Exotic hadrons at large $N_c$ and in QCD sum rules

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1. QCD at large $N_c$ and properties of the Green functions containing potential exotic poles
2. Implications for QCD sum rules for exotic states vs ordinary hadrons

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QCD at large $N_c$

$SU(N_c)$ gauge theory with $N_c \to \infty$ and $\alpha_s \sim 1/N_c$. At leading order, QCD Green functions have only non-interacting mesons as intermediate states; tetraquark bound states may emerge only in $N_c$-subleading diagrams. This fact was believed to provide the theoretical explanation of the non-existence of exotic tetraquarks.

However, even if the exotic tetraquark bound states appear only in subleading diagrams, the crucial question is their width: if narrow, they might be well observed in nature.

We discuss four-point Green functions of bilinear quark currents, depend on 6 variables $p_1^2, p_2^2, p_1'^2, p_2'^2, p = p_1 + p_2 = p_1' + p_2'$, and the two Mandelstam variables $s = p^2$ and $t = (p_1 - p_1')^2$.

Criteria for selecting diagrams which potentially contribute to the tetraquark pole at $s = M_i^2$:

1. The diagram should have a nontrivial (i.e., non-polynomial) dependence on the variable $s$.

2. The diagram should have a four-particle cut (i.e. threshold at $s = (m_1 + m_2 + m_3 + m_4)^2$), where $m_i$ are the masses of the quarks forming the tetraquark bound state. The presence or absence of this cut is established by solving the Landau equations for the corresponding diagram.
Flavour-exotic tetraquarks

Bilinear quark currents \( j_{ij} = \bar{q}_i q_j \) producing \( M_{ij} \) from the vacuum, \( \langle 0 | j_{ij} | M_{ij} \rangle = f_{M_{ij}}, f_{M} \sim \sqrt{N_c} \).

“Direct” 4-point functions \( \Gamma^{(\text{dir})}_I = \langle j_{12}^\dagger j_{34}^\dagger j_{12} j_{34} \rangle \) and \( \Gamma^{(\text{dir})}_II = \langle j_{14}^\dagger j_{32}^\dagger j_{14} j_{32} \rangle \):

\[
\begin{align*}
\sim N_c^2 & \quad \sim N_c^3 \alpha_s & \sim N_c^2 \alpha_s^2 \\
(a) & (b) & (c)
\end{align*}
\]

“Recombination” functions \( \Gamma^{(\text{rec})} = \langle j_{12}^\dagger j_{34}^\dagger j_{14} j_{32} \rangle \) and \( \Gamma^{(\text{rec})\dagger} = \langle j_{14}^\dagger j_{32}^\dagger j_{12} j_{34} \rangle \):

\[
\begin{align*}
\sim N_c & \quad \sim N_c^2 \alpha_s & \sim N_c \alpha_s^2 \\
(a) & (b) & (c)
\end{align*}
\]
\[ \sim N_c \] 
\[ \sim N_c^2 \alpha_s \] 
\[ \sim N_c^2 \alpha_s \]
\( \Gamma_{I,T}^{(\text{dir})} = \langle j_{12}^\dagger j_{34}^\dagger j_{12} j_{34} \rangle = O(N_c^0), \quad \Gamma_{II,T}^{(\text{dir})} = \langle j_{14}^\dagger j_{32}^\dagger j_{14} j_{32} \rangle = O(N_c^0), \quad \Gamma_{T}^{(\text{rec})} = \langle j_{12}^\dagger j_{34}^\dagger j_{14} j_{32} \rangle = O(N_c^{-1}). \)

The fact that “dir” and “rec” amplitudes have different behaviors in \( N_c \) requires at least two exotic poles:

\( T_A \) couples stronger to \( M_{12}M_{34} \) channel, \( T_B \) couples stronger to \( M_{14}M_{32} \) channel.

\[
\Gamma_{I,T}^{(\text{dir})} = O(N_c^0) = f^4_M \left( \frac{|A(M_{12}M_{34} \to T_A)|^2}{p^2 - M_{T_A}^2} + \frac{|A(M_{12}M_{34} \to T_B)|^2}{p^2 - M_{T_B}^2} \right) + \cdots,
\]

\[
\Gamma_{II,T}^{(\text{dir})} = O(N_c^0) = f^4_M \left( \frac{|A(M_{14}M_{32} \to T_A)|^2}{p^2 - M_{T_A}^2} + \frac{|A(M_{14}M_{32} \to T_B)|^2}{p^2 - M_{T_B}^2} \right) + \cdots,
\]

\[
\Gamma_{T}^{(\text{rec})} = O(N_c^{-1}) = f^4_M \left( \frac{A(M_{12}M_{34} \to T_A)A(T_A \to M_{14}M_{32})}{p^2 - M_{T_A}^2} + \frac{A(M_{12}M_{34} \to T_B)A(T_B \to M_{14}M_{32})}{p^2 - M_{T_B}^2} \right) + \cdots.
\]

We seek tetraquarks with finite mass at large \( N_c \):

\[
A(T_A \to M_{12}M_{34}) = O(N_c^{-1}), \quad A(T_A \to M_{14}M_{32}) = O(N_c^{-2}),
\]

\[
A(T_B \to M_{12}M_{34}) = O(N_c^{-2}), \quad A(T_B \to M_{14}M_{32}) = O(N_c^{-1}).
\]

The widths \( \Gamma(T_{A,B}) = O(N_c^{-2}) \).
Cryptoexotic tetraquarks

Diagrams of new topologies emerge.

For direct functions $\Gamma^{(\text{dir})}_{(I,II),T}$, new diagrams do not change leading large-$N_c$ behavior:

For recombination functions, the new diagram modifies leading large-$N_c$ behavior.
The new diagram modifies the leading large-$N_c$ behavior:

\[ \Gamma_{I,T}^{(\text{dir})} = \langle j_{12}^{\dagger} J_{23} j_{12} J_{23} \rangle = O(N_c^0), \quad \Gamma_{II,T}^{(\text{dir})} = \langle j_{13}^{\dagger} J_{22} j_{13} J_{22} \rangle = O(N_c^0), \quad \Gamma_T^{(\text{rec})} = \langle j_{12}^{\dagger} J_{23} j_{13} J_{22} \rangle = O(N_c^0). \]

“dir” and “rec” functions have the same behavior, and one exotic state $T$ is enough:

\[ A(T \to M_{12} M_{23}) = O(N_c^{-1}), \quad A(T \to M_{13} M_{22}) = O(N_c^{-1}). \]

Its width is $\Gamma(T) = O(N_c^{-2})$.

$T$ can mix with the ordinary meson $M_{13}$. The restriction on the mixing parameter $g_{TM_{13}}$:

\[ \Gamma_{I,T}^{(\text{dir})} = O(N_c^0) = f_M^4 \left( \frac{A(M_{12} M_{23} \to T)}{p^2 - M_T^2} g_{TM_{13}} \frac{A(M_{13} \to M_{12} M_{23})}{p^2 - M_{M_{13}}^2} \right) + \ldots. \]

\[ A(M_{13} \to M_{12} M_{23}) \sim 1/ \sqrt{N_c}, \text{ so } g_{TM_{13}} \leq O(1/ \sqrt{N_c}). \]

The analysis of Green functions in large-$N_c$ QCD allows one to restrict some properties of the possible exotic states.
Strong decays from 3–point vertex functions

• The basic object:

\[ \Gamma(p, p', q) = \int \langle \Omega | T(J(x) j(0) j'(x') | \Omega \rangle \exp(ipx - ip' x') dx dx' \]

This correlator contains the triple-pole in the Minkowski region: namely

\[ \Gamma(p, p', q) = \frac{ff'}{(p^2 - M^2)(p'^2 - M'^2)} F(q^2) + \cdots \]

where the form factor \( F(q^2) \) contains pole at \( q^2 = M_q^2 \):

\[ F(q^2) = \frac{f_q g_{MM'M_q}}{(q^2 - M_q^2)} + \cdots \]

\( g_{MM'M_q} \) describes the \( M \to M_1 M_2 \) strong transition;

\( f, f', \) and \( f_{M_q} \) are the decay constants of the mesons \( \langle 0 | j(0) | M \rangle = f_M \).
• Normal hadrons:
\[ \Gamma_{\text{OPE}}(p^2, p'^2, q^2) = \Gamma_0(p^2, p'^2, q^2) + \alpha_s \Gamma_1(p^2, p'^2, q^2) + \ldots \]

Already one-loop lowest-order diagram has a nonzero double-spectral density, and therefore provides a nonzero contribution to the form factor.

• Exotic hadrons: \[ \langle T(\theta(x)j_1(0)j_2(y)) \rangle. \]
Many possibilities to write interpolating current for \( X \), \[ \langle 0|\theta|X \rangle = f_X, \quad f_X \neq 0. \]
\[ \theta = M(x)M(x), \quad \theta = M^A(x)M^A(x), \quad \theta = \bar{D}^a(x)D^a(x). \]

Color singlet - color singlet:

\[ \Gamma_{\text{OPE}}(p^2, p'^2, q^2) = \Pi(p'^2)\Pi(q^2) + (\alpha_s)^2 \Gamma_{\text{connected}}(p^2, p'^2, q^2) \]

Thus the LO contribution is not related to the exotic-state decay.
Conclusions

• **QCD at large $N_c$:**

Large-$N_c$ QCD restricts properties of the exotic poles if such poles exist. E.g., two exotic $\bar{q}_1 q_2 \bar{q}_3 q_4$ narrow states $\Gamma \sim O(1/N_c^2)$, each decaying into one meson-meson channel.

One cryptoexotic state $\bar{q}_1 q_2 \bar{q}_2 q_3$ $\Gamma \sim O(1/N_c^2)$ decaying into various meson-meson channels with similar probabilities.

• **Dynamics of exotic-state decays and implications for QCD sum rules:**

Dynamics of fall-apart decays of exotic resonances has fundamental difference from dynamics of ordinary-meson decays: the appropriate contributions to Green functions describing decays of exotic states emerge only at subleading $\alpha_s$ orders; the leading order disconnected diagrams are not related to strong decays of exotic hadrons. This makes the calculation of $\alpha_s$-corrections mandatory.

*Many efforts for obtaining reliable predictions are necessary!*