Light Pseudoscalar Mesons: an Inverse Instantaneous Bethe–Salpeter Glimpse

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#Massless #QuarkAntiquarkBoundState

For ground-state pseudoscalar mesons their description as quark–antiquark bound states must reflect their near masslessness demanded by Goldstone's theorem for bosons connected to dynamical (and explicit) breaking of chiral symmetries of quantum chromodynamics, the theory of strong interactions.

The underlying effective interactions enabling such common picture may be elucidated by inversion [1,2] of the Bethe–Salpeter (BS) formalism, suitably simplified by adequate three-dimensional (3D) reductions [3,4]. From these, we gain information [5–7] in form of central potentials [8–10] $V(r)$, $r \equiv |\boldsymbol{x}|$.

Strictly respecting **Poincaré covariance**, the homogeneous BS equation describing bound states of a fermion f and an antifermion $\bar{\mathrm{f}}$ of momenta $p_{\mathrm{f},\bar{\mathrm{f}}}$,

$$\Phi(p) \propto S_{\mathrm{f}}(p_{\mathrm{f}}) \int \mathrm{d}^4 q\, K(p,q)\, \Phi(q)\, S_{\bar{\mathrm{f}}}(-p_{\bar{\mathrm{f}}}) \ ,$$

is defined by its integral kernel $K(p,q)$, subsuming the effective interactions forming these bound states, and two bound-state constituents' propagators

$$S(p) = \frac{\mathrm{i}\, Z(p^2)}{\not p - M(p^2) + \mathrm{i}\,\varepsilon} \ , \qquad \not p \equiv p^\mu\, \gamma_\mu \ , \qquad \varepsilon \downarrow 0 \ ,$$

each involving **two** functions: the fermion's mass $M(p^2)$ and wave-function renormalization $Z(p^2)$. The solutions, the BS amplitudes $\Phi(p)$, capture the distribution of the fermions' relative momenta p. For simplicity, we skip the state's total momentum and assume the fermions to carry the same flavour.

3D-reduced Bethe–Salpeter formalism . . .

If the BS kernel only involves spatial components of both relative momenta, $K(p,q)=K(\boldsymbol{p},\boldsymbol{q})$, and the dependence of the propagator functions $M(p^2)$ and $Z(p^2)$ on p_0^2 can be dropped, integrating w.r.t. p_0 casts the BS equation into a bound-state equation [3] for a Salpeter amplitude $\phi(\boldsymbol{p})\propto\int \mathrm{d}p_0\,\Phi(p)$,

$$\phi(\boldsymbol{p})=Z^2(p^2)\left(\frac{\Lambda^+(\boldsymbol{p})\,\gamma_0\,I(\boldsymbol{p})\,\Lambda^-(\boldsymbol{p})\,\gamma_0}{\widehat{M}-2\,E(p)}-\frac{\Lambda^-(\boldsymbol{p})\,\gamma_0\,I(\boldsymbol{p})\,\Lambda^+(\boldsymbol{p})\,\gamma_0}{\widehat{M}+2\,E(p)}\right),$$

with one-fermion free energies and projectors (henceforth $p\equiv|\boldsymbol{p}|$, $q\equiv|\boldsymbol{q}|$)

$$E(p)\equiv\sqrt{p^2+M^2(p^2)}\ ,\qquad \Lambda^\pm(\boldsymbol{p})\equiv\frac{E(p)\pm\gamma_0\,[\boldsymbol{\gamma}\cdot\boldsymbol{p}+M(p^2)]}{2\,E(p)}\ ,$$

and the (by assumption now instantaneous) interactions entering in form of

$$I(\boldsymbol{p})\propto\int \mathrm{d}^3q\,K(\boldsymbol{p},\boldsymbol{q})\,\phi(\boldsymbol{q})\ .$$

The action of $K(\boldsymbol{p},\boldsymbol{q})$ on $\phi(\boldsymbol{q})$ may be expanded in tensor products $\Gamma\otimes\Gamma$ of (generalized) Dirac matrices Γ identifying its Lorentz nature and associated Lorentz-scalar functions $V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ encoding all the momentum dependence:

$$K(\boldsymbol{p},\boldsymbol{q})\,\phi(\boldsymbol{q})=\sum_\Gamma V_\Gamma(\boldsymbol{p},\boldsymbol{q})\,\Gamma\,\phi(\boldsymbol{q})\,\Gamma\ .$$

In the free-constituent-propagator case $M(p^2)=\text{const}$ and $Z(p^2)=1$, this bound-state equation reduces to, hence generalizes, Salpeter's equation [11].

. . . and its Dyson–Schwinger-based inverse

In order to define a well-posed inversion problem, we have to be a little more specific about the physical systems we interested in: We study bound states of quark and associated antiquark. For **all** pseudoscalar bound states of two fermions of spin $\frac{1}{2}$, the total-spin **and** orbital-angular-momentum quantum numbers both inevitably vanish; as a consequence, their Salpeter amplitude is fixed by only **two** independent components $\varphi_{1,2}(\boldsymbol{p})$ and can be written as

$$\phi(\boldsymbol{p})=\left[\varphi_1(\boldsymbol{p})\,\frac{\gamma_0\,[\boldsymbol{\gamma}\cdot\boldsymbol{p}+M(p^2)]}{E(p)}+\varphi_2(\boldsymbol{p})\right]\gamma_5\ .$$

The **ansatz** $2\,\Gamma\otimes\Gamma=\gamma_\mu\otimes\gamma^\mu+\gamma_5\otimes\gamma_5-1\otimes 1$ ensures the Fierz symmetry of the effective interactions. Assuming the latter to be encoded in a spherically symmetric convolution-type potential $V((\boldsymbol{p}-\boldsymbol{q})^2)$ prompts us to discard all (then trivial) angular-variable dependences, allowing us to condense our 3D bound-state equation to two coupled equations for the radial factors $\varphi_{1,2}(p)$ of the independent components; one of the two relations is purely algebraic:

$$
\begin{aligned}
&2\,E(p)\,\varphi_2(p)+2\,Z^2(p^2)\int\limits_0^\infty\frac{\mathrm{d}q\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q)=\widehat{M}\,\varphi_1(p)\ ,\\
&2\,E(p)\,\varphi_1(p)=\widehat{M}\,\varphi_2(p)\ .
\end{aligned}
$$

wherein $\widehat{M}$ is the bound-state mass and the potential $V(r)$ enters in form of

$$
V(p,q)\propto\frac{1}{p\,q}\int\limits_0^\infty\mathrm{d}r\sin(p\,r)\sin(q\,r)\,V(r)\ .
$$

For mass $\widehat{M}=0$, characteristic of Goldstone bosons, the algebraic equation forces $\varphi_1(p)$ to vanish and the remaining component $\varphi_2(p)$ is determined by

$$
E(p)\,\varphi_2(p)+Z^2(p^2)\int\limits_0^\infty\frac{\mathrm{d}q\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q)=0\ .
$$

Denoting, for $Z(p^2)\neq 0$, the Fourier transform of the effective kinetic term $E(p)\,\varphi_2(p)/Z^2(p^2)$ by $T(r)$, $V(r)$ is easily extracted in configuration space:

$$
T(r)+V(r)\,\varphi_2(r)=0\qquad\Longrightarrow\qquad V(r)=-\frac{T(r)}{\varphi_2(r)}\ .
$$

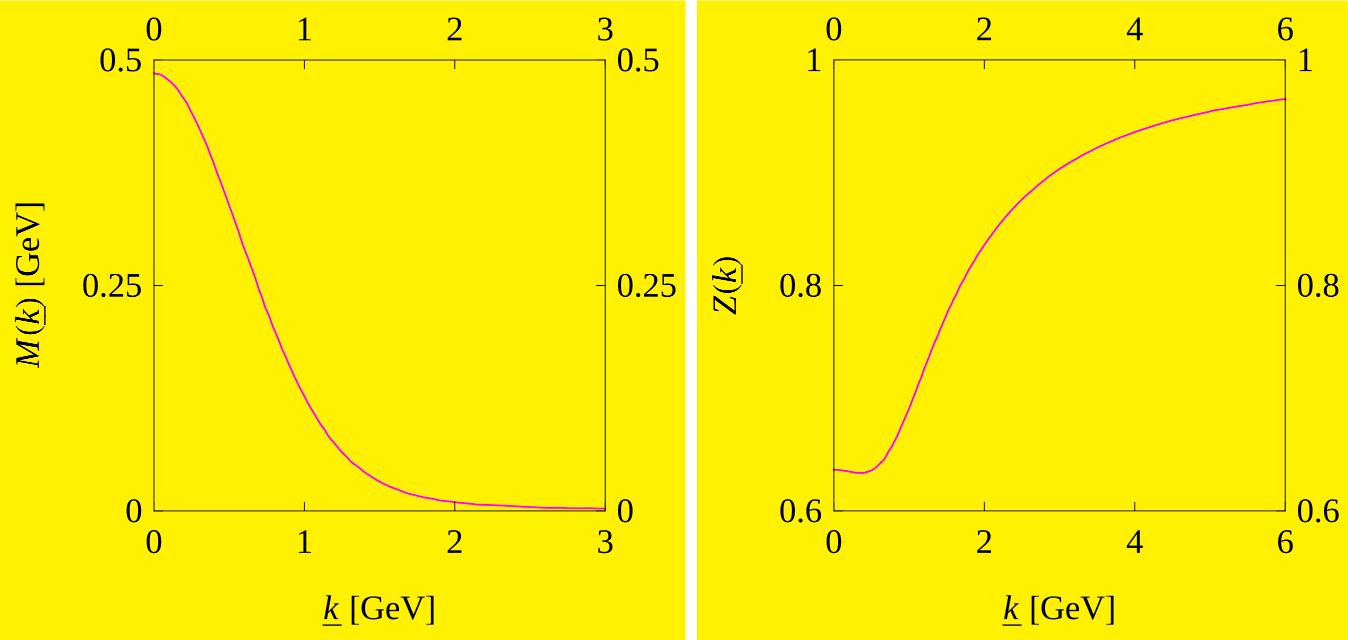
Application to Light Pseudoscalar Mesons

Gauge invariance of a quantum field theory entails identities that relate this theory's Green functions; one of these Euclidean-space identities relates the rest-frame BS amplitude for massless pions and the quark propagator [5,12]:

$$
\Phi(\underline{k})\propto\frac{Z(\underline{k}^2)\,M(\underline{k}^2)}{\underline{k}^2+M^2(\underline{k}^2)}\,\underline{\gamma}_5+\text{subleading contributions}\ .
$$

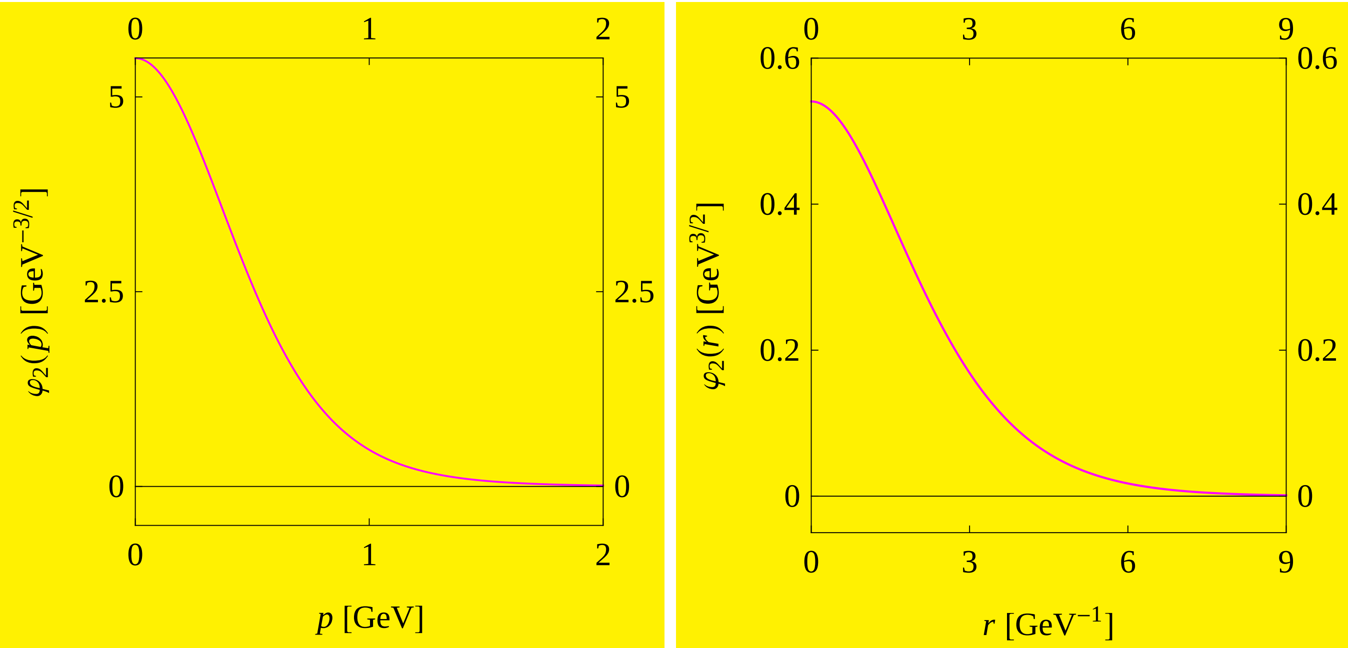
This serendipitous finding enables the construction of $\Phi(\underline{k})$ by knowledge of $M(\underline{k}^2)$ and $Z(\underline{k}^2)$. These functions may be found by solving the equation of motion of the quark propagator: for our purpose, we adopt numerical model outcomes [13] distilled pointwise from their behaviour presented in Ref. [14].

Quark propagator functions $M(\underline{k})$ (left) and $Z(\underline{k})$ (right) vs. $\underline{k} \equiv \sqrt{\underline{k}^2}$ [14]:



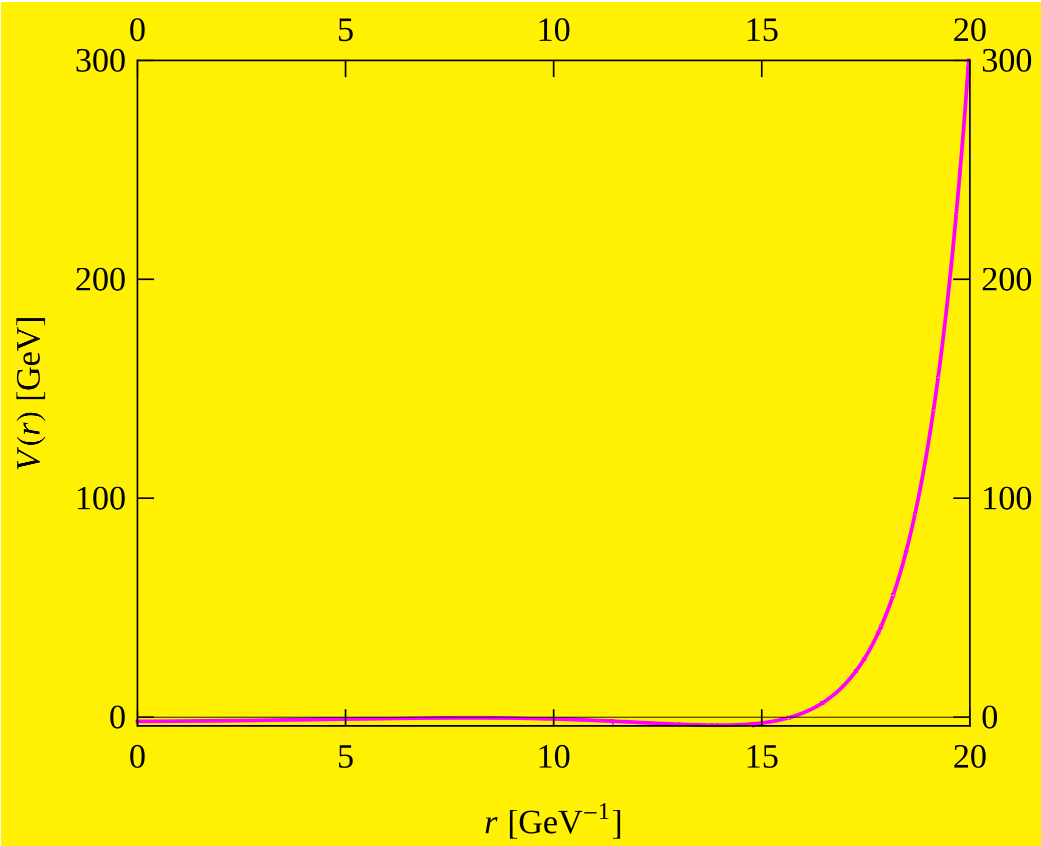
Mimicking the p_0 integration in the definition of $\phi(\boldsymbol{p})$ by integrating over $\underline{k}_4$ gives us $\varphi_2(p)$ and its Fourier counterpart $\varphi_2(r)$, the sought inversion input.

Salpeter component in momentum (left) and configuration (right) space [8]:



Division of the effective kinetic term $T(r)$ by $\varphi_2(p)$ takes us to the potential $V(r)$ aimed for. This potential's unexpected square-well shape resemblance we consider as this exercise's true quintessence: we find $V(0) = -1.92$ GeV (finite), $V(15.70\ \text{GeV}^{-1}) = 0$ (single zero), $V(r) \xrightarrow[r\to+\infty]{} \infty$ (confinement).

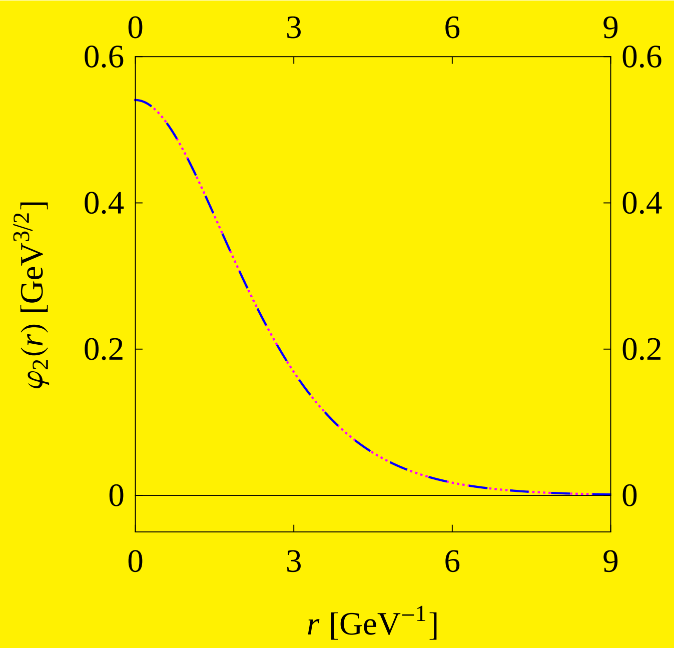
Potential $V(r)$ [8] derived by inverting our 3D bound-state equation [3] with Fierz-invariant interaction kernel, for a "lightest-quark" propagator [13,14]:

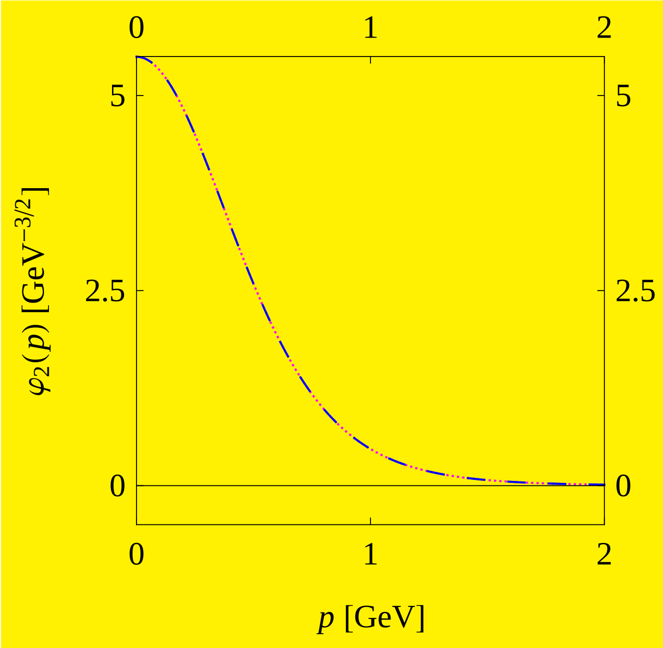


A brief inspection with the naked eye reveals that the ground-state solution of our 3D equation with $V(r)$ as inferred by inversion yields, **inevitably**, a reasonable size of the pion: its average interquark distance, $\langle r \rangle = 0.483$ fm, and root-mean-square radius, $\sqrt{\langle r^2 \rangle} = 0.535$ fm, predicted by our starting point $\varphi_2(r)$ of the inversion do match the experimentally measured value of the electromagnetic charge radius, $\sqrt{\langle r_\pi^2 \rangle} = (0.672 \pm 0.008)$ fm, of the pion.

Consistency of such inversion can be established by numerically solving, for the effective interaction potential, the bound-state equation variationally or by expansion over a suitable basis. Our approach passes with flying colours. In both cases, the overlap of wave-function input and outcome equals unity.

Ground-state solution (dotted) to our 3D bound-state equation, inferred by application of variational techniques (left) or by conversion to an equivalent matrix eigenvalue problem (right), vs. initial Salpeter component (dashed):





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