Reflections on conformal spectra

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CERN, 14 Dec 2015

Outline

1. Convergence bounds for large Δ_ϕ
2. Cardy-like formula for large Δ_ϕ
3. A convergence bound for finite Δ_ϕ

The problem

Consider the state in a Euclidean CFT_d

$$|\psi_r\rangle = \phi(r)\,|\phi\rangle\,,$$

or the four point function on the real line with $x = \bar{x} = r^2$

$$G_4(x) = \langle\phi|\,\phi(1)\phi(x,\bar{x})\,|\phi\rangle \propto \langle\,\psi_r|\,\psi_r\rangle\,.$$

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Using $\phi \times \phi$ OPE, one can decompose

$$|\psi_r\rangle = \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}} V_{\mathcal{O}}(r)\,|\mathcal{O}\rangle$$

$$G_4(x) = \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} x^{-2\Delta_\phi} F_{\Delta,\ell}(x) \propto \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\,\langle\mathcal{O}|\,V^\dagger_{\mathcal{O}}(r)V_{\mathcal{O}}(r)\,|\mathcal{O}\rangle$$

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Consider the CDF

$$\mathcal{F}(\Delta_*, x) = \frac{\langle \psi_r | \, P_{\Delta_{\mathcal{O}} < \Delta_*} \, | \psi_r \rangle}{\langle \psi_r | \, \psi_r \rangle} = 1 - \frac{G_4^{\Delta_*}(x)}{G_4(x)} ,$$

where

$$G_4^{\Delta_*}(x) = \sum_{\mathcal{O}, \Delta_{\mathcal{O}} > \Delta_*} C^2_{\phi\phi\mathcal{O}} x^{-2\Delta_\phi} F_{\Delta,\ell}(x)$$

[Pappadopulo,Rychkov,Espin,Rattazzi '12; Rychkov,Yvernay '15]

A simpler problem

Ignore conformal symmetry, use only scaling symmetry → "scaling blocks",

$$G_4(x) = \sum_O C^2_{\phi\phi O} x^{\Delta_O - 2\Delta_\phi} = \int_0^\infty x^{\Delta - 2\Delta_\phi} g^{(s)}(\Delta) d\Delta.$$

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From [PRER '12], in a given theory, for sufficiently large Δ_*

$$G_4^{\Delta_*}(x) \lesssim \frac{1}{\Gamma(2\Delta_\phi + 1)} \Delta_*^{2\Delta_\phi} x^{\Delta_* - 2\Delta_\phi}.$$

Can we obtain more information on the structure of $\mathcal{F}(\Delta_*, x)$?

Crossing symmetry

Using a different channel for OPE expansion one finds

$$G_4(x) = G_4(1-x),$$

and so

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At $x = 1/2$ one obtains

$$\int_0^\infty [\Delta - 2\Delta_\phi]^{(2k+1)} \gamma_{1/2}^{(s)}(\Delta) d\Delta = 0,$$

$$[\alpha]^{(n)} = x^{-\alpha+n} \partial^n x^\alpha = \alpha(\alpha-1)\dots(\alpha-n+1)$$
$$\gamma_x^{(s)}(\Delta) = x^{\Delta-2\Delta_\phi} g^{(s)}(\Delta)$$

Crossing symmetry

Suppose $\Delta_\phi \gg 1$ in

$$\int_0^\infty [\Delta - 2\Delta_\phi]^{(2k+1)} \gamma_{1/2}^{(s)}(\Delta) d\Delta = 0,$$

Then for $k \ll \sqrt{\Delta_\phi}$ approximate

$$[\Delta - 2\Delta_\phi]^{(2k+1)} \simeq (\Delta - 2\Delta_\phi)^{2k+1},$$

$$\int_{-2\Delta_\phi}^\infty w^{2k+1} \gamma_{1/2}^{(s)}(w + 2\Delta_\phi) dw \simeq 0.$$

This suggests that $\gamma_{1/2}^{(s)}(w + 2\Delta_\phi)$ is approximately symmetric around $w = 0$.

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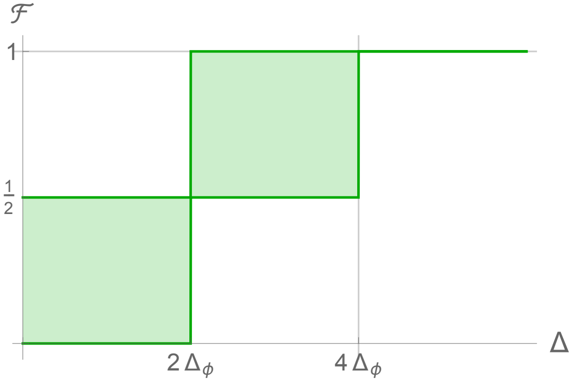

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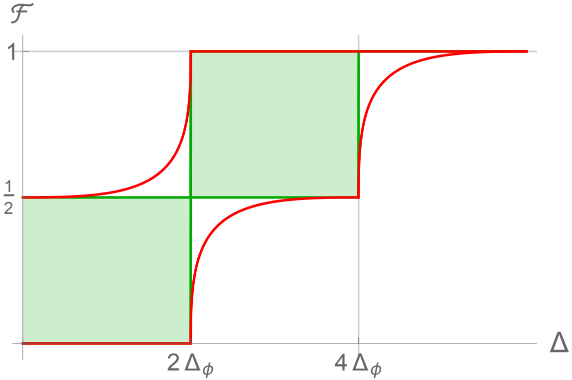

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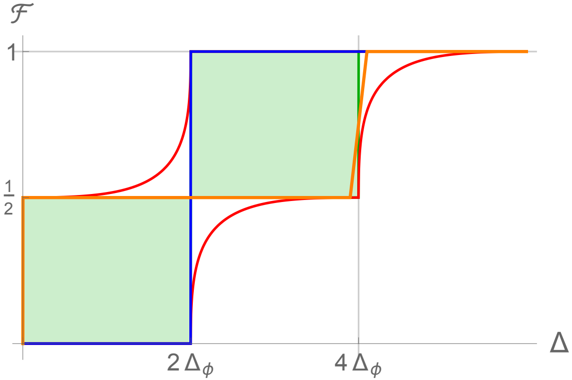

$G_4(x) = x^{2\Delta_\phi - 2\Delta_\phi}$ or $G_4(x) = x^{-2\Delta_\phi} + (1-x)^{-2\Delta_\phi}$

Reflection symmetry

For general x the reflection is between $\gamma_x^{(s)}$ and $\gamma_{1-x}^{(s)}$, relating

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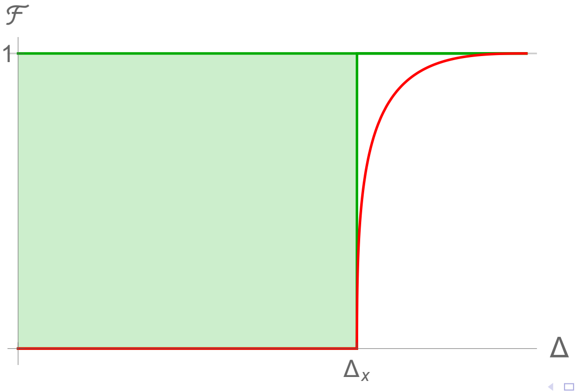
Let $x > 1/2$, $\Delta_x = \frac{2\Delta_\phi}{1-x}$. Then $\gamma_{1-x}^{(s)}(\Delta) = 0$ for $\Delta < 0 \Rightarrow$ $\gamma_x^{(s)}(\Delta) \simeq 0$ for $\Delta \geq \Delta_x$.



Saddle point interpretation

The same relation for general x can be obtained if one assumes that the four-point function is dominated by a saddle point at $\Delta = \Delta(x)$,

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$$G_4(x) = G_4(1-x) \Rightarrow \frac{\Delta(x) - 2\Delta_\phi}{x} = -\frac{\Delta(1-x) - 2\Delta_\phi}{1-x}$$

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$$\int_0^\infty [\Delta - 2\Delta_\phi]^{(2k+1)} \gamma^{(s)}_{1/2}(\Delta) d\Delta = 0$$

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This is of the form

$$A\vec{x} = \vec{b}, \quad x \geq 0$$
$$\max \vec{c} \cdot \vec{x} = ?$$

Linear programming duality

The linear programming problem of the form

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$$\vec{c} \cdot \vec{x} \leq \vec{y} \cdot A\vec{x} = \vec{y} \cdot \vec{b}.$$

Linear programming duality

Any feasible solution to the dual problem provides an upper bound for the primal problem. In our case the dual problem is

$$
\begin{aligned}
Q(\Delta) &= y_0 + \sum_k y_k [\Delta - 2\Delta_\phi]^{(2k-1)}, \\
Q(\Delta) &\geq 0, \quad \forall \Delta \geq 0, \\
Q(\Delta) &\geq 1, \quad \forall \Delta \geq \Delta_*, \\
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or alternatively

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\begin{aligned}
&Q(\Delta) = \sum_k \lambda_k [\Delta - 2\Delta_\phi]^{(2k-1)}, \\
&Q(\Delta) \geq -1, \quad \forall \Delta \geq 0, \\
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&\min \frac{1}{Q_0 + 1} = ?.
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Large Δ_ϕ

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For large Δ_ϕ we truncate at $k = n \ll \sqrt{\Delta_\phi}$ to find an approximate truncated version $(v = (\Delta - 2\Delta_0)/2\Delta_\phi)$

$$
\begin{aligned}
q(v) &\in P_n^{\mathrm{odd}}, \\
q(v) &\geq -1, \quad \forall v \geq -1, \\
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1 - \mathcal{F}(\Delta_*, 1/2) \leq \frac{1}{1 + T_{2n-1}\left(\frac{\Delta_* - 2\Delta_\phi}{2\Delta_\phi}\right)}, \quad \Delta_* \geq 4\Delta_\phi
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$$1 - \mathcal{F}(\Delta_*, x) \leq \frac{2}{1 + T_{2n-1}\left(\frac{\Delta_* - \Delta_x/2}{\Delta_x/2}\right)}, \quad \Delta_* \geq \Delta_x$$

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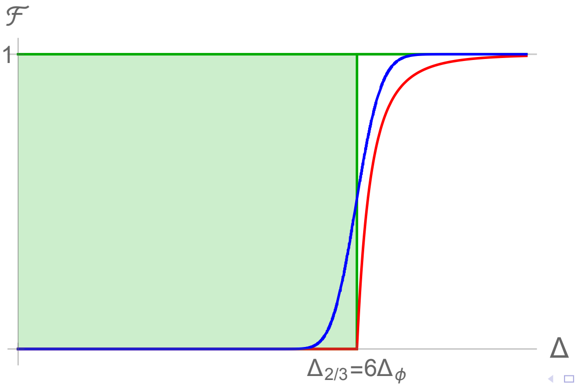
GFF with $\Delta_\phi = 100$ and T_5 bound



Other cases

The essential ingredient in the above analysis was the formula

$$\partial^n e^{\lambda f(x)} = [\lambda f'(x)]^n e^{\lambda f(x)} \left(1 + O(n^2 \lambda^{-1})\right) .$$

We had $\lambda = \Delta - 2\Delta_0$ and $f(x) = \log x$. More generally, the same approach works for many other cases when there is a UV-IR crossing-like equation and a large parameter limit. The analysis can be extended to

1. Conformal block expansion, ($\Delta_x = 2\Delta_\phi/\sqrt{1-x}$)
2. "Scaling block" expansion in ρ coordinate ($\Delta_x = 2\Delta_\phi/\sqrt{1-x}$)
3. Large space-time dimension limit of conformal block expansion (Δ_x is more complicated)
4. Large central charge limit of modular-invariant partition function in CFT_2 ($\Delta_\tau = (1 + |\tau|^{-2})c/12$)

Cardy-like formula

In [PRER '12] an asymptotic formula for the OPE coefficients was found, similar in spirit to Cardy formula. At large central charge and under an additional sparse light spectrum condition Cardy formula can be shown to work for operators $\Delta \sim c$ [Hartman,Keller,Stoica '14].

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Consider a situation in which Δ_ϕ is large and the OPE coefficients with light operators are sufficiently small so that for $x < 1/2$ we have

$$\log G_4(x) = -2\Delta_\phi \log x + O(1),$$

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It then follows from the approximate reflection symmetry that we should expect the dominant contribution for $x > 1/2$ to come from $\Delta = \Delta_x = 2\Delta_\phi/(1-x)$. This implies an asymptotic formula for the OPE coefficients for operators of dimension $\Delta > \Delta_{1/2} = 4\Delta_\phi$.

Cardy-like formula

Using the technology of [HKS '14] one shows for $\Delta > 4\Delta_\phi$

$$\bar{g}^{(s)}(\Delta) = \exp\left[-\Delta \log\left(1 - \frac{2\Delta_\phi}{\Delta}\right) + 2\Delta_\phi \log\left(\frac{\Delta}{2\Delta_0} - 1\right) + O(\Delta_\phi^\alpha)\right],$$

where $\bar{g}^{(s)}$ is $g^{(s)}$ averaged over interval of size $\sim \Delta_\phi^\alpha$ with $1/2 < \alpha < 1$.

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Take $Q(\Delta) \propto [\Delta - 2\Delta_\phi]^{(2k-1)}$ for some optimal k. This gives (interpolation weakens the bound)

$$1 - \mathcal{F}(\Delta, 1/2) \leq \frac{1}{1 + \frac{\Gamma(\Delta - 2\Delta_\phi - 1)\Gamma(2\Delta_\phi)}{\Gamma(\frac{\Delta+3}{2})\Gamma(\frac{\Delta-1}{2})}}$$

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or, defining $k(\Delta)=\lceil(\Delta-4\Delta_\phi-3)/4\rceil$, without interpolation

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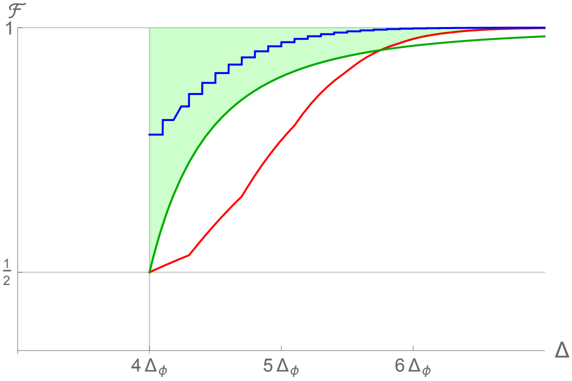
Compare to

$$G_4(1/2)\,(1-\mathcal{F}(\Delta_*,1/2))\lesssim \frac{1}{\Gamma(2\Delta_\phi+1)}\Delta_*^{2\Delta_\phi}\left(\frac{1}{2}\right)^{\Delta_*-2\Delta_\phi}$$

Finite Δ_ϕ

T_3, finite-Δ_ϕ bounds and GFF at $\Delta_\phi = 10$



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- What about spin? (Transverse derivatives)
- Virasoro symmetry?
- Can we solve the case of scaling blocks exactly? (e.g. the tail bound)
- Or at least guess the result?

Conformal block for large Δ

We can find the form of the conformal block on real line $x = \bar{x}$ in the limit of large intermediate scaling dimension using the quartic Casimir equation with WKB-like approximation. This gives

$$
\begin{aligned}
F_{\Delta,\ell}(x) =& (1-\rho^2)^{-\epsilon-1}(4\rho)^{\Delta} \\
& \times \exp\left[\frac{1}{\Delta}\frac{\rho^2}{1-\rho^2}\frac{(1+\epsilon-\epsilon^2)\Delta^2+\epsilon(\epsilon-1)\ell^2}{\Delta^2-\ell^2}+O(\Delta^{-2})\right]
\end{aligned}
$$

where

$$
\epsilon = \frac{d-2}{2}, \quad \rho = \frac{x}{(1+\sqrt{1-x})^2}
$$

Large spacetime dimension

Unitarity bounds

$$\Delta = 0 \quad \text{or} \quad \Delta \geq \frac{d-2}{2} \sim \frac{d}{2}$$

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For $\Delta_\phi = \frac{d}{2}$, $\Delta_x = \frac{d}{2} \Rightarrow \phi$ resembles free fied

Saturation of tail bound

2d global conformal blocks at Ising point $\Delta_\phi = 1/8$