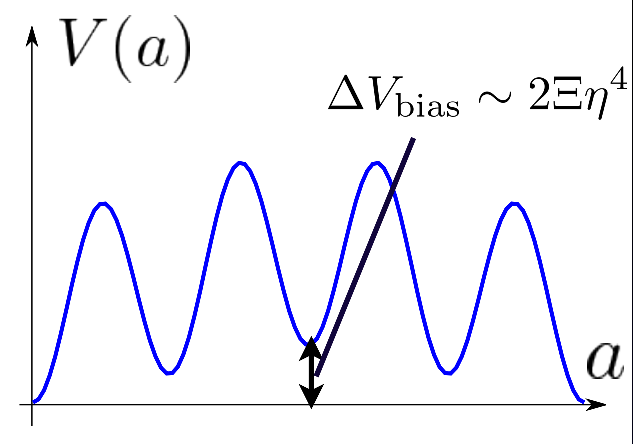
$$m_a \gtrsim (0.8\text{–}1.3) \times 10^{-4}\,\mathrm{eV}$$

$N_{DW} > 1$ (DFSZ-like models): long-lived domain walls

Hiramatsu, Kawasaki, KS and Sekiguchi (2013),
Kawasaki, KS and Sekiguchi (2015), Ringwald and KS (2016)

- Domain walls are long-lived and decay due to the bias term

$$\Delta V_{\rm bias} = -2\Xi\eta^4 \cos((Na)/\eta + \Delta_D)$$



- For small bias

  Long-lived domain walls emit a lot of axions which might exceed the observed matter density

  Cosmology → large bias is favored

- For large bias

  Bias term shifts the minimum of the potential and might spoil the original Peccei-Quinn solution to the strong CP problem

$$\bar{\theta} = \frac{2\Xi N N_{\rm DW}^3 F_a^2 \sin\Delta_D}{m_a^2 + 2\Xi N^2 N_{\rm DW}^2 F_a^2 \cos\Delta_D} < 7\times 10^{-12}$$

Δ_D : phase of the bias term

  CP → small bias is favored

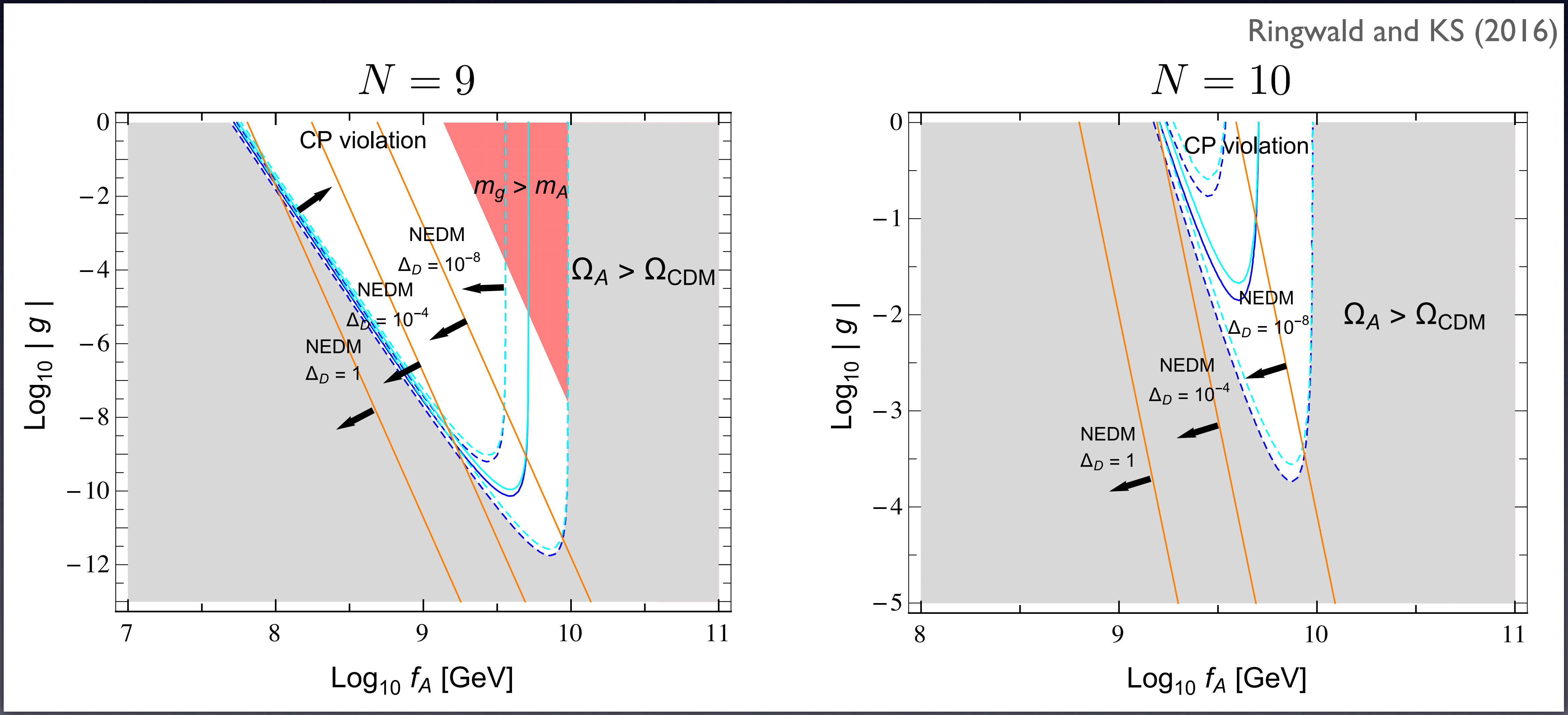
- Consistent parameters ?

- Constraints on the bias parameter ($=$ on the coefficient g)

$$\Xi = \frac{|g| N_{\rm DW}^{N-4}}{(\sqrt{2})^N} \left(\frac{F_a}{M_{\rm Pl}} \right)^{N-4} \Leftarrow \mathcal{L} \supset \frac{g}{M_{\rm Pl}^{N-4}} \Phi^N + {\rm h.c.}$$

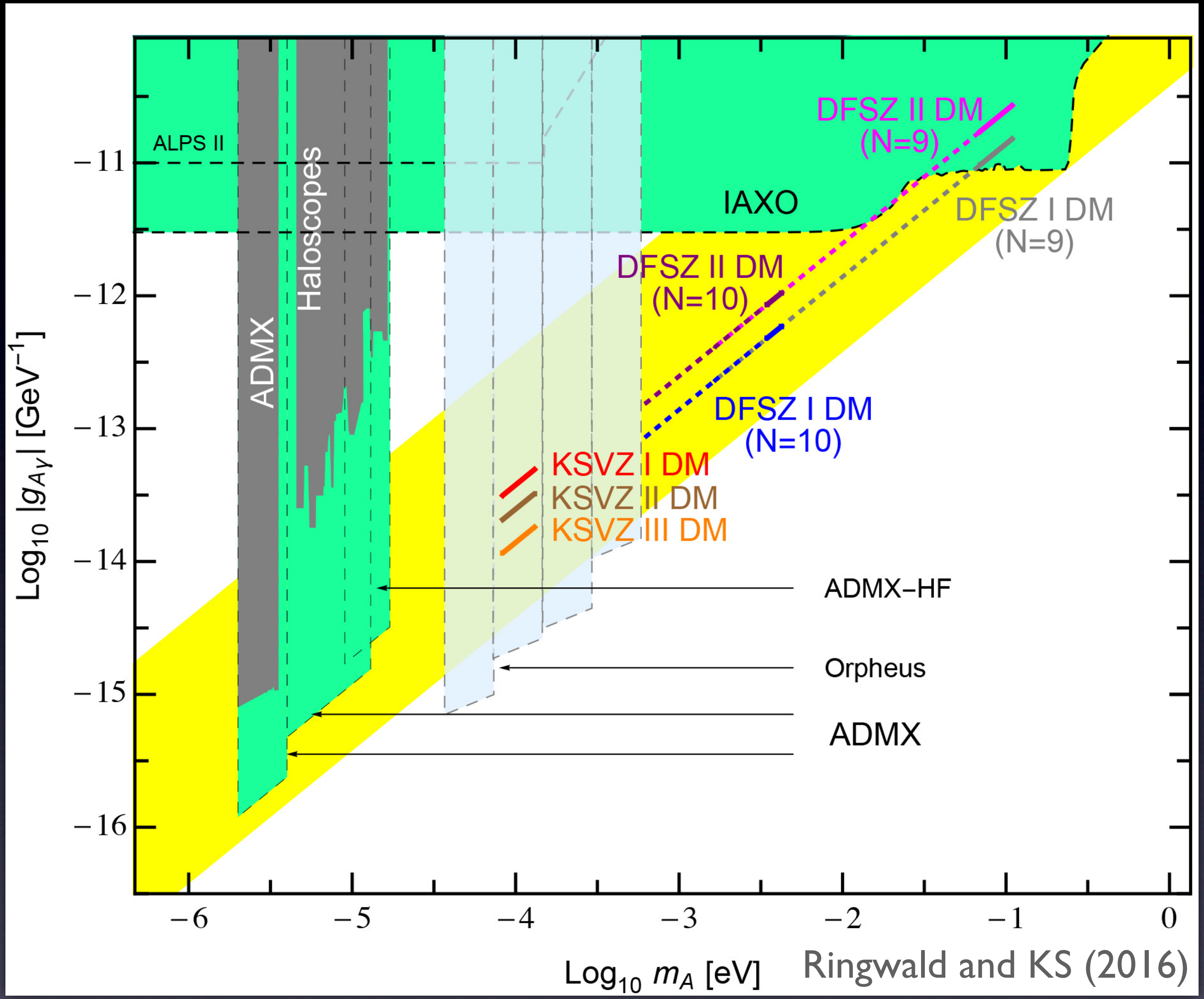
  - Axion energy density $\Omega_{a,{\rm real}} + \Omega_{a,{\rm string}} + \Omega_{a,{\rm dec}} \leq \Omega_{\rm CDM}$
  - Neutron electric dipole moment (NEDM) $\bar{\theta} < 0.7 \times 10^{-11}$

- Loopholes appear if the order of the discrete symmetry is N = 9 or 10, but some tuning of the phase parameter Δ_D is required



Search for axion DM

Search space in photon coupling $g_{a\gamma} \sim \alpha/(2\pi F_a)$ vs. mass m_a



- CDM abundance can be explained at higher m_a due to the additional contribution from long-lived string-wall systems for DFSZ models
- Every axion dark matter model gives a distinctive prediction for coupling parameters which can be probed by future experimental studies

Conclusion

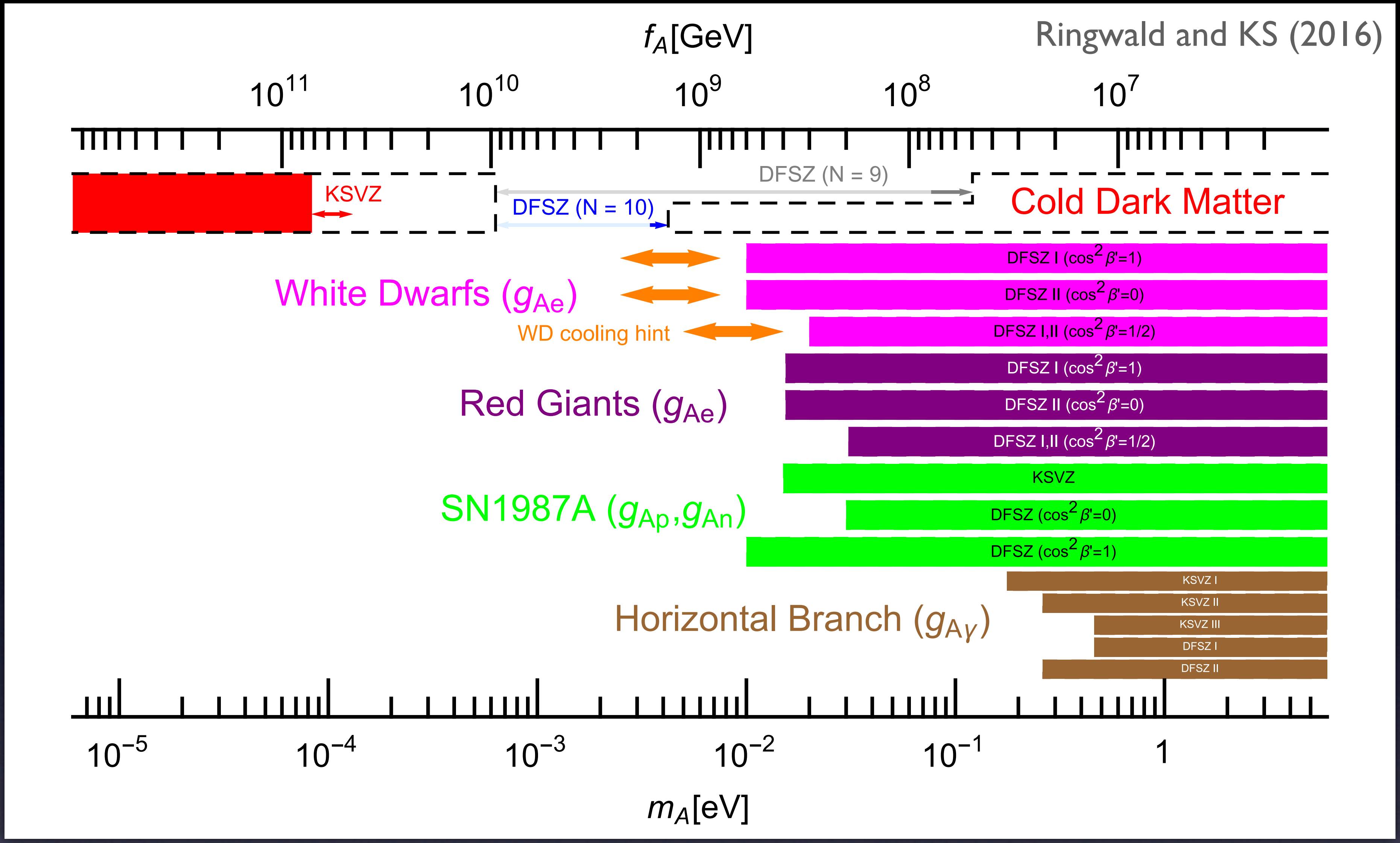
- If the PQ symmetry is broken after inflation, axions from string-wall systems give additional contributions to the CDM abundance
- Axion can be the dominant component of dark matter if

$m_a \simeq (0.8\text{–}1.3) \times 10^{-4}\,\mathrm{eV}$	for $N_{DW} = 1$ (KSVZ-like models)
$m_a \simeq \mathcal{O}(10^{-4}\text{–}10^{-2}\,\mathrm{eV})$	for $N_{DW} > 1$ (DFSZ-like models)

- These predictions depend strongly on the early history of the universe according to the detailed construction of the models (i.e. domain wall number N_{DW}, structure of the bias term, etc.)
- Future experimental searches will probe broad parameter ranges, which can provide rich information about underlying particle physics models, as well as the early history of the universe

Backup

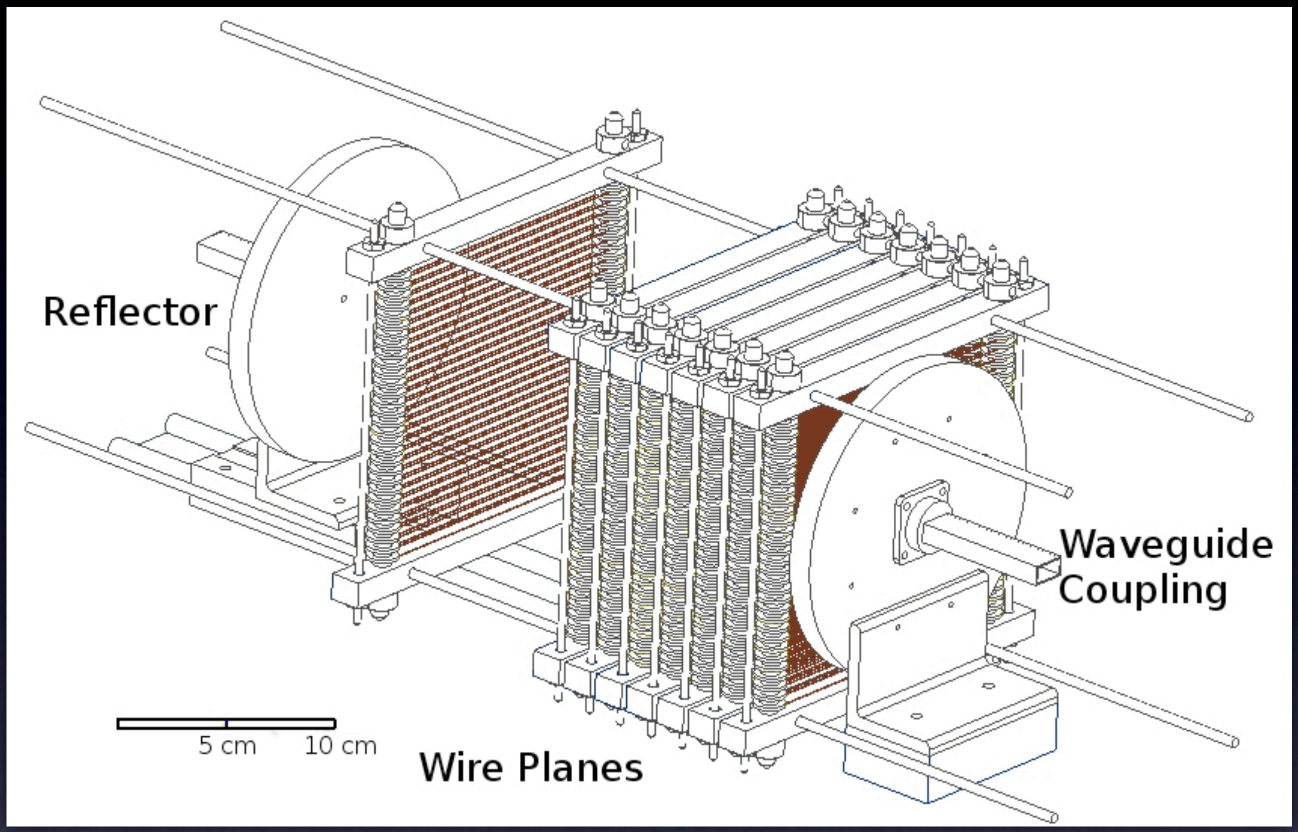
Astrophysical and cosmological constraints

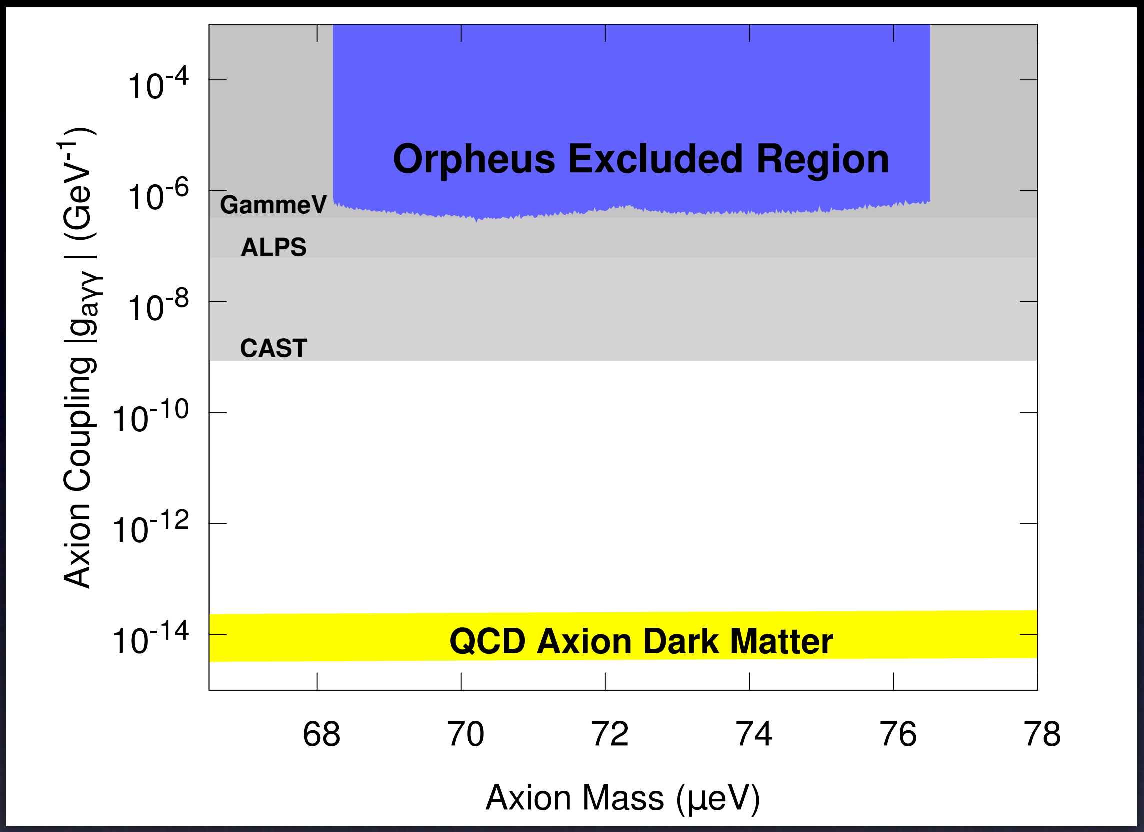


- Astrophysical observations give lower (upper) bounds on F_a (m_a)
- Dark matter abundance gives upper (lower) bounds on F_a (m_a) [and also a lower (upper) bound for DFSZ models]
- DFSZ models can explain CDM abundance at lower F_a (higher m_a) due to the additional contribution from long-lived string-wall systems

Orpheus

Rybka, Wagner, Patel, Percival, Ramos and Brill (2015)





- Open Fabry-Perot resonator and a series of current-carrying wire planes
- Searches for axion like particles in the 68.2-76.5μeV mass range were demonstrated
- Potentially searches in the mass range 40-400μeV in the future

KSVZ models

- Additional complex SM singlet scalar σ ($= \Phi$) and color triplet exotic quark Q

$$\mathcal{L}_{\rm yukawa} = Y_{ij}\bar{q}_{iL}\tilde{H}u_{jR} + \Gamma_{ij}\bar{q}_{iL}Hd_{jR} + G_{ij}\bar{L}_iHl_{jR} + \mathcal{L}_Q + \text{H.c.}$$

TABLE I. The Z_N charges (for $N = 9, 10$), where $\omega_N \equiv e^{i2\pi/N}$, of the KSVZ accion models, leaving the Yukawa interactions (2.1), (2.3), and (2.4) invariant.

	q_L	u_R	d_R	L	l_R	H	Q_L	Q_R	σ
Z_9	1	ω_9	ω_9^8	1	ω_9^8	ω_9	1	ω_9	ω_9^8
Z_{10}	1	ω_{10}^6	ω_{10}^4	1	ω_{10}^4	ω_{10}^6	ω_{10}^5	ω_{10}^6	ω_{10}^9

- Different possibilities according to the U(1)$_Y$ hypercharge Y_{Q_R} of Q_R

$$Y_{Q_R} = \begin{cases} 0 & \text{(KSVZ I)} \\ -\frac{1}{3} & \text{(KSVZ II)} \\ \frac{2}{3} & \text{(KSVZ III)} \end{cases}$$

TABLE II. The U(1)$_{\rm PQ}$ charge assignments leaving (2.1) (KSVZ I), (2.1) plus (2.3) (KSVZ II), or (2.1) plus (2.4) (KSVZ III) invariant.

Model	q_L	u_R	d_R	L	l_R	H	Q_L	Q_R	σ
KSVZ I	0	0	0	0	0	0	1/2	−1/2	1
KSVZ II	3/2	3/2	3/2	0	0	0	1/2	−1/2	1
KSVZ III	−1/2	−1/2	−1/2	0	0	0	1/2	−1/2	1

$$\mathcal{L}_Q = \begin{cases} y_Q\bar{Q}_L\sigma Q_R & \text{(KSVZ I)} \\ y_Q\bar{Q}_L\sigma Q_R + y'_Q\bar{Q}_L\sigma^* d_R & \text{(KSVZ II)} \\ y_Q\bar{Q}_L\sigma Q_R + y''_Q\bar{Q}_L\sigma u_R + y'''_Q\bar{q}_L\tilde{H}Q_R & \text{(KSVZ III)} \end{cases}$$

DFSZ models

- A complex SM singlet σ $(= \Phi)$ and two Higgs doublets H_u and H_d

$$\mathcal{L}_{\rm yukawa} = \begin{cases} \Gamma_{ij}\bar{q}_{iL}H_d d_{jR} + Y_{ij}\bar{q}_{iL}\tilde{H}_u u_{jR} + G_{ij}\bar{L}_i H_d l_{jR} + \text{H.c.} & \text{(DFSZ I)} \\ \Gamma_{ij}\bar{q}_{iL}H_d d_{jR} + Y_{ij}\bar{q}_{iL}\tilde{H}_u u_{jR} + G_{ij}\bar{L}_i H_u l_{jR} + \text{H.c.} & \text{(DFSZ II)} \end{cases}$$

$$V(H_u, H_d, \sigma) = \lambda H_d^\dagger H_u \sigma^{*2} + \text{H.c.} + (\text{Hermitian terms})$$

- The orthogonality of the axion field and the NG boson eaten by the Z^0 boson implies

$$X_u = x\xi_v \quad \text{and} \quad X_d = x^{-1}\xi_v$$

where

$$\xi_v = \frac{2}{x + x^{-1}}$$

$$x \equiv \frac{v_d}{v_u} \equiv \tan\beta'$$

$$\langle H_u^0 \rangle = v_u/\sqrt{2}$$

$$\langle H_d^0 \rangle = v_d/\sqrt{2}$$

TABLE III. The Z_N charges (for $N = 9, 10$) of the DFSZ accion models, leaving interactions (2.5) and (2.7) (DFSZ I) or (2.6) and (2.7) (DFSZ II) invariant.

	q_L	u_R	d_R	L	l_R	H_u	H_d	σ
Z_9 (DFSZ I)	1	ω_9^6	ω_9^5	1	ω_9^5	ω_9^6	ω_9^4	ω_9
Z_9 (DFSZ II)	1	ω_9^4	ω_9^7	1	ω_9^5	ω_9^4	ω_9^2	ω_9
Z_{10} (DFSZ I)	1	ω_{10}^3	ω_{10}^9	1	ω_{10}^9	ω_{10}^3	ω_{10}	ω_{10}
Z_{10} (DFSZ II)	1	ω_{10}^3	ω_{10}^9	1	ω_{10}^7	ω_{10}^3	ω_{10}	ω_{10}

TABLE IV. The $U(1)_{PQ}$ charge assignments, where X_u and X_d are some real numbers satisfying the condition $X_u + X_d = 2$, leaving (2.5) and (2.7) (DFSZ I) or (2.6) and (2.7) (DFSZ II) invariant.

Model	q_L	u_R	d_R	L	l_R	H_u	H_d	σ
DFSZ I	0	X_u	X_d	0	X_d	X_u	$-X_d$	1
DFSZ II	0	X_u	X_d	0	$-X_u$	X_u	$-X_d$	1

Couplings to other particles

$$\mathcal{L}_{\text{int}} = -\frac{\alpha}{8\pi} C_{A\gamma} \frac{A}{f_A} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} \sum_{N=p,n} C_{AN} \frac{\partial_\mu A}{f_A} \bar{\psi}_N \gamma^\mu \gamma_5 \psi_N + \frac{1}{2} \sum_{\ell=e,\mu,\tau} C_{A\ell} \frac{\partial_\mu A}{f_A} \bar{\ell} \gamma^\mu \gamma_5 \ell$$

where
$$C_{Ap} = (C_{Au} - \eta)\Delta u + (C_{Ad} - \eta z)\Delta d + (C_{As} - \eta w)\Delta s$$
$$C_{An} = (C_{Au} - \eta)\Delta d + (C_{Ad} - \eta z)\Delta u + (C_{As} - \eta w)\Delta s$$

$$\eta = (1 + z + w)^{-1}, \quad z = m_u/m_d = 0.38\text{–}0.58, \quad w = m_u/m_s$$

$$\Delta u = 0.84 \pm 0.02, \quad \Delta d = -0.43 \pm 0.02, \quad \Delta s = -0.09 \pm 0.02$$ Raffelt (2008)

Axion-photon coupling

$$g_{A\gamma} = \frac{\alpha}{2\pi} \frac{C_{A\gamma}}{f_A}$$

Axion-electron coupling

$$g_{Ae} = \frac{C_{Ae} m_e}{f_A}$$

Axion-nucleon coupling

$$g_{AN} = \frac{C_{AN} m_N}{f_A}$$

Model	$C_{A\gamma}$	C_{Au}	C_{Ad}	C_{As}	$C_{A\ell}$
KSVZ I	$-\frac{2}{3}\frac{4+z}{1+z}$	0	0	0	0
KSVZ II	$\frac{2}{3} - \frac{2}{3}\frac{4+z}{1+z}$	0	0	0	0
KSVZ III	$\frac{8}{3} - \frac{2}{3}\frac{4+z}{1+z}$	0	0	0	0
DFSZ I	$\frac{8}{3} - \frac{2}{3}\frac{4+z}{1+z}$	$\frac{1}{3}\sin^2\beta'$	$\frac{1}{3}\cos^2\beta'$	$\frac{1}{3}\cos^2\beta'$	$\frac{1}{3}\cos^2\beta'$
DFSZ II	$\frac{2}{3} - \frac{2}{3}\frac{4+z}{1+z}$	$\frac{1}{3}\sin^2\beta'$	$\frac{1}{3}\cos^2\beta'$	$\frac{1}{3}\cos^2\beta'$	$-\frac{1}{3}\sin^2\beta'$

Total axion abundance for N_{DW} = 1

Kawasaki, KS and Sekiguchi (2015)

$$\Omega_{a,\mathrm{real}}h^2 \simeq 4.63 \times 10^{-3} \left(\frac{F_a}{10^{10}\mathrm{GeV}}\right)^{1.19} \left(\frac{\Lambda_{\mathrm{QCD}}}{400\mathrm{MeV}}\right)$$

$$\Omega_{a,\mathrm{string}}h^2 \simeq (7.3 \pm 3.9) \times 10^{-3} \left(\frac{F_a}{10^{10}\mathrm{GeV}}\right)^{1.19} \left(\frac{\Lambda_{\mathrm{QCD}}}{400\mathrm{MeV}}\right)$$

$$\Omega_{a,\mathrm{dec}}h^2 \simeq (3.7 \pm 1.4) \times 10^{-3} \left(\frac{F_a}{10^{10}\mathrm{GeV}}\right)^{1.19} \left(\frac{\Lambda_{\mathrm{QCD}}}{400\mathrm{MeV}}\right)$$

$$\Omega_{a,\mathrm{tot}}h^2 = \Omega_{a,\mathrm{real}}h^2 + \Omega_{a,\mathrm{string}}h^2 + \Omega_{a,\mathrm{dec}}h^2$$

$$< \Omega_{\mathrm{CDM}}h^2 \simeq 0.11$$

→ $F_a \lesssim (4.6\text{–}7.2) \times 10^{10}\ \mathrm{GeV}$

- Wall decay contribution is comparable to others
- cf. bound from astrophysics: $F_a > 4 \times 10^8 \mathrm{GeV}$

Total axion abundance for $N_{DW} > 1$

Kawasaki, KS and Sekiguchi (2015)

$$\Omega_{a,\rm tot}h^2 = \Omega_{a,\rm real}h^2 + \Omega_{a,\rm string}h^2 + \Omega_{a,\rm dec}h^2$$

$$\Omega_{a,\rm real}h^2 \simeq 4.63\times 10^{-3}\left(\frac{F_a}{10^{10}\rm GeV}\right)^{1.19}\left(\frac{\Lambda_{\rm QCD}}{400\rm MeV}\right)$$

$$\Omega_{a,\rm string}h^2 \simeq (7.3\pm 3.9)\times 10^{-3}\times N_{\rm DW}^2\left(\frac{F_a}{10^{10}\rm GeV}\right)^{1.19}\left(\frac{\Lambda_{\rm QCD}}{400\rm MeV}\right)$$

$$\Omega_{a,\rm dec}h^2 \simeq 1.23\times 10^{-6}\times[7.22\times 10^3]^{\frac{3}{2p}}\times\frac{1}{\tilde{\epsilon}_a}\frac{2p-1}{3-2p}C_d^{\frac{3}{2}-p}$$

$$\times\mathcal{A}_{\rm form}^{\frac{3}{2p}}\left[N_{\rm DW}^4\left(1-\cos\left(\frac{2\pi N}{N_{\rm DW}}\right)\right)\right]^{1-\frac{3}{2p}}\left(\frac{\Xi}{10^{-52}}\right)^{1-\frac{3}{2p}}$$

$$\times\left(\frac{F_a}{10^{10}\rm GeV}\right)^{4+\frac{3(4p-16-3n)}{2p(4+n)}}\left(\frac{\Lambda_{\rm QCD}}{400\rm MeV}\right)^{-3+\frac{6}{p}}$$

where $n = 6.68$

$\Omega_{a,\rm dec}h^2$ is the contribution from long-lived string-wall systems, which depends on three (four) model parameters $(F_a, \Xi, N_{\rm DW})$ (and N). Other parameters $(A_{\rm form}, p, \tilde{\epsilon}_a, C_d)$ can be determined from numerical simulations.