# Adiabaticity and gravity independent conservation laws for cosmological perturbations

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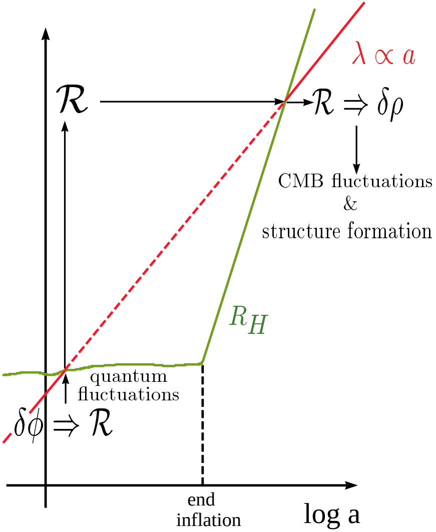
# Outline


$\lambda \propto a$
$\mathcal{R}$
$\mathcal{R} \Rightarrow \delta\rho$
CMB fluctuations
&
structure formation
$R_H$
quantum
fluctuations
$\delta\phi \Rightarrow \mathcal{R}$
end
inflation
log a

# Outline

- The definition of adiabaticity used in theory of cosmological perturbations (CPT)comes from the thermodynamics of a fluid admitting an equation of state $P(\rho, S)$,

$$\delta P = \frac{\partial P}{\partial \rho}\Big|_S \delta\rho + \frac{\partial P}{\partial S}\Big|_\rho \delta S \tag{1}$$

For an **adiabatic** fluid, also known as **barotropic** or **isentropic** fluid, we have an equation of state $P(\rho)$ independent of the entropy S so that

$$\delta P_{ad} = c_w^2 \delta\rho \quad , \quad c_w^2 \equiv \frac{\partial P}{\partial \rho}\Big|_S = \frac{P_0'}{\rho_0'} \tag{2}$$

where the subscript $_0$ stands for background quantities, and the prime denotes derivative respect to time. Adiabatic perturbations move with the speed of sound $c_w$, also called **adiabatic speed of sound**.

- It is natural to define the non-adiabatic pressure as

$$\delta P_{nad} \equiv \delta P - c_w^2 \delta\rho = \delta P_{\delta\rho=0} = \frac{\partial P}{\partial S}\bigg|_{\rho} \delta S, \tag{3}$$

which is **gauge invariant**and vanishes by construction for a perfect fluid. Note that it coincides with $\delta P_{\delta\rho=0}$, and for this reason conservations laws for adiabatic perturbations, i.e. satisfying $\delta_{nad} = 0$, are conveniently derived in the uniform density gauge. There are some **important exceptions** though!

# Outline

# Scalar metric perturbation

- Metric perturbation can be classified according to their behavior under spatial rotations as scalar, vector, and tensor
- According to inflation theory primordial scalar perturbations the seeds of the CMB temperature anisotropy and of large structure formation
- We set the perturbed metric as

$$
\begin{aligned}
ds^2 &= a^2\Big[-(1+2A)d\eta^2 + 2\partial_j B dx^j d\eta \\
&\quad + \Big\{\delta_{ij}(1+2\mathcal{R}) + 2\partial_i\partial_j E\Big\}dx^i dx^j\Big]\,, \qquad (4)
\end{aligned}
$$

# Perturbed energy-momentum tensor

- The perturbed energy-momentum tensor cab written as

$$T^0_0 = -(\rho + \delta\rho)\,, \quad T^0_j = (\rho + P)u^0 u_j = \frac{\rho + P}{a} u_j\,,$$
$$T^i_j = (P + \delta P)\delta^i_j + \Pi^i{}_j\,; \quad \Pi^k{}_k \equiv 0\,. \tag{5}$$

For a scalar perturbations

$$u_j = -a\partial_j(v - B) \qquad \rightarrow \qquad T^0_j = -(\rho + P)\partial_j(v - B) \tag{6}$$

$\Pi^k{}_j$ in the form can be written as

$$\Pi_{ij} = \delta_{ik}\Pi^k{}_j = \left[\partial_i\partial_j - \frac{1}{3}\delta_{ij} \overset{(3)}{\Delta}\right]\Pi\,, \tag{7}$$

where $\Delta^{(3)} = \delta^{ij}\partial_i\partial_j$.

# Gauge invariant variables

- The following gauge-invariant variables are relevant to set the initial conditions before inflation:

$$\mathcal{R}_c \equiv \mathcal{R} - \mathcal{H}(v - B)\,, \tag{8}$$

$$\zeta \equiv \mathcal{R} - \frac{\mathcal{H}}{\rho'}\delta\rho = \mathcal{R} + \frac{\delta\rho}{3(\rho + P)}\,, \tag{9}$$

$$V_f \equiv (v - B) - \frac{\mathcal{R}}{\mathcal{H}}\,. \tag{10}$$

# Geometrical meaning of gauge invariant variables

- $\mathcal{R}_c$ is the curvature perturbation on comoving slices $(v - B = 0)$
- $\zeta$ is the curvature perturbation on uniform density slices $(\delta\rho = 0)$
- $V_f$ is the velocity potential on flat slices $(\mathcal{R} = 0)$
- They are related to each other as

$$\mathcal{R}_c = -\mathcal{H} V_f \,, \tag{11}$$

$$\zeta \equiv \mathcal{R}_{ud} = \mathcal{R}_c + \frac{\delta\rho_c}{3(\rho + P)} \,. \tag{12}$$

# Conservation of the perturbed energy momentum-tensor in the comoving gauge

- In the comoving slices gauge, $v - B = 0$ ($\Leftrightarrow \; T^0{}_j = 0$) $\delta(\nabla_\mu T^\mu{}_j) = 0$ implies

$$(\rho + P)A_c + \delta P_c + \frac{2}{3}\nabla^2\Pi = 0\,. \tag{13}$$

If the perturbation is isotropic, by definition $\Pi = 0$. Thus we find

$$\delta P_c = -(\rho + P)A_c\,. \tag{14}$$

- Note that this relation between $\delta P_c$ and $A_c$ is completely **independent of the theory of gravity**.

# Outline

# Gravity theory independent synchronous, comoving, and uniform density gauge coincidence

- Combining the definition of $\delta P_{nad}$ with $\delta P_c = -(\rho + P)A_c$ and defining the **sound speed of perturbations** $c_s$ we get

$$\delta P_c = c_s^2 \delta\rho_c \tag{15}$$

$$\delta P_{nad} = (c_s^2 - c_w^2)\delta\rho_c = \frac{c_w^2 - c_s^2}{c_s^2}(\rho + P)A_c \,. \tag{16}$$

(17)

- For adiabatic perturbations the **synchronous**$(A = 0)$, **uniform density**$(\delta\rho = 0)$ and **comoving** gauge can coincide

$$\{\delta\mathcal{P}_{nad} \approx 0, c_s \neq c_w\} \Rightarrow \delta\rho_c \approx A_c \approx 0 \,. \tag{18}$$

# Implications for CMB anisotropy spectrum calculation

- This is why numerical codes such as CAMB which solve the perturbed Boltzman equations to calculate the CMB anisotropy spectrum use the **synchronous gauge** but the initial conditions are set using the spectrum for primordial curvature **perturbations on comoving slices** $\mathcal{R}_c$. In fact for standard slow-roll attractor models we have $c_s \neq c_w$. For example for a minimally coupled scalar field one has

$$c_w^2 = -1 + \frac{2\epsilon}{3} - \frac{\eta}{3}, \qquad c_s^2 = 1, \tag{19}$$

  with $\epsilon, \eta$ the usual slow-roll parameters.
- **But what happens when $c_s = c_w$ ?**

# Gravity independent relation between $\mathcal{R}_c$ and $\zeta$

- Combining the relation for $\delta\rho_c$, with the gauge transformation from uniform density to the comoving slices gauge we get another important relation which is independent of the gravity theory

$$\zeta = \mathcal{R}_c - \frac{H}{\dot\rho}\delta\rho_c = \mathcal{R}_c + \delta P_{nad}\frac{H}{\dot\rho(c_w^2 - c_s^2)} \tag{20}$$

- When $c_s \neq c_w$, as in attractor slow-roll models, adiabaticity implies that $\mathcal{R}_c = \zeta$.

# General relativity case for generic matter field

- In general relativity the perturbed Einstein's equations in the comoving slices gauge give for the $G^0_i$-component

$$\mathcal{R}'_c = \mathcal{H} A_c \tag{21}$$

which combined with the equation for $\delta P_{nad}$ gives the important relations

$$\delta P_{nad} = \left[\left(\frac{c_w}{c_s}\right)^2 - 1\right](\rho + P)\frac{\dot{\mathcal{R}}_c}{H} \tag{22}$$

$$\zeta = \mathcal{R}_c - \frac{\dot{\mathcal{R}}_c}{3c_s^2 H}. \tag{23}$$

- Thus $\delta P_{nad} = 0$ if either $c_w^2 = c_s^2$ or $\dot{\mathcal{R}}_c = 0$. In particular in the latter case, $\dot{\mathcal{R}}_c = 0$, we have $\zeta = \mathcal{R}_c$.

# Relation with previous results

- We can immediately deduce that in general relativity there are two possible scenarios for the non-conservation of $\mathcal{R}_c$,

$$(1) \quad c_s^2 = c_w^2 \,, \quad \delta P_{nad} = 0$$
$$(2) \quad c_s^2 \neq c_w^2 \,, \quad \delta P_{nad} \neq 0$$

- The first case corresponds to **globally adiabatic** (GA) models because $c_s^2 = c_w^2 \leftrightarrow \delta P_{nad} = 0$ on **any scale**
- The second case is the standard slow-roll inflation for example, where on superhorizon scales $\zeta = \mathcal{R}_c$ and they are both conserved if $\delta P_{nad} = 0$, but **only on super-horizon scales**

# Relation with previous results

- Demanding $\delta(\nabla_\mu T^\mu_0) = 0$ yields, in the uniform density slicing (astro-ph/0003278)

$$\zeta' = -\frac{\mathcal{H}\delta P_{nad}}{(\rho + P)} + \frac{1}{3}\overset{(3)}{\Delta}\left(v - E'\right)_{ud} \tag{24}$$

- The usual interpretation of the above equation is that for **adiabatic perturbations, $\zeta$ is conserved on super-horizon scales, as long as the gradient terms can be neglected**.
- As we have proved, when $c_s = c_w$ this is not necessarily true anymore, and indeed neglecting the gradient terms may not be justified, as for example happens in ultra-slow inflation(USR).

# Ultra-slow inflation

- Ultra-slow inflation is a single scalar field minimally coupled model with standard kinetic term and constant potential
The density and pressure perturbations are equal to each other, $\delta P = \delta\rho$, in arbitrary gauge.
- $L = X + V_0 \quad , \quad X = \frac{1}{2} g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$
- Therefore we have

$$c_w^2 - c_s^2 = \delta P_{nad} = 0 \quad , \quad \dot\phi \propto a^{-3} \qquad (25)$$

$$\epsilon \equiv -\frac{\dot H}{H^2} = \frac{\dot\phi^2}{2H^2} \propto a^{-6} \quad , \quad \eta \equiv \frac{\dot\epsilon}{H\epsilon} = -6 \qquad (26)$$

# Super-horizon growth of curvature perturbations for $\mathcal{R}_c$

- In this case adiabaticity does not imply the conservation of $\mathcal{R}_c$ neither of $\zeta$, and they can differ from each other because

$$\zeta = \mathcal{R}_c + \delta P_{nad} \frac{H}{\dot{\rho}(c_w^2 - c_s^2)} \quad (27)$$

$$\delta P_{nad} = \left[ \left( \frac{c_w}{c_s} \right)^2 - 1 \right] (\rho + P) \frac{\dot{\mathcal{R}}_c}{H} \quad (28)$$

- On super horizon scales we get The superhorizon solution for $\mathcal{R}_c$ is then

$$\mathcal{R}_c = c_1 \left( 1 + \mathcal{O}(k^2) \right) + c_2 a^3 \left( 1 + \frac{1}{2} \frac{k^2}{\mathcal{H}^2} + \mathcal{O}(k^4) \right) . \quad (29)$$

# Super-horizon growth of curvature perturbations for $\zeta$

- Inserting this into the equation relating $\mathcal{R}_c$ and $\zeta$ we get

$$\zeta = c_1 \left(1 + \mathcal{O}(k^2)\right) + \frac{c_2 a^3}{3} \left(\frac{k^2}{\mathcal{H}^2} + \mathcal{O}(k^4)\right) . \tag{30}$$

Thus we see that the time-dependent solution grows like $a$ even on superhorizon scales. More specifically, $\zeta(t) \approx \zeta(t_k) a(t)/a(t_k)$ where $t_k$ is the horizon crossing time $a(t_k) = kH$ of the wavenumber $k$.

# Outline

# Inversion method to find models violating super-horizon curvature perturbations

- We have developed an inversion method to find new models exhibiting super-horizon growth of curvature perturbations based on the equivalence between K-inflation globally adiabatic models (GA) and barotropic fluids.
- Imposing conditions on $\epsilon(a)$ we have found the corresponding equation of state for the barotropic fluid admitting such a background behavior and the correspondent equivalent scalar field Lagrangian, obtaining an infinite class of models which do no conserve curvature perturbations

# Inversion method to find models violating super-horizon curvature perturbations

- These single scalar field models **do not satisfy the non-Gaussianity consistency** conditions
- They could have **large local shape** $f_{NL}$, while satisfying other observational constraints such the spectral index,

- From the equation for the curvature perturbation on comoving slices

$$\frac{\partial}{\partial t}\left(\frac{a^3\epsilon}{c_s^2}\frac{\partial}{\partial t}\mathcal{R}_c\right) - a\epsilon\Delta\mathcal{R}_c = 0\,, \qquad (31)$$

we can deduce, after re-expressing the time derivative in terms of the derivative respect to the scale factor $a$, that on superhorizon scales there is (apart from a constant solution) a solution of the form,

$$\mathcal{R}_c \quad \propto \quad \int^a \frac{da}{a} f(a)\,; \quad f(a) \equiv \frac{c_s^2(a)}{Ha^3\epsilon(a)}\,, \qquad (32)$$

# Constructing GA models

- $$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\frac{\rho + P}{\rho}\,. \tag{33}$$

  In terms of the scale factor and $\epsilon$ the energy conservation equation reads

  $$\frac{d\rho}{da} + \frac{3}{a}(\rho + p) = \frac{d\rho}{da} + \frac{2\epsilon\rho}{a} = 0. \tag{34}$$

  We may now define the quantity $b(a) = 2\epsilon\rho$. The function $f(a)$ can be re-written in terms of it as

  $$f(a) \propto \frac{Hc_s^2}{a^3 b(a)}\,. \tag{35}$$

- It is possible to associate any barotropic perfect fluid with an equivalent K-inflation model with $L = P(Y)$ such that $c_s = c_w$ according to ( arXiv:1002.1376)

$$2\int^{P}\frac{du}{F(u)} = \log(Y)\,, \tag{36}$$

$$F(P) = \rho(P) + P \tag{37}$$

$$Y = g(\phi)X \quad , \quad X = \frac{1}{2}g_{\mu\nu}\partial^{\mu}\phi\partial^{\nu}\phi$$

# Examples

- Generalized Ultra-slow inflation : $2\epsilon\rho = b(a) \propto a^{-n}$

$$L = P(Y) = Y^{n/(2n-6)} - V_0 \,. \tag{38}$$

- Lambert Inflation : $\epsilon = \propto a^{-n}$

$$L = \rho_0 \left(\frac{n-3}{3} W(Y) - 1\right) \exp\left[\frac{n-3}{n} W(Y)\right] - V_0$$

$$h(z) = z e^z \quad , \quad z = h^{-1}(z e^z) \equiv W(z e^z)$$

- $Y = X g(\phi)$

# Globally adiabatic inflationary models

- They can be an alternative to attractor models
- These single scalar field models do not satisfy the non-Gaussianity consistency conditions and could have large local shape $f_{NL}$, while satisfying other observational constraints such the spectral index, and could be an alternative to attractor models
- **Adiabatic perturbations are not always conserved**, and on the contrary, GA models are good candidates for the violation of curvature perturbations on super-horizon scales together with non adiabatic models