Non-Linear Invariance of Black Hole Entropy

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see also (other applications)

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Summary

Maxwell-Einstein-Scalar Gravity Theories

Extremal Black Holes

Attractor Mechanism

Freudenthal Duality

Groups of type E_7

Hints for the Future…

Maxwell-Einstein-Scalar Theories

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}\left(\varphi\right) \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}\left(\varphi\right) F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}\left(\varphi\right) \epsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma}_{\rho\sigma}$$

D=4 Maxwell-Einstein-scalar system (with no potential)

[may be the bosonic sector of **D=4** (ungauged) sugra]

$$H := \left(F^\Lambda, G_\Lambda\right)^T;$$

$$^*G_{\Lambda|\mu\nu} := 2\frac{\delta\mathcal{L}}{\delta F^{\Lambda|\mu\nu}}.$$

Abelian 2-form field strengths

static, spherically symmetric, asymptotically flat, **extremal Black Hole (BH)**

$$ds^2 = -e^{2U(\tau)} dt^2 + e^{-2U(\tau)} \left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(d\theta^2 + \sin\theta d\psi^2\right)\right]$$

$$\tau := -1/r$$

$$\mathcal{Q} \ : \ = \int_{S^2_\infty} H = \left(p^\Lambda, q_\Lambda\right)^T;$$

$$p^\Lambda \ : \ = \frac{1}{4\pi}\int_{S^2_\infty} F^\Lambda, \ q_\Lambda = \frac{1}{4\pi}\int_{S^2_\infty} G_\Lambda.$$

dyonic vector of e.m. fluxes (**BH charges**)

$$S_{D=1} = \int [(U')^2 + g_{ij}\varphi'^i\varphi'^j + e^{2U}V_{BH}(\varphi(\tau), \mathcal{Q})]d\tau \qquad ' \equiv \frac{d}{d\tau}$$

reduction D=4 →D=1 :effective 1-dimensional (radial) Lagrangian

$$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\mathcal{Q},$$

BH effective potential

Ferrara,Gibbons,Kallosh ‘96

eoms
$$\begin{cases} \frac{d^2U}{d\tau^2} = e^{2U}V_{BH}; \\ \frac{d^2\varphi^i}{d\tau^2} = g^{ij}e^{2U}\frac{\partial V_{BH}}{\partial\varphi^j}. \end{cases}$$

in N=2 ungauged sugra, hyper mults. decouple, and we thus disregard them : scalar fields belong to vector mults.

Attractor Mechanism : $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau\to-\infty}\varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

conformally flat geometry $AdS_2 \times S^2$ near the horizon

$$ds^2_{\text{B-R}} = \frac{r^2}{M^2_{\text{B-R}}}dt^2 - \frac{M^2_{\text{B-R}}}{r^2}(dr^2 + r^2 d\Omega)$$

near the horizon, the scalar fields are **stabilized** purely in terms of charges

$$S = \frac{A_H}{4} = \pi\, V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T\mathcal{M}_H\mathcal{Q}$$

Bekenstein-Hawking entropy-area formula for extremal dyonic BH

Particular Class : Symmetric Scalar Manifolds

A remarkable class of Einstein-Maxwell-scalar theories is endowed with scalar manifolds which are **symmetric cosets G/H**

[in presence of local SUSY :**N>2** : general, **N=2** : particular, **N=1** : special cases]

H = isotropy group = *linearly* realized; scalar fields sit in an **H**-repr.

G = (global) **electric-magnetic duality** group, on-shell symmetry

General Features in D=4

The 2-form field strengths (F,G) vector and the BH e.m. charges sit in a **G**-repr. **R** which is **symplectic** :

$$\exists ! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R};$$

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \equiv \mathcal{Q}_1^M \mathcal{Q}_2^N \mathbb{C}_{MN} = - \langle \mathcal{Q}_2, \mathcal{Q}_1 \rangle$$

symplectic product

$$\mathbb{C} = \begin{pmatrix} \mathbf{0}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbf{0}_n \end{pmatrix}$$

$$\begin{array}{rcl} G & \subset & Sp(2n, \mathbb{R}); \\ \mathbf{R} & = & \mathbf{2n} \end{array}$$

in physics : **Gaillard-Zumino** embedding (generally maximal, but not symmetric)

[application of a Th. of Dynkin]

Let's reconsider the starting **Maxwell-Einstein-scalar** Lagrangian density

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu} + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma}_{\rho\sigma}$$

…and introduce the following real 2n x 2n matrix :

$$\mathcal{M} = \begin{pmatrix} \mathbb{I} & -R \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -R & \mathbb{I} \end{pmatrix} = \begin{pmatrix} I + R I^{-1} R & -R I^{-1} \\ -I^{-1} R & I^{-1} \end{pmatrix}$$

$$\mathcal{M} = \mathcal{M}(R, I) = \mathcal{M}(\mathrm{Re}(\mathcal{N}), \mathrm{Im}(\mathcal{N})).$$

$$\mathcal{M}^T = \mathcal{M} \qquad \mathcal{M}\mathbb{C}\mathcal{M} = \mathbb{C}$$

$$\mathcal{M} = -\left(\mathbf{L}\mathbf{L}^T\right)^{-1} = -\mathbf{L}^{-T}\mathbf{L}^{-1},$$

L = element of the **Sp(2n,R)**-bundle over the scalar manifold
(= *coset representative* for homogeneous spaces **G/H**)

By virtue of this matrix, one can introduce a (scalar-dependent) **anti-involution in *any*** Maxwell-Einstein-scalar gravity theory in **D=4**:

$$\mathcal{S}(\varphi) \quad : \quad = \mathbb{C}\mathcal{M}(\varphi)$$

$$\mathcal{S}^2(\varphi) \;=\; \mathbb{C}\mathcal{M}(\varphi)\mathbb{C}\mathcal{M}(\varphi) = \mathbb{C}^2 = -\mathbb{I},$$

Ferrara,AM,Yeranyan; Borsten,Duff,Ferrara,AM

In turn, this allows to define an **anti-involution** on the dyonic charge vector Q, which has been named (scalar-dependent) **Freudenthal duality (F-duality)**

$$\mathfrak{F}(\mathcal{Q}) := -\mathcal{S}(\varphi)(\mathcal{Q}).$$

$$\mathfrak{F}^2 \;=\; -Id.$$

By recalling $V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\mathcal{Q},$

F-duality is the **symplectic gradient** of the **effective BH potential** :

$$\boxed{\mathfrak{F} : \mathcal{Q} \rightarrow \mathfrak{F}(\mathcal{Q}) := \mathbb{C}\frac{\partial V_{BH}}{\partial \mathcal{Q}}.}$$

All this enjoys a nice physical interpretation when evaluated **at the BH horizon** :

Attractor Mechanism $\partial_\varphi V_{BH} = 0 \Leftrightarrow lim_{\tau \to -\infty} \varphi^a(\tau) = \varphi^a_H(\mathcal{Q})$

Bekenstein-Hawking entropy $S = \frac{A_H}{4} = \pi \left. V_{BH} \right|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2} \mathcal{Q}^T \mathcal{M}_H \mathcal{Q}$

By evaluating the matrix M at the horizon : $\lim_{\tau \to -\infty} \mathcal{M}(\varphi(\tau)) = \mathcal{M}_H(\mathcal{Q})$

one can define the **horizon F-duality** as:

$$\lim_{\tau \to -\infty} \mathfrak{F}(\mathcal{Q}) =: \mathfrak{F}_H(\mathcal{Q}) = -\mathbb{C}\mathcal{M}_H \mathcal{Q} = \frac{1}{\pi} \mathbb{C} \frac{\partial S_{BH}}{\partial \mathcal{Q}} =: \tilde{\mathcal{Q}},$$

$$\mathfrak{F}^2_H(\mathcal{Q}) = \mathfrak{F}_H(\tilde{\mathcal{Q}}) = -\mathcal{Q}$$

It is a **non-linear (scalar-independent) anti-involutive map** on Q (hom of degree 1)

Bek.-Haw. entropy is **invariant** under its own **non-linear symplectic gradient** (*i.e.*, **F-duality**) :

$$\boxed{S(\mathcal{Q}) = S(\mathfrak{F}_H(\mathcal{Q})) = S\left(\frac{1}{\pi} \mathbb{C} \frac{\partial S}{\partial \mathcal{Q}}\right) = S(\tilde{\mathcal{Q}})}$$

This can be extended to include *at least* **all quantum corrections** with **homogeneity 2** or **0** in the BH charges Q

Ferrara, AM, Yeranyan
(and late Raymond Stora)

Lie groups **of type E_7** : (G,**R**)

Brown (1967);
Garibaldi; Krutelevich;
Borsten,Duff *et al.*
Ferrara,Kallosh,AM;
AM,Orazi,Riccioni

❖ the (ir)repr. **R** is **symplectic** :

$$\exists ! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R};$$

$$\langle Q_1, Q_2 \rangle \equiv Q_1^M Q_2^N \mathbb{C}_{MN} = - \langle Q_2, Q_1 \rangle ;$$

symplectic product

❖ the (ir)repr. admits a completely symmetric **invariant rank-4** tensor

$$\exists \; K_{MNPQ} = K_{(MNPQ)} \equiv \mathbf{1} \in (\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R})_s$$ (K-tensor)

G-invariant quartic polynomial

$$I_4 := K_{MNPQ} Q^M Q^N Q^P Q^Q =: \epsilon \, |I_4| \, ,$$

$$\boxed{S_{BH} = \pi \sqrt{|I_4|}}$$

❖ defining a **triple map** in **R** as

$$T \colon \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

$$\langle T(Q_1, Q_2, Q_3), Q_4 \rangle \equiv K_{MNPQ} Q_1^M Q_2^N Q_3^P Q_4^Q$$

it holds
$$\langle T(Q_1, Q_1, Q_2), T(Q_2, Q_2, Q_2) \rangle = \langle Q_1, Q_2 \rangle \, K_{MNPQ} Q_1^M Q_2^N Q_2^P Q_2^Q$$

this third property makes a **group of type E_7** amenable to a description as automorphism group of a **Freudenthal triple system (FTS)**

Evidence : all electric-magnetic duality groups of D=4 ME(S)GT's with **symmetric** scalar manifolds (and *at least* **8** supersymmetries) are **of type E_7**

$N = 2$

G	R
$U(1,n)$	$(\mathbf{1+n})$
$SL(2,\mathbb{R}) \times SO(2,n)$	$(\mathbf{2,2+n})$
$SL(2,\mathbb{R})$	$\mathbf{4}$
$Sp(6,\mathbb{R})$	$\mathbf{14'}$
$SU(3,3)$	$\mathbf{20}$
$SO^*(12)$	$\mathbf{32}$
$E_{7(-25)}$	$\mathbf{56}$

N	G	R
3	$U(3,n)$	$(\mathbf{3+n})$
4	$SL(2,\mathbb{R}) \times SO(6,n)$	$(\mathbf{2,6+n})$
5	$SU(1,5)$	$\mathbf{20}$
8	$E_{7(7)}$	$\mathbf{56}$

(**E_7, 912 – embedding tensor N=8/N=2 exc, D=4**)
satisfies the first two Brown's axioms,
*but **not** the third one!*

"degenerate" groups **of type E_7**

$$I_4(p,q) = (I_2(p,q))^2$$

$$S_{BH} = \pi\sqrt{|I_4(p,q)|} = \pi\,|I_2(p,q)|\,.$$

In D=4 sugras with the previous electric-magnetic duality group **of type E_7**, the **G**-invariant **K-tensor** determining the extremal BH Bekenstein-Hawking entropy

$$\boxed{S_{BH} = \pi\sqrt{|I_4|}}$$

$$I_4 := K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q =: \epsilon\,|I_4|\,,$$

can generally be expressed as adjoint-trace of the product of G-generators (dim **R** = 2n, and dim **Adj** = d)

$$K_{MNPQ} = -\frac{n\,(2n+1)}{6d}\left[t^{\alpha}_{MN}t_{\alpha|PQ} - \frac{d}{n(2n+1)}\mathbb{C}_{M(P}\mathbb{C}_{Q)N}\right]$$

The **horizon F-duality** can be expressed in terms of the **K-tensor**

$$\mathfrak{F}_H(\mathcal{Q})_M = \tilde{\mathcal{Q}}_M = \frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{|I_4(\mathcal{Q})|}}K_{MNPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q$$

Borsten,Dahanayake,Duff,Rubens

and the **invariance** of the BH entropy under **horizon F-duality** can be recast as

$$\boxed{I_4\left(\mathcal{Q}\right) = I_4(\mathbb{C}\tilde{\mathcal{Q}}) = I_4\left(\mathbb{C}\frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}}\right)}$$

Hints for the Future…

- extension to **“small”** [intrinsically quantum] **Black Holes** :
  how to define **F-duality *?***

- extension to **multi-centered** (extremal) BH solutions:
  Yeranyan; Ferrara,AM,Shcherbakov,Yeranyan

- into the **quantum regime** of gravity [e.m. duality over **discrete** fields]:
  F-duality for **integer, quantized charges ?** Borsten, Duff *et al.*

- application in **flux compactifications** : U-duality inv. expression of the cosmological constant [Cassani, Ferrara, AM, Morales, Samtleben]

- interpretation in **string theory** / **M-theory** ? ***Still mysterious***…
  Work in progress….

Thank You!