A precise and high-quality determination of $\alpha_s(m_Z)$

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talk based on results [1, 2, 3] *of the* ALPHA Collaboration

PRL 117 (2016) 182001, PRD 95 (2017) 014507 & [1706.03821]

Quantum Chromodynamics

In Euclidean space with gauge group $\mathrm{SU}(3)$ and N_f quark flavours:

$$\mathcal{L}^{\mathrm{QCD}} = -\frac{1}{2g^2}\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] + \sum_{i=1}^{N_\mathrm{f}} \overline{\psi}_i[\gamma_\mu D_\mu + m_i]\psi_i$$

describes a plethora of strongly interacting processes

- gauge invariant
- $N_\mathrm{f} + 1$ free parameters $\left\{ \begin{array}{ll} \text{strong coupling} & g^2 \\ \text{quark masses} & m_i, i = 1, \ldots, N_\mathrm{f} \end{array} \right\}$ require physical input

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- Reg. & Renormalization required $\rightsquigarrow \bar{g}(\mu), \bar{m}_i(\mu)$
- scale dependence follows *massless Renormalization Group eq. (RGE)*, defining the mass anomalous dimension τ & the β-function:

$$\tau(\bar{g}) \equiv \frac{\mu}{\bar{m}_i(\mu)}\frac{\partial \bar{m}_i(\mu)}{\partial \mu}\,, \qquad \beta(\bar{g}) \equiv \mu\frac{\partial \bar{g}(\mu)}{\partial \mu}$$

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see [1611.06102][4]

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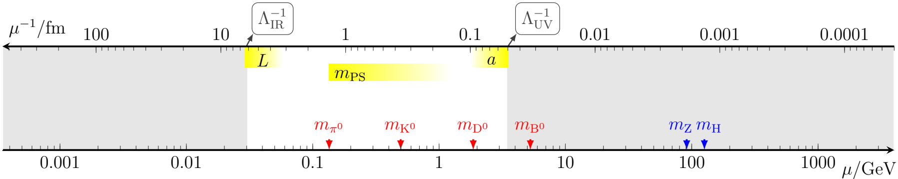
hadronic input

$m_\pi, f_\pi, \ldots$

$$\tau(\bar{g}) \equiv \frac{\mu}{\bar{m}_i(\mu)}\frac{\partial \bar{m}_i(\mu)}{\partial\mu}\,, \qquad \beta(\bar{g}) \equiv \mu\frac{\partial \bar{g}(\mu)}{\partial \mu}$$

- **Challenge:** *precise & accurate (high-quality) determination of α_s from 1st principles (Lattice QCD)*
- **Pitfall:** $\alpha_s(\mu)$ traditionally quoted at $\mu = m_\mathrm{Z}$ in $\overline{\mathrm{MS}}$ scheme

Running coupling and Lattice QCD



$$\beta(\bar{g}) \equiv Q \frac{\partial}{\partial Q} \bar{g}(Q)$$

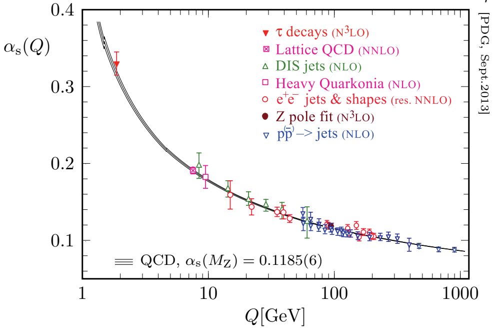
$$\Updownarrow$$

$$\ln\left[\frac{\mu}{\mu_0}\right] = \int_{\bar{g}(\mu_0)}^{\bar{g}(\mu)} \frac{\mathrm{d}g}{\beta(g)}$$

$$\Updownarrow$$

$$\Lambda^{(N_{\rm f})} = \lim_{\mu\to\infty} \mu \left[b_0 \bar{g}^2(\mu)\right]^{\frac{-b_1}{2b_0^2}}$$

Eqs. valid & exact beyond PT



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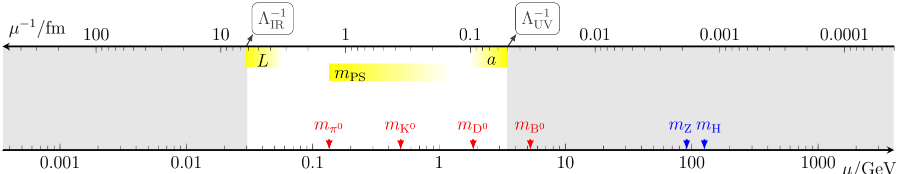

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$$\Updownarrow$$

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Asymptotic series at low-Q/large-α_s?

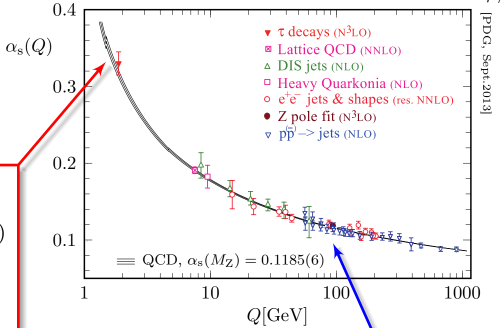
$$\beta(g) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3(b_0 + b_1\bar{g}^2 + b_2\bar{g}^4 + \ldots)$$

a) series truncation

b) non-perturbative effects
(instantons, renormalons, you name it)

$\Downarrow$

Bias when lattice obs. matched to PT.



Match to PT at high-Q/small-α_s!

Need to solve RGE non-perturbatively

Disentangle Large Volume (simulations) & Renormalization

Breakdown of the general approach:

- use non-perturbative (massless) renormalization scheme: Schrödinger functional (SF)
- cook up appropriate lattice observable for $\alpha_s(\mu)$: $\bar{g}^2_{\mathrm{qq}}, \bar{g}^2_{\mathrm{SF}}, \bar{g}^2_{\mathrm{GF}}, \ldots$
- relate scale μ to physical box size L: $\mu = 1/L$
- employ finite-size rescaling technique to map out

$$-\ln\left[\frac{L_1}{L_0}\right] = \int_{\bar{g}(L_0)}^{\bar{g}(L_1)} \frac{\mathrm{d}g}{\beta(g)} \quad \Leftrightarrow \quad \bar{g}^2(L_1) \equiv \sigma\Big(\bar{g}^2(L_0)\Big) = \lim_{a\to 0} \Sigma\Big(\bar{g}^2(L_0), a/L_0\Big)$$

for various scales L_0, L_1 with $L_1/L_0 = \mathrm{const}$

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for various scales L_0, L_1 with $L_1/L_0 = \text{const}$

$\Rightarrow$ non-perturbative $\beta(\bar{g})$ covering a wide range of scales $[\mu_{\rm min}, \mu_{\rm max}] \sim [0.2, 100]\,\mathrm{GeV}$

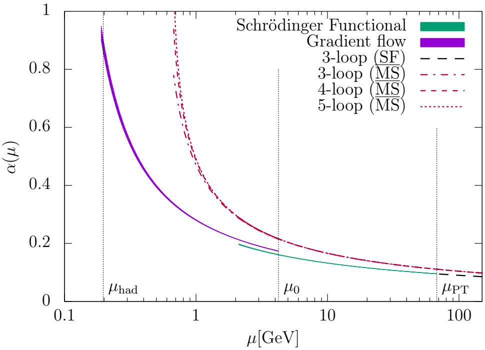
$$\Lambda \equiv \mu\left[b_0\bar{g}^2(\mu)\right]^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0\bar{g}^2(\mu))} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}g \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- *exact equation* $\forall\mu$ — for any $N_{\rm f}$ ($M_{\rm q} = 0$)
- *trivial scheme dependence* — Λ_a/Λ_b is 1-loop exact!

$$\bar{g}_a^2(\mu) = \bar{g}_b^2(\mu) + c_{ab}\bar{g}_b^4(\mu) + \ldots \quad \Rightarrow \quad \frac{\Lambda_a}{\Lambda_b} = \exp\left(c_{ab}/2b_0\right) \quad \text{, e.g., } \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm SF}}$$

Our strategy for $N_f = 3$ [5]

Two strong coupling definitions (schemes) in the SF setup



- PDG input enters $f_{\text{had}}^{\text{PDG}}$

  m_π, m_K, f_π, f_K

- $\bar{g}^2_{\text{GF}}(\mu_{\text{had}}) \equiv 11.31$

  $\mu_0/\mu_{\text{had}} = 21.86(42)$

- switch: $\bar{g}^2_{\text{GF}}(2\mu_0) = 2.6723(64)$
- $\bar{g}^2_{\text{SF}}(\mu_0) = 2.012$

  $\Lambda^{(3)}_{\overline{\text{MS}}}/\mu_0 = 0.0791(21)$

- $\mu_{\text{PT}}/\mu_{\text{had}} = 349.7(6.8)$

$$\underbrace{f_{\text{had}}^{\text{PDG}} \times \frac{\mu_{\text{had}}}{f_{\text{had}}}}_{\text{LV scale setting}} \times \underbrace{\frac{2\mu_0}{\mu_{\text{had}}}}_{\text{GF running}} \times \underbrace{\frac{\mu_0}{2\mu_0}}_{\text{scheme change}} \times \underbrace{\frac{\mu_{\text{PT}}}{\mu_0}}_{\text{SF running}} \times \underbrace{\frac{\Lambda^{(3)}_{\text{SF}}}{\mu_{\text{PT}}}}_{\text{PT@70GeV}} \times \underbrace{\frac{\Lambda^{(3)}_{\overline{\text{MS}}}}{\Lambda^{(3)}_{\text{SF}}}}_{\text{exact}} \equiv \Lambda^{(3)}_{\overline{\text{MS}}}$$

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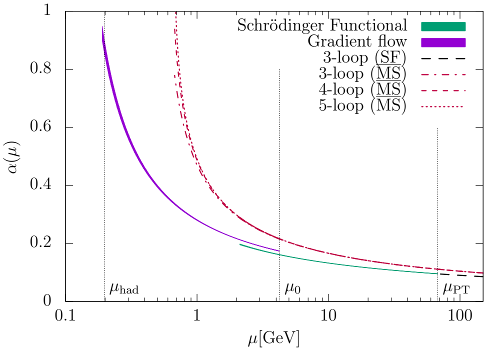

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$$\Lambda^{(3)}_{\overline{\text{MS}}} = 341(12)\ \text{MeV}$$

$\Lambda_{\overline{\mathrm{MS}}}^{(3)} \to \alpha_s^{(5)}(m_Z)$ via perturbative decoupling

Decoupling relation

$$\bar{g}^{N_f}(\mu) = \bar{g}^{N_f+1}(\mu) \times \xi\left(g^{N_f}(\mu), \overline{m}_h/\mu\right) \quad + \quad O(\overline{m}_h^{-2})$$

or equivalently relation for $\Lambda^{(N_f)}/\Lambda^{(N_f+1)}$

- requires further PDG input ($\overline{\mathrm{MS}}$ scheme)

  $\overline{m}_c(\overline{m}_c)$ $\qquad$ $\overline{m}_b(\overline{m}_b)$

- $O(\overline{m}_h^{-2})$ already very small[6, 7] for $h = c$
- ξ known in PT to 4 loops[8, 9]
- for decoupling perturbation theory looks surprisingly well-behaved already at $\mu = \overline{m}_c$
- Future: include charm non-perturbatively

n (loops)	$\alpha_{\overline{\mathrm{MS}}}^{(N_f=5)}$	$\alpha_n - \alpha_{n-1}$
1	0.11699	
2	0.11827	0.00128
3	0.11846	0.00019
4	0.11852	0.00006

$$\Delta\alpha = \alpha_4 - \alpha_2 \approx 0.00025$$

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$$\alpha_s^{(5)}(m_Z) = 0.11852(80)(25)\,\mathrm{MeV}$$

PDG-16:[10] $\alpha_s^{(5)}(m_Z) = 0.1174(16)$ w/o lattice

FLAG-16:[11] $\alpha_s^{(5)}(m_Z) = 0.1182(12)$

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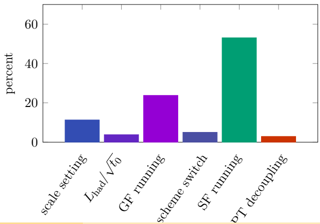
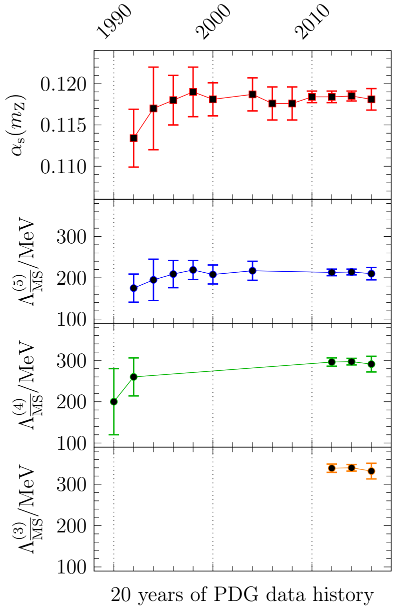
Contribution to relative error squared



Historic $\alpha_s(m_Z)$ averages



Note:

- recent *increase of uncertainty*
- quoting $\alpha_s^{\overline{MS}}$ at $\mu = m_Z$ is a *convention*

PDG-2016 values:

$$\alpha_s(m_Z) = 0.1181(11) \qquad \sim 0.9\%$$

$$\Lambda^{(5)}_{\overline{MS}} = 210(14)\,\text{MeV} \qquad \sim 6.7\%$$

$$\Lambda^{(4)}_{\overline{MS}} = 292(16)\,\text{MeV} \qquad \sim 5.5\%$$

$$\Lambda^{(3)}_{\overline{MS}} = 332(17)\,\text{MeV} \qquad \sim 5.1\%$$

Our results:

$$\Lambda^{(3)}_{\overline{MS}} = 341(12)\,\text{MeV} \qquad \sim 3.5\%$$

$$\alpha_s(m_Z) = 0.11852(84) \qquad \sim 0.7\%$$

Experimental support for QCD

RG running of α_s, past and present

tremendous progress over the years

RG scale evolution consistent with data

1989 Altarelli[12]

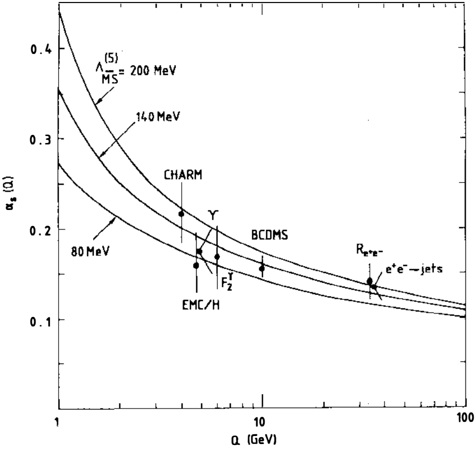
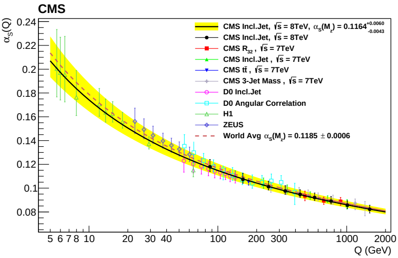


Figure 7 A summary of the determinations of the running coupling constant α_s discussed in the text. The curves for $\Lambda^{(5)}_{\overline{MS}} = 140 \pm 60$ MeV are obtained following the matching procedure at the b threshold explained in Equations 14–19 (with $a \approx 1$).

2016 CMS[13]

$$\alpha_s^{(6)}(1.508\ \text{TeV}) = 0.0840(35)$$



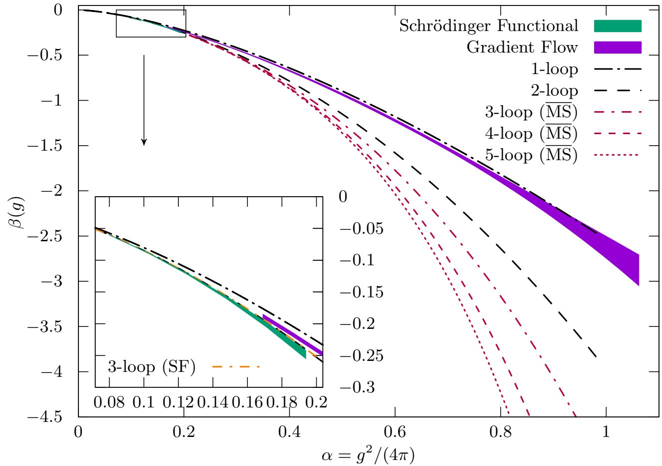
ALPHA Collaboration: $\alpha_s^{(6)}(1.508\ \text{TeV}) = 0.0852(4)$

And many thanks to my collaborators:

- M. Bruno
- M. Dalla Brida
- T. Korzec
- A. Ramos
- S. Schaefer
- H. Simma
- S. Sint
- R. Sommer

Backup slides

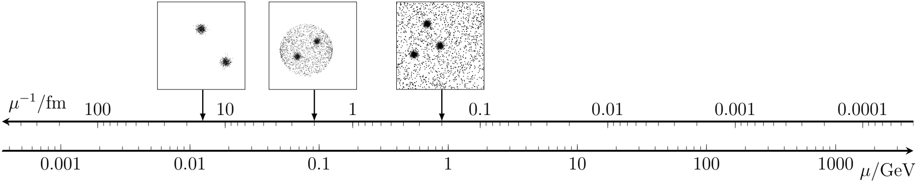
Non-perturbative β functions



QCD at different length scales

QCD in a box:

Bjorken[14]



$\leftsquigarrow$ confinement region

asymptotic freedom ($\mu \to \infty$)

QCD at different length scales:

- 2 hadrons in a box of size $L = 15\,\mathrm{fm}$
- a $q\overline{q}$ meson in a $2\,\mathrm{fm}$ box
- interacting quarks and gluons at $L = 0.2\,\mathrm{fm}$

The Femtouniverse ($L \ll 0.5\,\mathrm{fm}$):

- in principle, QCD phenomena can be probed at any scale
- a single experiment cannot achieve that
- ideal playground for lattice QCD as a non-perturbative toolbox at all scales
  (boundary conditions very important at intermediate or small box sizes)

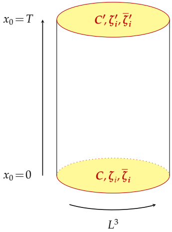
The Schrödinger functional coupling

- Euclidean partition function

$$\mathcal{Z} \equiv \int_{T\times L^3} \mathcal{D}\left[U,\overline{\psi},\psi\right] \mathrm{e}^{-S[U,\overline{\psi},\psi]} = \langle 0|\, \mathrm{e}^{-TH} P|0\rangle$$

  with *periodic BC in L^3*
  and *Dirichlet BC in T* (breaking translational inv. in time)

- renormalization scale $\mu \propto L^{-1}$ (for step-scaling)
- mass-independent scheme, …



Abelian boundary fields: $C_k = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix}$; $C'_k = \begin{pmatrix} \phi'_1 & 0 & 0 \\ 0 & \phi'_2 & 0 \\ 0 & 0 & \phi'_3 \end{pmatrix}$

SF coupling

defined as variation of effective action $\Gamma = -\ln \mathcal{Z}[C, C']$,

$$\left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=0} = \frac{\mathrm{const}}{\bar{g}^2_{\mathrm{SF}}(L)}$$

for non-vanishing boundary gauge fields $C_k \neq 0 \neq C'_k$

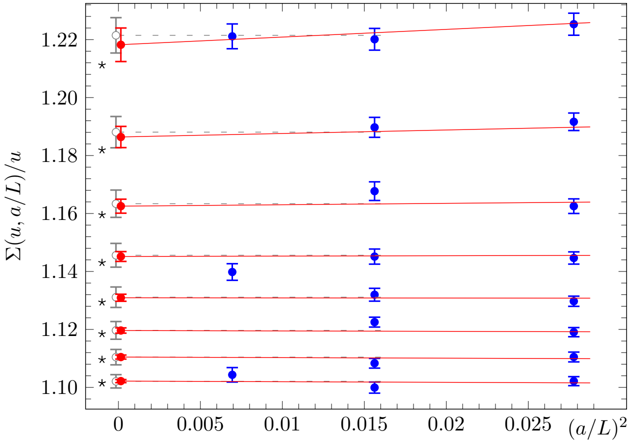
[15] Lüscher+ '92,…

Non-perturbative running at high energies

Continuum extrapolation of SSF $\Sigma_{\rm SF}$[1]

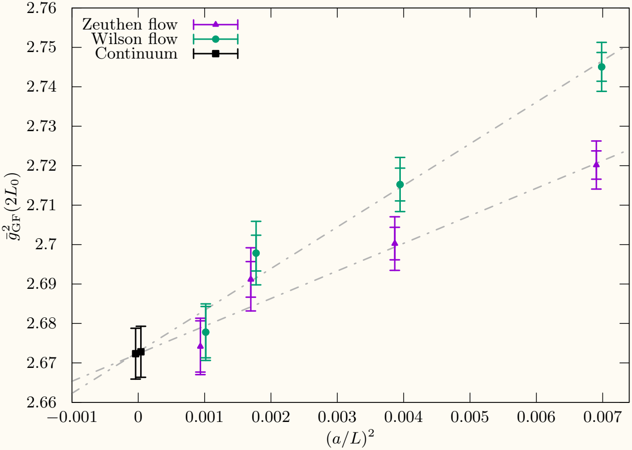
Example for global fit ansatz: (4 parms., 19 pts., $\chi^2/N_{\rm dof} \approx 1$)

$$\Sigma(u, a/L) = u + s_0 u^2 + s_1 u^3 + c_1 u^4 + c_2 u^5 + \rho_1 u^4 (\tfrac{a}{L}) + \rho_2 u^5 (\tfrac{a}{L})^2$$



s_0, s_1 fixed to perturbative values: $s_0 = 2b_0 \ln(s)$, $s_1 = s_0^2 + 2b_1 \ln(s)$, $s = 2$

$$\bar{g}^2_{\mathrm{SF}}(L_{\mathrm{swi}}) = 2.0120 \quad \Rightarrow \quad \bar{g}^2_{\mathrm{GF}}(2L_{\mathrm{swi}}) = 2.6723(64)$$

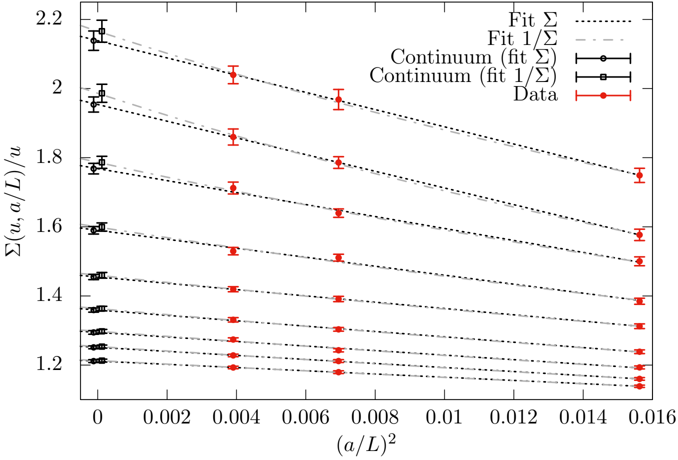


- 0.2% total uncertainty (continuum)[2]

$2L_{\mathrm{swi}}/a \in \{12, 16, 24, 32\}$

Non-perturbative running at low energies

Taking the continuum limit[2]



- sizeable discretization effects → requires **careful** extrapolations $\quad 2L/a \in \{16, 24, 32\}$
- nonetheless, continuum results are very **precise**!

Why not just a single coupling?

Two of the most important reasons:

- need (exact) 1-loop matching to $\bar{g}^2_{\overline{\rm MS}}$ to obtain $\Lambda_{\overline{\rm MS}}/\Lambda_{\rm GF}$
- precision goal difficult to reach otherwise
- PDG criteria (only NNLO QCD results enter avg.) would not be met

$$I[\bar{g}] = \int_0^{\bar{g}(\mu)} \mathrm{d}g \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] = \left(\frac{b_2}{2b_0^2} - \frac{b_1^2}{2b_0^3} \right) \bar{g}^2(\mu) + \ldots$$

$\Rightarrow \quad \exp(-I[\bar{g}]) \equiv 1$ until b_2 is known

Bibliography I

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[2] M. Dalla Brida, P. Fritzsch, T. Korzec, A. Ramos, S. Sint and R. Sommer, *Slow running of the Gradient Flow coupling from 200 MeV to 4 GeV in $N_\mathrm{f} = 3$ QCD*, *Phys. Rev.* **D95** (2017) 014507, [1607.06423].

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