Group Theory Application to Multiquark States; N(1440), N(1520), N(1535) and Pentaquarks

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Outlooks

- Introduction
- Basic Knowledge of $SU(m)$ Group
- Basic Knowledge of Permutation Group
- Decomposition of Tensor Representations of $SU(m)$
- Application to q^3 States
- Application to Pentaquarks
- Masses of Baryons

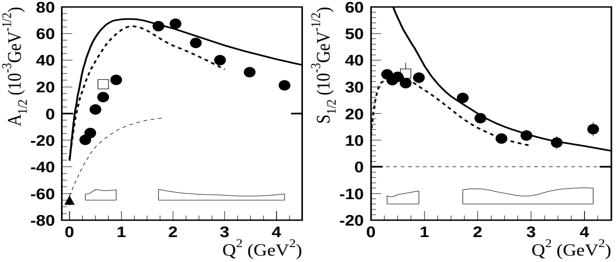
Roper Resonance $N_{1/2+}(1440)$ and $N_{1/2-}(1535)$

A good review paper, "Baryon Spectroscopy", Rev. Mod. Phys. 82, 1095

- In the traditional q^3 picture, the Roper $N_{1/2+}(1440)$ usually gets a mass ~ 100 MeV above the $N_{1/2-}(1535)$, but not 100 MeV below it.
- $N_{1/2-}(1535)$ is observed at a mass expected in quark models
- Mainstream understandings:
  - Roper resonance is usually blamed sitting at a wrong place or intruding the q^3 spectrum. It could be $q^4\bar{q}$ pentaquark, q^3g hybrid, $q^3(q\bar{q})$ resonance...
  - $$N_{1/2-}(1535) \text{ is a normal } q^3 \text{ first excited state of } l = 1.$$
- In $q^4\bar{q}$ picture, $N_{1/2+}(1440)$ is usually interpreted as a mixture of $q^3(l=0)$ and $q^4\bar{q}(l=1)$ components.

Roper Resonance $N_{1/2+}(1440)$ and $N_{1/2-}(1535)$

Helicity amplitudes for the $\gamma^* p \to N(1440)$ transition. The full circles are from CLAS (Aznauryan, 2008). The thick curves correspond to quark models assuming that $N(1440)$ is a q^3 first radial excitation: dashed (Capstick and Keister, 1995), solid (Aznauryan, 2007). The thin dashed curves are obtained assuming that $N(1440)$ is a $q^3 g$ hybrid state (Li et al., 1992). Figure courtesy of RRC78:045209.



- The sign change in the helicity amplitude as a function of Q^2 suggests a node in the wave function and thus a radially excited state.

Roper Resonance $N_{1/2+}(1440)$ and $N_{1/2-}(1535)$

- Large couplings to the $N\eta$, $N\eta'$, $N\phi$ and $K\Lambda$ but small couplings to the $N\pi$ and $K\Sigma$ are claimed.

- A large $N\eta$ coupling invites speculation that it might be created dynamically as $N\eta - \Sigma K$ coupled channel effect.

- A large $N\phi$ coupling leads to the proposal that the $N_{1/2-}(1535)$ may have a large component of $uuds\bar{s}$ pentaquark states.

$\Rightarrow$ It might be more natural to propose in the $q^4\bar{q}$ picture that

- $N_{1/2+}(1440)$ may mainly be a q^3 first radial excitation;
- $N_{1/2-}(1535)$ may have a large ground-state $(l = 0)$ $uuds\bar{s}$ component.

$SU(m)$ Group

All the $m \times m$ matrices, which are unitary and unimodular, form a group called $SU(m)$ group, that is

$$SU(m) = \{U\} \tag{1}$$

with

$$\begin{aligned} &U^\dagger U = UU^\dagger = 1 \quad \text{or} \quad U^\dagger = U^{-1} \\ &\text{Det}(U) = 1 \end{aligned} \tag{2}$$

Due to the restriction of the unitary and unimodular conditions, $SU(m)$ group has $(m^2 - 1)$ real parameters, hence $SU(m)$ is a Lie group of degree $(m^2 - 1)$.
It can be proven that any elements of $SU(m)$ can be expressed as

$$U = e^{iH} \tag{3}$$

where H is a $m \times m$ hermitian matrix.

Fundamental Representation of $SU(m)$

$SU(m)$ has a fundamental representation which is just the group element. Let $\{\psi_i\}$ $(i = 1, 2, \cdots, m)$ be the basis of the m dimensional space, then we have

$$U\psi_i = \psi_i' = \sum_j D_{ji}(U)\psi_j \tag{4}$$

and hence the fundamental representation

$$D = e^{i\lambda_j \theta_j}, \quad (j = 1, 2, \cdots, m^2 - 1) \tag{5}$$

Here λ_j are the so-called infinitesimal operators of the fundamental representation of $SU(m)$.

For the SU(2) group there are two basis functions ψ_i $(i = 1, 2)$, and one may write them in the matrix form,

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6}$$

The fundamental representation of $SU(2)$ is two-dimensional, taking the form

$$D = e^{\frac{i}{2}\sigma_j \alpha_j} \tag{7}$$

where the infinitesimal operators are just the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Fundamental Representation of $SU(m)$

For the SU(3) group there are three basis functions ψ_i $(i = 1, 2, 3)$, and one may write them in the matrix form,

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{8}$$

The fundamental representation of $SU(3)$ is

$$D = e^{i\lambda_j \theta_j}, \quad (j = 1, 2, \cdots, 8) \tag{9}$$

which is three-dimensional. The infinitesimal operators λ_j are usually the 8 Gell-Mann matrices,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} \tag{10}$$

Tensor (Product) Representations of $SU(m)$

- A product of fundamental and conjugacy representations is also a representation of $SU(m)$. The corresponding basis functions are written in the form

$$\psi_{ijk\cdots}^{i'j'k'\cdots} = \psi_i\psi_j\psi_k\cdots\psi^{i'}\psi^{j'}\psi^{k'}\cdots \quad (11)$$

- A direct product representation of $SU(m)$ is generally reducible. It can be proven that the direct product representation can be reduced according to the irreducible representations of permutation groups.

- Suppose that the n rank tensors are the basis functions of a tensor representation of $SU(m)$, then the m^n space can be decomposed into an addition of different sub-spaces according to all the possible irreducible representations of the permutation group S_n.

- Each sub-space derived in such a way forms the complete basis of a irreducible representation of $SU(m)$.

- In practise, we use the technique of Young tableaux to carry out the reduction.

Permutations

- n objects with labels $1, 2, \cdots, n$ arranged in the series $X = (1, 2, \cdots, n)$. If one rearranges those objects to make $X = (1, 2, \cdots, n)$ to $X_{S_i} = (S_1, S_2, \cdots, S_n)$, we call this rearrangement is a permutation to n objects, represented as

$$S = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ S_1 & S_2 & S_3 & \cdots & S_n \end{pmatrix} = \begin{pmatrix} i \\ S_i \end{pmatrix} \tag{12}$$

  which means the rearrangement of $1 \to S_1$, $2 \to S_2$, $\cdots$, and $n \to S_n$.

- Any permutation may be decomposed into a product of disjoint cycles. A cycle is a special permutation defined as

$$(S_1 S_2 \cdots S_k) \equiv \begin{pmatrix} S_1 & S_2 & \cdots & S_k \\ S_2 & S_3 & \cdots & S_1 \end{pmatrix} \tag{13}$$

  that is $S_i \to S_{i+1}$ $(i = 1, 2, \cdots, k-1)$ and $S_k \to S_1$.

- The decomposition is unique apart from the sequence of the cyclic factors.

Permutations

For example

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 1 & 3 & 5 & 2 \end{pmatrix} = (143)(26)(5)$$
$$= (431)(62)(5) = (62)(431) \quad (14)$$

Here are some properties:

(1) Two independent cycles commute each other, for example $(123)(45) = (45)(123)$.

(2) A cycle keeps the same when its elements are cyclically moved, for example $(123) = (231) = (312)$.

(3) A cycle may be resolved into a product of transpositions, for example $(123)(456)(78) = (13)(12)(46)(45)(78)$

Permutation Group S_n

All the possible permutations to n objects together form a group called permutation group S_n. n is the degree of the group S_n.

- In S_n the unit element is

$$e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \tag{15}$$

- If there exists an element S in the group S_n

$$S = \begin{pmatrix} 1 & 2 & \cdots & n \\ S_1 & S_2 & \cdots & S_n \end{pmatrix} \tag{16}$$

  then the inverse of the element S is

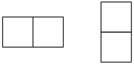
$$S^{-1} = \begin{pmatrix} S_1 & S_2 & \cdots & S_n \\ 1 & 2 & \cdots & n \end{pmatrix} \tag{17}$$

- It is clear that there are $n!$ elements in S_n. We say the order of S_n is $n!$.
- e, (12), (13), (23), (123) and (132) form the S_3 group.

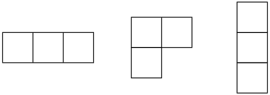
Young Tabloids

Young tabloid [1] of S_1,

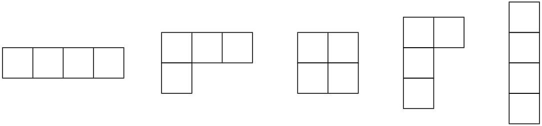
Young tabloids [2] and [11] of S_2,



Young tabloids [3], [21] and [111] of S_3,



Young tabloids [4], [31], [22], [211] and [1111] of S_4,



Young Tabloids, Representations of S_n

Young tabloids are worked out following the rules,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m, \quad \sum_{i=1}^{m} \lambda_i = n \tag{18}$$

- λ_i is the number of squares in the ith row.
- The number of all squares together must be n for S_n.
- Each Young tabloid corresponds to an irreducible representation of S_n. For instance, S_1, S_2, S_3 and S_4 have 1, 2, 3 and 5 irreducible representations.
- We may use $[\lambda] \equiv [\lambda_1, \lambda_2, \ldots, \lambda_m]$ to stand for an irreducible representation of S_n.
- For S_3 we have three irreducible representations [3], [21] and [111]. For S_4 we have [4], [31], [22], [211] and [1111] irreducible representations.

Young Tableaux, Representations of S_n

The dimension of an irreducible representation can be determined the way as below:

- First, draw the corresponding Young tabloid of the irreducible representation $[\lambda]$.

- Fill the squares of the Young tabloid with n numbers $(1, 2, \cdots, n)$ by following the rules below:
  - The number in a square differs from any number in other squares;
  - The numbers in a row must increase from left to right;
  - The numbers in a column must increase from top to bottom.

- What are worked out in such a way are called Young tableaux. The number of the Young tableaux derived in the method above is just the dimension of the irreducible representation $[\lambda]$.

Irreducible representations of S_4

S_4 has five irreducible representations, the corresponding Young tableaux and dimensions are

$$
\begin{array}{llllr}
[4] & \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline\end{array} & & & r=1 \\[2ex]
[31] & \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4 \\ \cline{1-1}\end{array} & r=3 \\[2ex]
[22] & \begin{array}{|c|c|}\hline 1&2\\ \hline 3&4\\ \hline\end{array} & \begin{array}{|c|c|}\hline 1&3\\ \hline 2&4\\ \hline\end{array} & & r=2 \\[2ex]
[211] & \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&3\\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&4\\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} & r=3 \\[2ex]
[1111] & \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline 4\\ \hline\end{array} & & & r=1
\end{array}
\tag{19}
$$

Weyl Tableaux

A Young tabloid filled with single particle states α_1, α_2, ... instead of numbers results in a Weyl tableau under the restrictions that:
(1) The same state must not appear twice in any column;
(2) The α_i values must read in increasing order as reading from the left to right in any row as well as from top to bottom in any column.
For example, for the four particle configuration $\alpha_1\alpha_2\alpha_3\alpha_4 \equiv \alpha\alpha\beta\gamma$, one derives the following Weyl tableaux for the permutation group S_4:

$$
\begin{array}{ll}
[4]: & \begin{array}{|c|c|c|c|}\hline \alpha & \alpha & \beta & \gamma \\ \hline\end{array} \\
[31]: & \begin{array}{|c|c|c|}\hline \alpha & \alpha & \beta \\ \hline \gamma \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline \alpha & \alpha & \gamma \\ \hline \beta \\ \cline{1-1}\end{array} \\
[22]: & \begin{array}{|c|c|}\hline \alpha & \alpha \\ \hline \beta & \gamma \\ \hline\end{array} \\
[211]: & \begin{array}{|c|c|}\hline \alpha & \alpha \\ \hline \beta \\ \cline{1-1} \gamma \\ \cline{1-1}\end{array}
\end{array}
\tag{20}
$$

Decomposition of Product Representations of $SU(m)$ Groups

- n-quark states $|q_1\rangle|q_2\rangle\cdots|q_n\rangle$ form a m^n dimensional direct product basis of $SU(m)$ ($m = 3,\ 3,\ 2$ for the color, flavor, and spin).
- The direct product representations of $SU(m)$ can be decomposed according to the irreducible representations of the permutation group S_n

$$\ytableaushort{1} \otimes \ytableaushort{2} = \ytableaushort{12} \oplus \ytableaushort{1,2}$$

$$\ytableaushort{1} \otimes \ytableaushort{2} \otimes \ytableaushort{3} = \ytableaushort{123} \oplus \ytableaushort{12,3} \oplus \ytableaushort{13,2} \oplus \ytableaushort{1,2,3}$$

$$\ytableaushort{1} \otimes \ytableaushort{2} \otimes \ytableaushort{3} \otimes \ytableaushort{4} = \ytableaushort{1234} \oplus \ytableaushort{123,4} \oplus \ytableaushort{124,3} \oplus \ytableaushort{134,2}$$

$$\oplus \ytableaushort{12,3,4} \oplus \ytableaushort{13,2,4} \oplus \ytableaushort{14,2,3} \oplus \ytableaushort{12,34} \oplus \ytableaushort{13,24} \oplus \ytableaushort{1,2,3,4}$$

Littlewood Rules

The outer product $[\lambda_1] \otimes [\lambda_2]$ of $[\lambda_1]$ of S_{n_1} and $[\lambda_2]$ of S_{n_2} are usually reducible representations of $S_{n=n_1+n_2}$, which can be resolved into irreducible representations of S_n according to the so-called Littlewood rules as follows:

- Work out the Young tabloids of $[\lambda_1]$ and $[\lambda_2]$. Choose the larger Young tabloid as the base and the smaller one as the multiplier. Fill the squares of the first row of the multiplier with a, the squares of the second row with b and so on.
- Piece the squares a of the multiplier to the base Young tabloid to get all the possible extended Young tabloids which can not have two a appears in the same column, then piece the squares b of the multiplier to the extended Young tabloids the same way and so on, until all the squares of the multiplier are used.
- Choose the Young tabloids in which, when one reads out the letters in the extended Young tabloids from right to left and from top to bottom, the number of b never surpasses the number of a and the number of c never surpasses the number of b and so on.
- The Young tabloids constructed following the above rules represent the possible irreducible representations of the outer product $[\lambda_1] \otimes [\lambda_2]$.

Yamanouchi Basis for Multiquark Systems

Yamanouchi Basis is also called the standard basis in permutation group. For q^2, q^3 and q^4 systems, for example, the basis functions are defined as

$$\psi_S = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} = |[2]\,(11)\rangle, \quad \psi_A = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} = |[11]\,(21)\rangle$$

$$\psi_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = |[21]\,(211)\rangle, \quad \psi_\rho = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} = |[21]\,(121)\rangle$$

$$\psi_S = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} = |[3]\,(111)\rangle, \quad \psi_A = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} = |[111]\,(321)\rangle$$

$$\psi_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} = |[211]\,(3211)\rangle, \quad \psi_\rho = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} = |[211]\,(3121)\rangle, \quad \psi_\eta = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} = |[211]\,(1321)\rangle$$

Yamanouchi Basis in General

- In general, a Yamanouchi basis function is written as

$$|[\lambda_1, \lambda_2, \cdots] \, (r_n, r_{n-1}, \cdots, r_2, r_1)\rangle \tag{21}$$

λ_i: the number of squares in the ith row of a Young tabloid;
r_i: from the ith row a square is removed.

- Each Young tableau leads to one Yamanouchi basis function, for example,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} = |[321](322111)\rangle, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} = |[321](321121)\rangle$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} = |[321](312121)\rangle, \quad \cdots$$

- All such defined functions for a Young tabloid together form a complete basis.

Representations of S_n in Yamanouchi basis

It is to evaluate the matrices for all permutations of S_n under Yamanouchi basis

- Suppose that the irreducible representations of S_{n-1} are known, then we have the matrices for any element which is in both the S_{n-1} and S_n, for example, the permutation $(i, n-1)$.
- Any element of S_n can be resolved into a product of transpositions (i, j) (for example, $(123) = (13)(12)$), thus what we need to evaluate are the matrices of the elements (i, n).
- But due to

$$(i, n) = (n-1, n)(i, n-1)(n-1, n) \tag{22}$$

we need to evaluate only the matrices for the element $(n-1, n)$.

Operations of $(n-1,n)$

The operation of the element $(n-1,n)$ on the standard basis satisfies the followings:

$$\mathbf{A}: \qquad (n-1,n)|[\lambda](r,r,\cdots)\rangle = +\,|[\lambda](r,r,\cdots)\rangle. \tag{23}$$

$$\mathbf{B}: \qquad (n-1,n)|[\lambda](r,r-1,\cdots)\rangle = -\,|[\lambda](r,r-1,\cdots)\rangle \tag{24}$$

when $|[\lambda](r-1,r,r_{n-2},\cdots,r_2,1)\rangle$ not exist

$$\mathbf{C}: \ (n-1,n)|[\lambda](r,s,\cdots)\rangle = \sigma_{rs}|[\lambda](r,s,\cdots)\rangle + \sqrt{1-\sigma_{rs}^2}|[\lambda](s,r,\cdots)\rangle \tag{25}$$

when $r \neq s$. For $[\lambda] = [\lambda_1,\lambda_2,\cdots,\lambda_r\cdots\lambda_s\cdots\lambda_n]$, we have

$$\sigma_{rs} = \frac{1}{(\lambda_r - r) - (\lambda_s - s)} \tag{26}$$

Matrices for Elements in both S_n and S_{n-1}

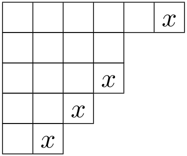
- S_{n-1} is a subgroup of S_n, and the elements of S_n which are also in S_{n-1} keeps the nth object unchanged.
- It can be proven, according to the representation theory of finite group, that

$$D(R) = \sum_r D_r(R), \quad \lambda_r - 1 \geq \lambda_{r+1} \tag{27}$$

  where $D(R)$ is the representation matrix of $[\lambda]$ of S_n, $D_r(R)$ are the representation matrices of irreducible representations $[\lambda_1, \cdots, \lambda_{r-1}, \lambda_r - 1, \lambda_{r+1}, \cdots, \lambda_m]$ of S_{n-1}.
- In the language of Young tabloids, the direct sum on the right side of eq. (27) corresponds to the summation of all the Young tabloids which are derived by removing one square from the Young tabloid $[\lambda]$ in all the possible ways.
- From the above expression we can evaluate the representation matrix $D(R)$ $(R \in S_{n-1})$ for the $[\lambda]$ of S_n if we know all the representation matrices $D_r(R)$ of S_{n-1}.

Matrices for Elements in both S_n and S_{n-1}

As an example, we consider the irreducible representation $[64432]$ of S_{19} for the elements $R \in S_{18}$.



- The squares with an "x" inside are those which could be removed.
- According to eq. (27), we have

$$D(R) = D_1(R) \oplus D_3(R) \oplus D_4(R) \oplus D_5(R) \tag{28}$$

- $D_1(R)$ $(R \in S_{n-1})$ is the representation matrix of $[54432]$, derived by removing the last square in the first row of $[64432]$.
- $D_3(R)$ $(R \in S_{n-1})$ is the representation matrix of $[64332]$, derived by removing the last square in the third row of $[64432]$ and so on.

Representations of S_2

S_2 has two irreducible representations $[2]$ and $[11]$.

- For $[2]$, the Young tableau is

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$$

  and the corresponding Yamanouchi basis is $|[2](11)\rangle$. We have

$$(12)|[2](11)\rangle| = |[2](11)\rangle \tag{29}$$

  and then the representation matrix of the element (12) is $D^{[2]}(12) = 1$.

- For $[11]$, the Young tableau is

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

  and the corresponding Yamanouchi basis is $|[2](21)\rangle$. We have

$$(12)|[2](21)\rangle = -|[2](21)\rangle \tag{30}$$

  and then the representation matrix of the element (12) is $D^{[11]}(12) = -1$.

Representations of S_3

Now we come to get the matrices of the representation $[21]$ of S_3.

- $[21]$ has two Young tableaux

$$\phi_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = |[21](211)\rangle$$
$$\phi_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} = |[21](121)\rangle \qquad (31)$$

  Thus the representation is two-dimensional.

- The Young tableaux derived by removing the box "3" from the Young tableaux ϕ_1 and ϕ_2 correspond respectively to the irreducible representations $[2]$ and $[11]$ of S_2.
- Thus we have

$$\begin{aligned} D^{[21]}(12) &= D^{[2]}(12) \oplus D^{[11]}(12) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \qquad (32)$$

Representations of S_3

From eq. (25), we have

$$(23)|[21](211)\rangle = \sigma_{21}|[21](211)\rangle + \sqrt{1-\sigma_{21}^2}|[21](121)\rangle$$
$$(23)|[21](121)\rangle = \sigma_{12}|[21](121)\rangle + \sqrt{1-\sigma_{12}^2}|[21](211)\rangle \tag{33}$$

with

$$\begin{aligned}\sigma_{21} &= \frac{1}{(\lambda_2-2)-(\lambda_1-1)} = -\frac{1}{2} \\ \sigma_{12} &= \frac{1}{(\lambda_1-1)-(\lambda_2-2)} = \frac{1}{2}\end{aligned} \tag{34}$$

Thus, under the basis of ϕ_1 and ϕ_2, the $[21]$ matrix for the element (23) is

$$D^{[21]}(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \tag{35}$$

Representations of S_3

For other elements of S_3, we have

$$\begin{aligned} D^{[21]}(13) &= D^{[21]}(23)D^{[21]}(12)D^{[21]}(23) \\ &= \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \\ D^{[21]}(123) &= D^{[21]}(13)D^{[21]}(12) \\ &= \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \\ D^{[21]}(132) &= D^{[21]}(12)D^{[21]}(13) \\ &= \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \end{aligned} \tag{36}$$

Note that we have used

$$\begin{aligned} (i,n) = (i,n-1)(n-1,n)(i,n-1), \\ (i,j,k) = (i,k)(i,j) \end{aligned} \tag{37}$$

Representations of S_4

Construct the representation matrices of the representation $[211]$ of S_4.

- There are three Young tableaux for $[211]$

$$
\begin{aligned}
\phi_1 &= \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} = |[211](3211)\rangle \equiv \phi_\lambda \\
\phi_2 &= \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} = |[211](3121)\rangle \equiv \phi_\rho \\
\phi_3 &= \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} = |[211](1321)\rangle \equiv \phi_\eta
\end{aligned} \qquad (38)
$$

- $\phi_1 = |[211](3211)\rangle$ and $\phi_2 = |[211](3121)\rangle$ are respectively the basis functions $|[21](211)\rangle$ and $|[21](121)\rangle$ of the representation $[21]$ of S_3.
- ϕ_3 is the basis function $|[21](321)\rangle$ of $[111]$ of S_3.
- For the elements (12), (13) and (23), we can directly write down the representation matrices since these elements are also the elements of S_3.

Representations of S_4

$$
\begin{aligned}
D^{[211]}(13) &= D^{[21]}(13) \oplus D^{[111]}(13) \\
&= \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
D^{[211]}(12) &= D^{[21]}(12) \oplus D^{[111]}(12) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
D^{[211]}(23) &= D^{[21]}(23) \oplus D^{[111]}(23) \\
&= \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{aligned} \tag{39}
$$

Representations of S_4

For the element (34) of S_4, we have

$$\begin{aligned}(34)|[211](3211)\rangle &= -|[211](3211)\rangle \\ (34)|[211](3121)\rangle &= \sigma_{31}|[211](3121)\rangle + \sqrt{1-\sigma_{31}^2}|[211](1321)\rangle \\ (34)|[211](1321)\rangle &= \sigma_{13}|[211](1321)\rangle + \sqrt{1-\sigma_{13}^2}|[211](3121)\rangle \end{aligned} \quad (40)$$

with

$$\sigma_{31} = \frac{1}{(\lambda_3 - 3) - (\lambda_1 - 1)} = -\frac{1}{3} = -\sigma_{13} \quad (41)$$

Thus in the basis of ϕ_1, ϕ_2 and ϕ_3, the $[211]$ matrix of the element (34) is

$$D^{[211]}(34) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix} \quad (42)$$

The representation matrices for other elements of S_4 can be derived from the matrices above.

Projection Operators

Projection operators of S_n are defined in the form

$$W_{(r)}^{[\lambda]} = \sum_i \langle [\lambda](r)|R_i|[\lambda](r)\rangle R_i \quad (43)$$

- R_i: all the permutations of S_n
- $|[\lambda](r)\rangle$: Yamanouchi basis function
- $\langle [\lambda](r)|R_i|[\lambda](r)\rangle$: Diagonal elements of matrix $D_{(r)}^{[\lambda]}(R_i)$
- $W_{(r)}^{[\lambda]}$: Projection operator corresponding to the irreducible representation $[\lambda]$ and the Yamanouchi basis function $|[\lambda](r)\rangle$ of S_n.
- Operating $W_{(r)}^{[\lambda]}$ on any function $f_1 f_2 \cdots f_n$, one could derive the corresponding standard basis function explicitly.

Projection Operators for q^3

For q^3 systems, the projection operators according to Young tableaux of λ and ρ types,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}: \quad P^\lambda = 1 + (12) - \tfrac{1}{2}(13) - \tfrac{1}{2}(23) - \tfrac{1}{2}(123) - \tfrac{1}{2}(132)$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}: \quad P^\rho = 1 - (12) + \tfrac{1}{2}(13) + \tfrac{1}{2}(23) - \tfrac{1}{2}(123) - \tfrac{1}{2}(132) \tag{44}$$

Note that we have used

$$D^{[12]}(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D^{[12]}(13) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}, \tag{45}$$

$$D^{[12]}(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad D^{[12]}(123) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{46}$$

$$D^{[12]}(132) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \tag{47}$$

Flavor Configurations of q^3

- Considering only the u, d and s quarks which together form the SU(3) fundamental representation, one could have totally 27 flavor combinations.
- These 27 states may be classified into several sets using the techniques of the $SU(3)$ group, with each set having similar symmetry properties.
- In the language of group theory, this is simply to reduce the direct product of three fundamental representations into a direct sum of irreducible representations, that is

$$\boxed{1} \otimes \boxed{2} \otimes \boxed{3} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

with the corresponding dimensions being

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \quad (48)$$

The symmetries of the flavor wave functions of baryons are

$$\phi_S = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}, \quad \phi_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}, \quad \phi_\rho = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}, \quad \phi_A = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \quad (49)$$

Spin Configurations of q^3

- The spin states (spin up $\uparrow$ and down $\downarrow$) of the quarks form the fundamental representation of the $SU(2)$ group. The total spin states of the three quark system can be derived the same way as for the flavor part. We have

$$\boxed{1} \otimes \boxed{2} \otimes \boxed{3} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

  with the corresponding dimensions being

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \;=\; \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}. \tag{50}$$

- The symmetries of the baryon spin wave functions are

$$\chi_S = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}, \quad \chi_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \quad \chi_\rho = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \tag{51}$$

- No antisymmetric representation for the spins of the three quark system.
- Fully symmetric spin-flavor wave functions may be formed as

$$\phi_S \chi_S, \quad \frac{1}{\sqrt{2}} \left(\phi_\lambda \chi_\lambda + \phi_\rho \chi_\rho \right) \tag{52}$$

q^3 Spin-Flavor States

Spin-flavor wave functions of various permutation symmetries may be written in the general form,

$$\Psi_{S,A,\lambda,\rho} = \sum_{i=S,A,\lambda,\rho} \sum_{j=S,A,\lambda,\rho} a_{ij}\,\psi_i\,\chi_j \tag{53}$$

The coefficient a_{ij} can be determined by applying the permutation operators of S_3 to the general form. Check the simplest case,

$$\begin{aligned} &(23)\,(a\,\psi_\lambda\,\chi_\lambda + b\,\psi_\rho\,\chi_\rho) \\ &= a\left(-\tfrac{1}{2}\psi_\lambda + \tfrac{\sqrt{3}}{2}\psi_\rho\right)\left(-\tfrac{1}{2}\chi_\lambda + \tfrac{\sqrt{3}}{2}\chi_\rho\right) + b\left(\tfrac{1}{2}\psi_\rho + \tfrac{\sqrt{3}}{2}\psi_\lambda\right)\left(\tfrac{1}{2}\chi_\rho + \tfrac{\sqrt{3}}{2}\chi_\lambda\right) \\ &= \left(\tfrac{1}{4}a + \tfrac{3}{4}b\right)\psi_\lambda\chi_\lambda + \left(\tfrac{3}{4}a + \tfrac{1}{4}b\right)\psi_\rho\chi_\rho - \tfrac{\sqrt{3}}{4}(a-b)\,(\psi_\lambda\chi_\rho + \psi_\rho\chi_\lambda) \end{aligned} \tag{54}$$

$a = b$ leads to the fully symmetric spin-flavor wave function. Here we have used $D^{[21]}(23)$, the $[21]$ representation matrix for the element (23) of S_3,

$$D^{[21]}(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \tag{55}$$

q^3 Spin-Flavor States

The spin-flavor wave functions of other permutation symmetries can be worked out the same way. The operations

$$(23)\,(\phi_\lambda\,\chi_\rho + \phi_\rho\,\chi_\lambda)$$

$$= \left(-\tfrac{1}{2}\phi_\lambda + \tfrac{\sqrt{3}}{2}\phi_\rho\right)\left(\tfrac{1}{2}\chi_\rho + \tfrac{\sqrt{3}}{2}\chi_\lambda\right) + \left(\tfrac{1}{2}\phi_\rho + \tfrac{\sqrt{3}}{2}\phi_\lambda\right)\left(-\tfrac{1}{2}\chi_\rho + \tfrac{\sqrt{3}}{2}\chi_\lambda\right)$$

$$= \tfrac{1}{2}\left(\phi_\lambda\chi_\rho + \phi_\rho\chi_\lambda\right) + \tfrac{\sqrt{3}}{2}\left(\phi_\rho\chi_\rho - \phi_\lambda\chi_\lambda\right) \tag{56}$$

and

$$(23)\,(\phi_\rho\,\chi_\rho - \phi_\lambda\,\chi_\lambda)$$

$$= \left(\tfrac{1}{2}\phi_\rho + \tfrac{\sqrt{3}}{2}\phi_\lambda\right)\left(\tfrac{1}{2}\chi_\rho + \tfrac{\sqrt{3}}{2}\chi_\lambda\right) - \left(-\tfrac{1}{2}\phi_\lambda + \tfrac{\sqrt{3}}{2}\phi_\rho\right)\left(-\tfrac{1}{2}\chi_\lambda + \tfrac{\sqrt{3}}{2}\chi_\rho\right)$$

$$= -\tfrac{1}{2}\left(\phi_\rho\chi_\rho - \phi_\lambda\chi_\lambda\right) + \tfrac{\sqrt{3}}{2}\left(\phi_\rho\chi_\lambda + \phi_\lambda\chi_\rho\right) \tag{57}$$

verify that $(\phi_\lambda\,\chi_\rho + \phi_\rho\,\chi_\lambda)$ and $(\phi_\rho\chi_\rho - \phi_\lambda\chi_\lambda)$ are respectively ρ-type and λ-type spin-flavor wave functions.Here we have used

$$D^{[21]}(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \tag{58}$$

q^3 Spin-Flavor States

And the operations

$$\begin{aligned}(23)\phi_A\,\chi_\rho &= -\phi_A\left(\frac{1}{2}\chi_\rho+\frac{\sqrt{3}}{2}\chi_\lambda\right)\\ &= -\frac{1}{2}\phi_A\chi_\rho+\frac{\sqrt{3}}{2}\left(-\phi_A\chi_\lambda\right) \qquad (59)\end{aligned}$$

and

$$\begin{aligned}(23)\left(-\phi_A\,\chi_\lambda\right) &= \phi_A\left(-\frac{1}{2}\chi_\lambda+\frac{\sqrt{3}}{2}\chi_\rho\right)\\ &= \frac{1}{2}\left(-\phi_A\chi_\lambda\right)+\frac{\sqrt{3}}{2}\phi_A\chi_\rho \qquad (60)\end{aligned}$$

tell that $\phi_A\,\chi_\rho$ and $-\phi_A\,\chi_\lambda$ are respectively λ-type and ρ-type spin-flavor wave functions.Here we have used

$$D^{[21]}(23)=\begin{pmatrix}-1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix} \qquad (61)$$

q^3 Spin-Flavor States

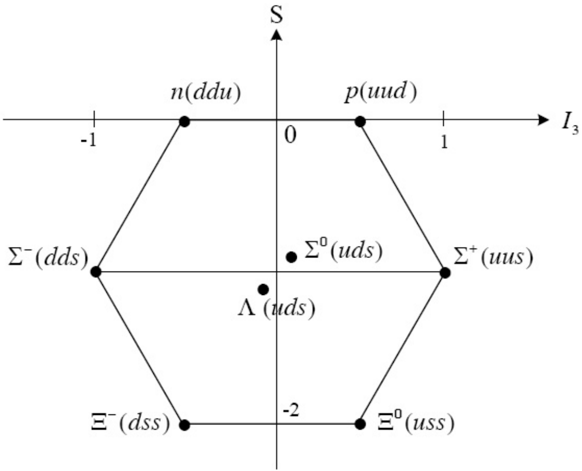
56 (S)	
(10,4): $\phi^S\chi^S$	**(8,2)**: $(\phi^\rho\chi^\rho + \phi^\lambda\chi^\lambda)/\sqrt{2}$
20 (A)	
(1,4) : $\phi^A\chi^S$	**(8,2)**: $(\phi^\lambda\chi^\rho - \phi^\rho\chi^\lambda)/\sqrt{2}$
70 (ρ)	
(10,2): $\phi^S\chi^\rho$	**(8,4)**: $\phi^\rho\chi^S$
(8,2) : $(\phi^\lambda\chi^\rho + \phi^\rho\chi^\lambda)/\sqrt{2}$	**(1,2)**: $-\phi^A\chi^\lambda$
70 (λ)	
(10,2): $\phi^S\chi^\lambda$	**(8,4)**: $\phi^\lambda\chi^S$
(8,2) : $(\phi^\rho\chi^\rho - \phi^\lambda\chi^\lambda)/\sqrt{2}$	**(1,2)**: $\phi^A\chi^\rho$

Out of the $3^3 \times 2^3 = 216$ states, derived are 56 symmetric, 70 ρ-type, 70 λ-type, and 20 antisymmetric states listed as follows:

$$\begin{aligned} S: &\quad (\mathbf{10},\mathbf{4}) + (\mathbf{8},\mathbf{2}) &=& \quad \mathbf{56} \\ \lambda: &\quad (\mathbf{10},\mathbf{2}) + (\mathbf{8},\mathbf{4}) + (\mathbf{8},\mathbf{2}) + (\mathbf{1},\mathbf{2}) &=& \quad \mathbf{70} \\ \rho: &\quad (\mathbf{10},\mathbf{2}) + (\mathbf{8},\mathbf{4}) + (\mathbf{8},\mathbf{2}) + (\mathbf{1},\mathbf{2}) &=& \quad \mathbf{70} \\ A: &\quad (\mathbf{8},\mathbf{2}) + (\mathbf{1},\mathbf{4}) &=& \quad \mathbf{20} \end{aligned} \tag{62}$$

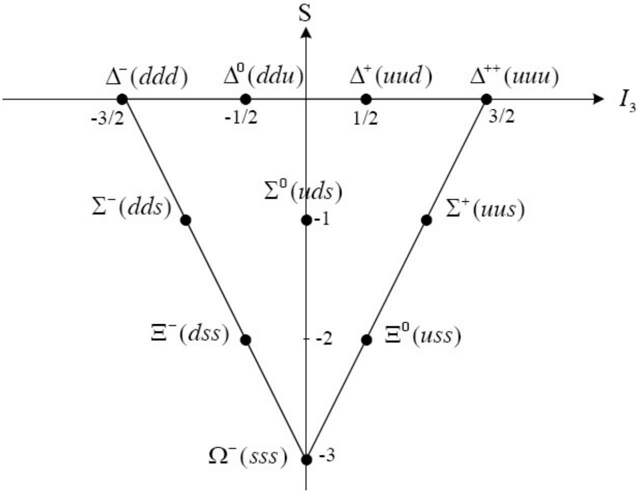
Baryons

Baryon octet: $\psi = \frac{1}{\sqrt{2}}\left(\phi_\lambda \chi_\lambda + \phi_\rho \chi_\rho\right)$, $S = 1/2$



Baryons

Baryon decuplet: $\psi = \phi_S \chi_S$, $S = 3/2$



q^3 Flavor Wave Functions

The explicit form of spin and flavor states can be derived in the framework of the Yamanouchi basis. What one needs to do is to work out the projection operator for a Young tableau, and then act the operator on the product state labeled by the corresponding Weyl tableau. For example,

$$\begin{aligned}
\left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline u & u \\ \hline d & \\ \cline{1-1}\end{array} \right\rangle &= P^{\lambda} uud \\
&= uud + uud - \frac{1}{2}duu - \frac{1}{2}udu - \frac{1}{2}udu - \frac{1}{2}duu \\
&= 2uud - udu - duu \\
\Longrightarrow \quad \phi_{\lambda} &= \frac{1}{\sqrt{6}}\left[2uud - duu - udu\right] \qquad (63)
\end{aligned}$$

$$\begin{aligned}
\left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline u & u \\ \hline d & \\ \cline{1-1}\end{array} \right\rangle &= P^{\rho} udu \\
&= udu - duu + \frac{1}{2}udu + \frac{1}{2}uud - \frac{1}{2}duu - \frac{1}{2}uud \\
&= \frac{3}{2}udu - \frac{3}{2}duu \\
\Longrightarrow \quad \phi_{\rho} &= \frac{1}{\sqrt{2}}\left[udu - duu\right] \qquad (64)
\end{aligned}$$

q^3 Flavor Wave Functions

For the configuration uds there are two Weyl tableaux,

$$\begin{array}{|c|c|}\hline u & d \\ \hline s & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c|}\hline u & s \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{65}$$

Correspondingly one gets two λ-type flavor functions,

$$\left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline u & d \\ \hline s & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\rangle \tag{66}$$

$$\left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline u & s \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\rangle \tag{67}$$

and two ρ-type ones,

$$\left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline u & d \\ \hline s & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\rangle \tag{68}$$

$$\left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline u & s \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\rangle \tag{69}$$

The linear combinations of the above functions lead to the λ-type and ρ-type flavor functions of the Σ and Λ. In the same way one may derive all the flavor and spin wave functions of baryon decuplet and octet.

q^3 Flavor Wave Functions

- The relative signs of the normalization coefficients of the octet wave functions is not trivial and must be given consistently. One way to choose the phase is to let the so-called principle term take a positive sign.
- The principle term is the product state derived by reading out the single particle state from the Weyl tableau according to the Young tableau.
- For example, the principle terms are uds for the basis vector

$$\left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline u & d \\ \hline s & \\ \cline{1-1} \end{array} \right\rangle, \tag{70}$$

usd for

$$\left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline u & d \\ \hline s & \\ \cline{1-1} \end{array} \right\rangle, \tag{71}$$

and $abdcfe$ for the six particle state

$$\left| \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \\ \cline{1-2} f & & \\ \cline{1-1} \end{array} \right\rangle. \tag{72}$$

Pentaquarks

- Pentaquarks are states of four quarks and one antiquark, $q^4\,\overline{q}$
- Pentaquark wave functions contain contributions of the spatial degrees of freedom and the internal degrees of freedom of colour, flavour and spin. The internal degrees of freedom are taken to be the three light flavors u, d and s with spin $s = 1/2$ and three possible colors r, g and b.
- The quark transforms under the fundamental representation of $SU(n)$, whereas the antiquark transforms under the conjugate representation of $SU(n)$, with $n = 2, 3, 3, 6$ for the spin, flavor, color and spin-flavor degree of freedom, respectively.
- The corresponding algebraic structure consists of the usual spin-flavor and color algebras

$$SU_{\rm sf}(6) \otimes SU_{\rm c}(3) \tag{73}$$

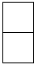
with

$$SU_{\rm sf}(6) = SU_{\rm f}(3) \otimes SU_{\rm s}(2) \tag{74}$$

$q^4\,\overline{q}$ Systems

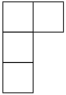
- The construction of pentaquark states is guided by
  - The pentaquark wave function should be a color singlet;
  - The pentaquark wave function should be antisymmetric under any permutation of the four quark configuration.
- In the language of group theory, the permutation symmetry of the four-quark configuration is characterized by the S_4 Young tabloids $[4]$, $[31]$, $[22]$, $[211]$ and $[1111]$.
- That the pentaquark wave function should be a color singlet demands that the color part of the pentaquark wave function must be a $[222]_1$ singlet.
- Since the color part of the antiquark in pentaquark states is a $[11]_3$ antitriplet

$$\psi^c_{[11]}(\overline{q}) = \quad (75)$$



the color wave function of the four-quark configuration must be a $[211]_3$ triplet

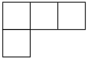
$$\psi^c_{[211]}(q^4) = \quad (76)$$



$q^4\,\overline{q}$ Systems

- The total states of q^4 is antisymmetric implies that the orbital-spin-flavour part must be a [31] state

$$\psi^{osf}_{[31]}(q^4) = \tag{77}$$



which is obtained from the Young tabloid of the colour part by interchanging rows and columns.

- Total wave function of the q^4 configuration may be written in the general form

$$\psi = \sum_{i,j=\lambda,\rho,\eta} a_{ij}\psi^{c}_{[211]_i}\psi^{osf}_{[31]_j} \tag{78}$$

The coefficients can be determined by operating the permutations of S_4 on the general form, using the $[31]$ and $[211]$ representation matrices.

- For example, applying the permutation (12) first by using

$$D^{[31]}(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$D^{[211]}(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{79}$$

$q^4\,\overline{q}$ Systems

One gets

$$\begin{aligned}(12)\psi &= +a_{\lambda\lambda}\psi^{c}_{[211]_\lambda}\psi^{osf}_{[31]_\lambda} - a_{\lambda\rho}\psi^{c}_{[211]_\lambda}\psi^{osf}_{[31]_\rho} + a_{\lambda\eta}\psi^{c}_{[211]_\lambda}\psi^{osf}_{[31]_\eta}\\ &\quad -a_{\rho\lambda}\Psi^{c}_{[211]_\rho}\psi^{osf}_{[31]_\lambda} + a_{\rho\rho}\psi^{c}_{[211]_\rho}\psi^{osf}_{[31]_\rho} - a_{\rho\eta}\psi^{c}_{[211]_\rho}\psi^{osf}_{[31]_\eta}\\ &\quad -a_{\eta\lambda}\Psi^{c}_{[211]_\eta}\psi^{osf}_{[31]_\lambda} + a_{\eta\rho}\psi^{c}_{[211]_\eta}\psi^{osf}_{[31]_\rho} - a_{\eta\eta}\psi^{c}_{[211]_\eta}\psi^{osf}_{[31]_\eta}\end{aligned}$$

An antisymmetric ψ requires $a_{\lambda\lambda} = a_{\lambda\eta} = a_{\rho\rho} = a_{\eta\rho} = 0$. Therefore, we have

$$\begin{aligned}\psi &= a_{\lambda\rho}\psi^{c}_{[211]_\lambda}\psi^{osf}_{[31]_\rho} + a_{\rho\lambda}\psi^{c}_{[211]_\rho}\psi^{osf}_{[31]_\lambda} + a_{\rho\eta}\psi^{c}_{[211]_\rho}\psi^{osf}_{[31]_\eta}\\ &\quad +a_{\eta\lambda}\psi^{c}_{[211]_\eta}\psi^{osf}_{[31]_\lambda} + a_{\eta\eta}\psi^{c}_{[211]_\eta}\psi^{osf}_{[31]_\eta}\end{aligned}$$

The action of the permutation (13) of S_4 on the above equation and the application of the antisymmetric restriction, $(13)\psi = -\psi$ lead to $a_{\eta\lambda} = a_{\rho\eta} = 0$ and $a_{\rho\lambda} = -a_{\lambda\rho}$, and hence

$$\psi = a_{\lambda\rho}\psi^{c}_{[211]_\lambda}\psi^{osf}_{[31]_\rho} - a_{\lambda\rho}\psi^{c}_{[211]_\rho}\psi^{osf}_{[31]_\lambda} + a_{\eta\eta}\psi^{c}_{[211]_\eta}\psi^{osf}_{[31]_\eta}$$

$q^4 \bar{q}$ Systems

Applying the permutation (34) of S_4 to the above equation, we have

$$\begin{aligned}(34)\psi &= -a_{\lambda\rho}\psi^{c}_{[211]_\lambda}\psi^{osf}_{[31]_\rho} \\ &+ a_{\rho\lambda}\left(-\frac{1}{3}\psi^{c}_{[211]_\rho}+\frac{2\sqrt{2}}{3}\psi^{c}_{[211]_\eta}\right)\left(\frac{1}{3}\psi^{osf}_{[31]_\lambda}+\frac{2\sqrt{2}}{3}\psi^{osf}_{[31]_\eta}\right) \\ &+ a_{\eta\eta}\left(\frac{2\sqrt{2}}{3}\psi^{c}_{[211]_\rho}+\frac{1}{3}\psi^{c}_{[211]_\eta}\right)\left(\frac{2\sqrt{2}}{3}\psi^{osf}_{[31]_\lambda}-\frac{1}{3}\psi^{osf}_{[31]_\eta}\right). \qquad (80)\end{aligned}$$

Here we have used the $[31]$ and $[211]$ representation matrices for the permutation (34),

$$D^{[31]}(34)=\begin{pmatrix} 1/3 & 0 & 2\sqrt{2}/3 \\ 0 & 1 & 0 \\ 2\sqrt{2}/3 & 0 & -1/3 \end{pmatrix}, \quad D^{[211]}(34)=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix} \qquad (81)$$

An antisymmetric ψ demands $a_{\lambda\rho}=a_{\eta\eta}$. Finally, we derive a fully antisymmetric wave function for the q^4 configuration

$$\psi = \frac{1}{\sqrt{3}}\left(\psi^{c}_{[211]_\lambda}\psi^{osf}_{[31]_\rho}-\psi^{c}_{[211]_\rho}\psi^{osf}_{[31]_\lambda}+\psi^{c}_{[211]_\eta}\psi^{osf}_{[31]_\eta}\right) \qquad (82)$$

q^4 Spatial-Flavor-Spin Configurations

The spatial-flavor-spin wave function of the q^4 cluster may be written in the general form

$$\Psi^{osf}_{[31]} = \sum_{i,j=S,A,\lambda,\rho,\eta} b_{ij} \Psi^{o}_{[X]_i} \Psi^{sf}_{[Y]_j} \quad (83)$$

Applying the permutations of S_4 on the general form leads to all the possible spatial-spin-flavor configurations:

$[31]_{OSF}$	
$[4]_O$	$[31]_{SF}$
$[1111]_O$	$[211]_{SF}$
$[22]_O$	$[31]_{SF}, [211]_{SF}$
$[211]_O$	$[31]_{SF}, [211]_{SF}, [22]_{SF}$
$[31]_O$	$[4]_{SF}, [31]_{SF}, [211]_{SF}, [22]_{SF}$

The flavor-spin wave function of the q^4 cluster may be written in the general form

$$\Psi^{sf}_{[Z]} = \sum_{i,j=S,A,\lambda,\rho,\eta} c_{ij} \Phi^{f}_{[X]_i} \chi^{s}_{[Y]_j} \quad (84)$$

Applying the permutations of S_4 on the general form leads to all the possible spin-flavor configurations.

q^4 Spin-Flavor Configurations

	$[4]_{FS}$		
$[4]_{FS}[22]_F[22]_S$	$[4]_{FS}[31]_F[31]_S$	$[4]_{FS}[4]_F[4]_S$	
	$[31]_{FS}$		
$[31]_{FS}[31]_F[22]_S$	$[31]_{FS}[31]_F[31]_S$	$[31]_{FS}[31]_F[4]_S$	$[31]_{FS}[211]_F[22]_S$
$[31]_{FS}[211]_F[31]_S$	$[31]_{FS}[22]_F[31]_S$	$[31]_{FS}[4]_F[31]_S$	
	$[22]_{FS}$		
$[22]_{FS}[22]_F[22]_S$	$[22]_{FS}[22]_F[4]_S$	$[22]_{FS}[4]_F[22]_S$	$[22]_{FS}[211]_F[31]_S$
$[22]_{FS}[31]_F[31]_S$			
	$[211]_{FS}$		
$[211]_{FS}[211]_F[22]_S$	$[211]_{FS}[211]_F[31]_S$	$[211]_{FS}[211]_F[4]_S$	$[211]_{FS}[22]_F[31]_S$
$[211]_{FS}[31]_F[22]_S$	$[211]_{FS}[31]_F[31]_S$		

q^4 Spin-Flavor Wave Functions

For the pentaquark states with isospin $I = 0$ and strangeness $S = 1$, the q^4 flavor-spin wave function of must be as follows:

$$\begin{array}{ccccc} [31] & = & [22] & \otimes & [31] \\ SU_{sf}(6) & & SU_f(3) & & SU_s(2) \end{array} \quad (85)$$

Again, the spin-flavor wave functions of various permutation symmetries take the general form,

$$\psi^{\rm sf} = \sum_{i=\lambda,\rho} \sum_{j=\lambda,\rho,\eta} a_{ij}\, \phi_{[22]_i} \chi_{[31]_j} \quad (86)$$

a_{ij} can be determined by acting the permutations of S_4 on the general form. The spin-flavor wave functions for the q^4 cluster are derived as,

$$\begin{aligned} \psi^{\rm sf}_{[31]_\rho} &= -\frac{1}{2}\phi_{[22]_\rho}\chi_{[31]_\lambda} - \frac{1}{2}\phi_{[22]_\lambda}\chi_{[31]_\rho} + \frac{1}{\sqrt{2}}\phi_{[22]_\rho}\chi_{[31]_\eta} \\ \psi^{\rm sf}_{[31]_\lambda} &= -\frac{1}{2}\phi_{[22]_\rho}\chi_{[31]_\rho} + \frac{1}{2}\phi_{[22]_\lambda}\chi_{[31]_\lambda} + \frac{1}{\sqrt{2}}\phi_{[22]_\lambda}\chi_{[31]_\eta} \\ \psi^{\rm sf}_{[31]_\eta} &= \frac{1}{\sqrt{2}}\phi_{[22]_\rho}\chi_{[31]_\rho} + \frac{1}{\sqrt{2}}\phi_{[22]_\lambda}\chi_{[31]_\lambda} \end{aligned} \quad (87)$$

q^4 Spin and Flavor Wave Functions

The explicit form of the spin and flavor wave functions of the q^4 configuration of pentaquark states can be easily worked out in the Yamanouchi technique, following the process:

- (1) Work out first the representation matrices in the Yamanouchi basis of the irreducible representations of S_4;
- (2) Construct the corresponding projection operators;
- (3) Act the operators on arbitrary four quark states to obtain the spin and flavor wave functions with the corresponding symmetries;

We worked out here for the q^4 subsystem

- the flavor wave functions of the $[22]$ symmetry;
- the spin wave functions of the [31] symmetry.

q^4 Flavor Wave Functions

The λ-type and ρ-type projection operators for the representation $[22]$ are derived as

$$\begin{aligned}
P_{[22]_\lambda} &= \sum_{i=1}^{24} \langle [22](2211)|R_i|[22](2211)\rangle R_i \\
&= 2+2(12)-(13)-(14)-(23)-(24)+2(34) \\
&\quad +2(12)(34)+2(14)(23)+2(13)(24) \\
&\quad -(123)-(124)-(132)-(134)-(142)-(143)-(234)-(243) \\
&\quad -(1234)-(1243)+2(1324)-(1342)+2(1423)-(1432) \qquad (88)
\end{aligned}$$

$$\begin{aligned}
P_{[22]_\rho} &= \sum_{i=1}^{24} \langle [22](2121)|R_i|[22](2121)\rangle R_i \\
&= 2-2(12)+(13)+(14)+(23)+(24)-2(34) \\
&\quad +2(12)(34)+2(14)(23)+2(13)(24) \\
&\quad -(123)-(124)-(132)-(134)-(142)-(143)-(234)-(243) \\
&\quad +(1234)+(1243)-2(1324)+(1342)-2(1423)+(1432) \qquad (89)
\end{aligned}$$

q^4 Flavor Wave Functions

The flavor wave functions of the four-quark subsystem with the [22] symmetry can be derived by operating $P_{[22]_{\lambda,\rho}}$ on any q^4 state. For example,

$$\left| \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline u & u \\ \hline d & d \\ \hline \end{array} \right\rangle = P_{[22]_\rho}(udud)$$
$$\Longrightarrow \phi_{[22]_\rho} = \frac{1}{2}(dudu - duud + udud - uddu) \tag{90}$$

$$\left| \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline u & u \\ \hline d & d \\ \hline \end{array} \right\rangle = P_{[22]_\lambda}(uudd)$$
$$\Longrightarrow \phi_{[22]_\lambda} = \frac{1}{2\sqrt{3}}(2uudd + 2dduu - duud - udud - uddu - dudu) \tag{91}$$

The flavor wave functions for the $I = I_3 = 0$, $S = 1$ pentaquark are given by

$$\begin{aligned} \Phi_{[22]_\rho} &= \phi_{[22]_\rho}\bar{s} \\ \Phi_{[22]_\lambda} &= \phi_{[22]_\lambda}\bar{s}\ . \end{aligned} \tag{92}$$

The flavor states with other values of the isospin I, its projection I_3 and hypercharge Y can be derived the same way.

q^4 Spin Wave Functions

The λ-type, ρ-type and η-type projection operators for the representation $[31]$ are derived as

$$\begin{aligned} P_{[31]_\lambda} &= 6 + 6(12) - 3(13) + 5(14) - 3(23) + 5(24) + 2(34) \\ &+2(12)(34) - 4(14)(23) - 4(13)(24) \\ &-3(123) + 5(124) - 3(132) - (134) + 5(142) - (143) - (234) - (243) \\ &-(1234) - (1243) - 4(1324) - (1342) - 4(1423) - (1432) \end{aligned} \tag{93}$$

$$\begin{aligned} P_{[31]_\rho} &= 2 - 2(12) + (13) + (14) + (23) + (24) + 2(34) \\ &-2(12)(34) \\ &-(123) - (124) - (132) + (134) - (142) + (143) + (234) + (243) \\ &-(1234) - (1243) - (1342) - (1432) \end{aligned} \tag{94}$$

$$\begin{aligned} P_{[31]_\eta} &= 3 + 3(12) + 3(13) - (14) + 3(23) - (24) - (34) \\ &-(12)(34) - (14)(23) - (13)(24) \\ &+3(123) - (124) + 3(132) - (134) - (142) - (143) - (234) - (243) \\ &-(1234) - (1243) - (1324) - (1342) - (1423) - (1432) \end{aligned} \tag{95}$$

q^4 Spin Wave Functions

The spin wave functions of the four-quark subsystem with the [31] symmetry can be derived by operating $P_{[31]_{\lambda,\rho,\eta}}$ on any q^4 spin state, for example, the state $\uparrow\uparrow\uparrow\downarrow$,

$$\begin{aligned}
&\left|\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline \uparrow & \uparrow & \uparrow \\ \hline \downarrow & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \right\rangle = P_{[31]_\eta}(\uparrow\uparrow\uparrow\downarrow) \\
&\Longrightarrow \chi_{[31]_\eta}(s_{q^4}=1, m_{q^4}=1) = \frac{1}{2\sqrt{3}}\ |\, 3\uparrow\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow\uparrow - \uparrow\downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow\uparrow \,\rangle \\
&\left|\ \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline \uparrow & \uparrow & \uparrow \\ \hline \downarrow & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \right\rangle = P_{[31]_\rho}(\uparrow\downarrow\uparrow\uparrow) \\
&\Longrightarrow \chi_{[31]_\rho}(s_{q^4}=1, m_{q^4}=1) = \frac{1}{\sqrt{2}}\ |\, \uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow \,\rangle \\
&\left|\ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline \uparrow & \uparrow & \uparrow \\ \hline \downarrow & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \right\rangle = P_{[31]_\lambda}(\uparrow\uparrow\downarrow\uparrow) \\
&\Longrightarrow \chi_{[31]_\lambda}(s_{q^4}=1, m_{q^4}=1) = \frac{1}{\sqrt{6}}\ |\, 2\uparrow\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\uparrow - \uparrow\downarrow\uparrow\uparrow \,\rangle
\end{aligned} \tag{96}$$

$q^4\bar{q}$ Spin-Flavor Wave Functions

The total spin wave function of the pentaquark states with $s = 1/2$ and [31] symmetry is the combination of the spin wave function of the four-quark subsystem with $[31]$ symmetry (with $s = 1$) and that of the antiquark with $s = 1/2$, that is

$$\chi(q^4\bar{s})_{[31]_\alpha} = \sqrt{\frac{2}{3}}\,\chi_{[31]_\alpha}(m_{q^4} = 1)\chi_{\bar{s}}(-\frac{1}{2}) - \sqrt{\frac{1}{3}}\,\chi_{[31]_\alpha}(m_{q^4} = 0)\chi_{\bar{s}}(\frac{1}{2}) \tag{97}$$

with $\alpha = \rho,\ \lambda,\ \eta$.
Combining the flavor wave functions in eq. (92) and the spin wave functions in eq. (97), we derive the total spin-flavor wave function of the pentaquark state with isospin $I = 0$, strangemess $S = 1$ and spin $s = 1/2$,

$$\begin{aligned}
\Psi^{\rm sf}_{[31]_\rho} &= -\frac{1}{2}\Phi_{[22]_\rho}\chi(q^4\bar{s})_{[31]_\lambda} - \frac{1}{2}\Phi_{[22]_\lambda}\chi(q^4\bar{s})_{[31]_\rho} + \frac{1}{\sqrt{2}}\Phi_{[22]_\rho}\chi(q^4\bar{s})_{[31]_\eta} \\
\Psi^{\rm sf}_{[31]_\lambda} &= -\frac{1}{2}\Phi_{[22]_\rho}\chi(q^4\bar{s})_{[31]_\rho} + \frac{1}{2}\Phi_{[22]_\lambda}\chi(q^4\bar{s})_{[31]_\lambda} + \frac{1}{\sqrt{2}}\Phi_{[22]_\lambda}\chi(q^4\bar{s})_{[31]_\eta} \\
\Psi^{\rm sf}_{[31]_\eta} &= \frac{1}{\sqrt{2}}\Phi_{[22]_\rho}\chi(q^4\bar{s})_{[31]_\rho} + \frac{1}{\sqrt{2}}\Phi_{[22]_\lambda}\chi(q^4\bar{s})_{[31]_\lambda}
\end{aligned} \tag{98}$$

q^4 Color Wave Functions

The color state of the antiquark in pentaquarks is a $[11]$ antitriplet, thus the color wave function of the four-quark configuration must be a $[211]_3$ triplet,

$$\psi^c_{[211]_\lambda}(q^4) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \psi^c_{[211]_\rho}(q^4) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \psi^c_{[211]_\eta}(q^4) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \tag{99}$$

The q^4 color wave functions can be derived by applying the [211] λ-type, ρ-type and η-type projection operators of the permutation group S_4 onto single particle color states. For the product state $RRGB$, for example, we have,

$$\psi^c_{[211]_\lambda}(q^4) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \psi^c_{[211]_\rho}(q^4) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \quad \psi^c_{[211]_\eta}(q^4) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \tag{100}$$

The q^4 color wave functions can be derived by applying the λ-type, ρ-type and η-type projection operators on single particle color states. For the product state $RRGB$, for example, we have,

$$\begin{aligned} P_{[211]_\lambda}(RRGB) &\Longrightarrow \psi^c_{[211]_\lambda}(R) \\ P_{[211]_\rho}(RGRB) &\Longrightarrow \psi^c_{[211]_\rho}(R) \\ P_{[211]_\eta}(RGBR) &\Longrightarrow \psi^c_{[211]_\eta}(R) \end{aligned} \tag{101}$$

$q^4\bar{q}$ Color Wave Functions

The explicit forms are

$$\begin{aligned}
\chi^c_{[211]_\lambda}(R) &= \frac{1}{\sqrt{16}}(2|RRGB\rangle - 2|RRBG\rangle \\
&-|GRRB\rangle - |RGRB\rangle - |BRGR\rangle - |RBGR\rangle \\
&+|BRRG\rangle + |GRBR\rangle + |RBRG\rangle + |RGBR\rangle),
\end{aligned}$$

$$\begin{aligned}
\chi^c_{[211]_\rho}(R) &= \frac{1}{\sqrt{48}}(3|RGRB\rangle - 3|GRRB\rangle \\
&+3|BRRG\rangle - 3|RBRG\rangle + 2|GBRR\rangle - 2|BGRR\rangle \\
&-|BRGR\rangle + |RBGR\rangle + |GRBR\rangle - |RGBR\rangle),
\end{aligned}$$

$$\begin{aligned}
\chi^c_{[211]_\eta}(R) &= \frac{1}{\sqrt{6}}(|BRGR\rangle + |RGBR\rangle + |GBRR\rangle \\
&-|RBGR\rangle - |GRBR\rangle - |BGRR\rangle). \qquad (102)
\end{aligned}$$

Thus, the corresponding singlet color wave function of the pentaquark at color symmetry pattern $j = \lambda, \rho, \eta$ is given by

$$\psi^c_{[211]_j} = \frac{1}{\sqrt{3}}\left[\chi^c_{[211]_j}(R)\,\bar{R} + \chi^c_{[211]_j}(G)\,\bar{G} + \chi^c_{[211]_j}(B)\,\bar{B}\right]. \qquad (103)$$

$q^4\bar{q}$ Spatial Wave Functions

A complete basis of certain symmetry may be constructed with pentaquark systems in the harmonic oscillator interaction.

$$H = \frac{p_\lambda^2}{2m} + \frac{p_\rho^2}{2m} + \frac{p_\eta^2}{2m} + \frac{p_\xi^2}{2m} + \frac{1}{2}C\left(\lambda^2 + \rho^2 + \eta^2 + \xi^2\right) \tag{104}$$

where

$$\begin{aligned}
\vec{\rho} &= \frac{1}{\sqrt{2}}(\vec{r_1} - \vec{r_2}) \\
\vec{\lambda} &= \frac{1}{\sqrt{6}}(\vec{r_1} + \vec{r_2} - 2\vec{r_3}) \\
\vec{\eta} &= \frac{1}{\sqrt{12}}(\vec{r_1} + \vec{r_2} + \vec{r_3} - 3\vec{r_4}) \\
\vec{\xi} &= \frac{1}{\sqrt{20}}(\vec{r_1} + \vec{r_2} + \vec{r_3} + \vec{r_4} - 4\vec{r_5})
\end{aligned} \tag{105}$$

$q^4\bar{q}$ Spatial Wave Functions

Spatial wave functions take the general form,

$$\begin{aligned}\Psi^{o}_{NLM} = & \sum_{n_\lambda,n_\rho,n_\eta,n_\xi,l_\lambda,l_\rho,l_\eta,l_\xi} A(n_\lambda,n_\rho,n_\eta,n_\xi,l_\lambda,l_\rho,l_\eta,l_\xi) \\ & \cdot\Psi_{n_\lambda l_\lambda m_\lambda}(\vec{\lambda})\Psi_{n_\rho l_\rho m_\rho}(\vec{\rho})\Psi_{n_\eta l_\eta m_\eta}(\vec{\eta})\Psi_{n_\xi l_\xi m_\xi}(\vec{\xi}) \\ & \cdot C(l_\lambda,l_\rho,m_\lambda,m_\rho,l_{\lambda\rho},m_{\lambda\rho}) \\ & \cdot C(l_{\lambda\rho},l_\eta,m_{\lambda\rho},m_\eta,l_{\lambda\rho\eta},m_{\lambda\rho\eta}) \\ & \cdot C(l_{\lambda\rho\eta},l_\xi,m_{\lambda\rho\eta},m_\xi,LM) \end{aligned} \tag{106}$$

with $N = 2(n_\lambda + n_\rho + n_\eta + n_\xi) + l_\lambda + l_\rho + l_\eta + l_\xi$
The coefficients A are determined by applying the Yamanouchi basis representations of the S_4. Various types of spatial wave functions with the $[4], [31], [22], [211]$ and $[1111]$ symmetries are worked out.

Constituent Quark Models for bare quark mass consistent

- Hamiltonian for a N-quark system:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i^0} + C\sum_{i<j}^{N} (\vec{r}_i - \vec{r}_j)^2 + \sum_{i=1}^{N} m_i^0 + H_{hyp} \tag{107}$$

$$H_{hyp}^{OGE} = -C_G \sum_{i<j} \frac{\lambda_i^C \cdot \lambda_j^C}{m_i m_j} \vec{\sigma}_i \cdot \vec{\sigma}_j \tag{108}$$

$$H_{hyp}^{OGE} = -C_G \sum_{i<j} \frac{\lambda_i^C \cdot \lambda_j^C}{m_i m_j} \vec{\sigma}_i \cdot \vec{\sigma}_j \delta^3(\vec{r}_{ij}) \tag{109}$$

- m_i^0 stands for "bare" quark masses, m_i are dressed quark masses resulted from m_i^0 and the ground-state energy of harmonic oscillation.
- Model parameters determined by fitting theoretical results to 8 baryon isospin states, 7 charm baryons and 6 bottom baryons:
  $m_u^0(m_u) = 91(362)$ MeV, $m_s^0(m_s) = 398(528)$ MeV
  $m_c^0(m_c) = 1598(1662)$ MeV, $m_b^0(m_b) = 4916(4953)$ MeV
  $w_0 = \sqrt{\frac{2C}{m_u}} = 157$ MeV, $C_m = C_G/m_u^2 = 19.1$ MeV

Constituent Quark Models for both bare and dressed quark mass consistent

- Hamiltonian for a N-quark system:

$$H_N = \sum_{i=1}^{N-1} \frac{\vec{\eta}_i^2}{2u_i} + \frac{\vec{P}_0^2}{2M_0} + \sum_{i=1}^{N-1} \frac{1}{2} u_i \, w_i^2 \vec{\xi}_i^2 + \sum_{i=1}^{N} m_j^0 \qquad (110)$$

- The unit oscillation energy w which represents three quark baryon oscillation energy w_3 while meson oscillation energy $w_2 = 4/3 w_3$, pentaquark oscillation energy $w_5 = 5/6 w_3$
- Model parameters are determined by fitting theoretical results to 38 states consisting of 21 baryon states and 17 meson states with both bare and dressed quark mass consistent:
  $m_u = 365 \text{ MeV}, \quad m_s = 531 \text{ MeV}$
  $m_c = 1691 \text{ MeV}, \quad m_b = 5019 \text{ MeV}$
  $w = \sqrt{\frac{6C}{m_u}} = 242 \text{ MeV}, \quad C_m = C_G/m_u^2 = 18.6 \text{ MeV}$

Mass of excited non-strange q^3 states with hyperfine interaction including $\delta^3(\vec{r}_{ij})$ function

States $\Psi(N,L)$	J^P	Mass MeV
$\Psi_{\text{Singlet}}(1,1)$	$\frac{1}{2}^-, \frac{3}{2}^-$	1371
$\Psi^{(1)}_{\text{Octet}}(1,1)$	$\frac{1}{2}^-, \frac{3}{2}^-$	1283
$\Psi^{(2)}_{\text{Octet}}(1,1)$	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$	1435
$\Psi_{\text{Decuplet}}(1,1)$	$\frac{1}{2}^-, \frac{3}{2}^-$	1435
$\Psi_{\text{Singlet}}(2,0)$	$\frac{1}{2}^+$	1569
$\Psi^{(1)}_{\text{Octet}}(2,0)$	$\frac{1}{2}^+$	1440
$\Psi^{(2)}_{\text{Octet}}(2,0)$	$\frac{3}{2}^+$	1535
$\Psi^{(3)}_{\text{Octet}}(2,0)$	$\frac{1}{2}^+$	1726
$\Psi^{(1)}_{\text{Decuplet}}(2,0)$	$\frac{1}{2}^+$	1726
$\Psi^{(2)}_{\text{Decuplet}}(2,0)$	$\frac{3}{2}^+$	1822

Ground state pentaquark $q^4\bar{q}$ masses for quark freedom case

$q^4\bar{q}$ Configurations	Spin (or J)	$M(q^4\bar{q})$ (MeV)
$\Psi^{sf}_{[31]_{FS}[4]_F[31]_S}(q^4\overline{q})$	$\frac{1}{2}$, $\frac{3}{2}$	3121, 2815
$\Psi^{sf}_{[31]_{FS}[31]_F[4]_S}(q^4\overline{q})$	$\frac{3}{2}$, $\frac{5}{2}$	2560, 2815
$\Psi^{sf}_{[31]_{FS}[31]_F[31]_S}(q^4\overline{q})$	$\frac{1}{2}$, $\frac{3}{2}$	2662, 2586
$\Psi^{sf}_{[31]_{FS}[31]_F[22]_S}(q^4\overline{q})$	$\frac{1}{2}$	2560
$\Psi^{sf}_{[31]_{FS}[22]_F[31]_S}(q^4\overline{q})$	$\frac{1}{2}$, $\frac{3}{2}$	2203, 2586

Ground state pentaquark $q^3 s\bar{s}$ masses in $q^4\bar{q}$ configuration for quark freedom case

$q^4\bar{q}$ Configurations	Spin (or J)	$M(q^4\overline{q})$ (MeV)
$\Psi^{sf}_{[31]_{FS}[4]_F[31]_S}(q^3 s\bar{s})$	$\frac{1}{2}$, $\frac{3}{2}$	3268, 3016
$\Psi^{sf}_{[31]_{FS}[31]_F[4]_S}(q^3 s\bar{s})$	$\frac{3}{2}$, $\frac{5}{2}$	2798, 2996
$\Psi^{sf}_{[31]_{FS}[31]_F[31]_S}(q^3 s\bar{s})$	$\frac{1}{2}$, $\frac{3}{2}$	2865, 2808
$\Psi^{sf}_{[31]_{FS}[31]_F[22]_S}(q^3 s\bar{s})$	$\frac{1}{2}$	2783
$\Psi^{sf}_{[31]_{FS}[211]_F[31]_S}(q^3 s\bar{s})$	$\frac{1}{2}$, $\frac{3}{2}$	2331, 2642
$\Psi^{sf}_{[31]_{FS}[211]_F[22]_S}(q^3 s\bar{s})$	$\frac{1}{2}$	2535
$\Psi^{sf}_{[31]_{FS}[22]_F[31]_S}(q^3 s\bar{s})$	$\frac{1}{2}$, $\frac{3}{2}$	2464, 2779

Ground state pentaquarks $q^3c\bar{c}$ in $q^3Q\bar{Q}$ configuration in mode freedom case

$q^3Q\bar{Q}$ Configurations	J^P	$M(q^3c\bar{c})$(MeV)
$\Psi^{csf}_{[111]_C[21]_F[21]_S}(q^3c\bar{c})$	$\frac{1}{2}^-$, $\frac{3}{2}^-$	4314, 4333
$\Psi^{csf}_{[111]_C[21]_F[3]_S}(q^3c\bar{c})$	$\frac{3}{2}^-$, $\frac{5}{2}^-$	4522, 4630
$\Psi^{csf}_{[21]_C[21]_F[21]_S}(q^3c\bar{c})$	$\frac{1}{2}^-$, $\frac{3}{2}^-$	4442, 4445
$\Psi^{csf}_{[21]_C[3]_F[21]_S}(q^3c\bar{c})$	$\frac{1}{2}^-$, $\frac{3}{2}^-$	4665, 4662
$\Psi^{csf}_{[21]_C[21]_F[3]_S}(q^3c\bar{c})$	$\frac{3}{2}^-$, $\frac{5}{2}^-$	4516 4530

Ground state pentaquarks $q^3b\bar{b}$ in $q^3Q\bar{Q}$ configuration in mode freedom case

$q^3Q\bar{Q}$ Configurations	J^P	$M(q^3b\bar{b})$(MeV)
$\Psi^{csf}_{[111]_C[21]_F[21]_S}(q^3b\bar{b})$	$\frac{1}{2}^-$, $\frac{3}{2}^-$	10983, 10985
$\Psi^{csf}_{[111]_C[21]_F[3]_S}(q^3b\bar{b})$	$\frac{3}{2}^-$, $\frac{5}{2}^-$	11182, 11282
$\Psi^{csf}_{[21]_C[21]_F[21]_S}(q^3b\bar{b})$	$\frac{1}{2}^-$, $\frac{3}{2}^-$	11096, 11098
$\Psi^{csf}_{[21]_C[3]_F[21]_S}(q^3b\bar{b})$	$\frac{1}{2}^-$, $\frac{3}{2}^-$	11319, 11319
$\Psi^{csf}_{[21]_C[21]_F[3]_S}(q^3b\bar{b})$	$\frac{3}{2}^-$, $\frac{5}{2}^-$	11170, 11176

Discussion about Roper without $\delta^3(\vec{r}_{ij})$ function

- The work gives a mass about 1478 MeV for non-strange q^3 first radial excited states. It may imply the Roper resonance is mainly a q^3 state.
- Assuming $N(1535)$ and $N(1520)$ having a large $q^3 s\bar{s}$ and $q^4\bar{q}$ component of $\Psi^{sf}_{[31]_{FS}[22]_F[31]_S}$, then we have from $M(q^3) = 1206$ MeV ,$M(q^3 s\bar{s})_{s=1/2} = 2331$ MeV and $M(q^4\bar{q})_{s=3/2} = 2586$ MeV,

States	J^P	$q^3\%$	$q^4\bar{q}\%$
$N(1535)$	$\frac{1}{2}^-$	70.8	29.2
$N(1520)$	$\frac{3}{2}^-$	77.2	22.8

- The work gives masses of $4310 \sim 4660$ MeV for ground state pentaquarks $q^3 c\bar{c}$. It is consistent with the LHCb observation of $P_c^+(4380)$ and $P_c^+(4450)$.
- The work predicts that ground state pentaquarks $q^3 b\bar{b}$ may have masses around 11 GeV.

Discussion about Roper with $\delta^3(\vec{r}_{ij})$ function

- The work gives a mass about 1440 MeV for non-strange q^3 first radial excited of spatial symmetric states. It may imply the Roper resonance is mainly a q^3 state.
- Assuming $N(1535)$ and $N(1520)$ having a large $q^3 s\bar{s}$ component of $\Psi^{sf}_{[31]_{FS}[211]_F[31]_S}$ and $q^4\bar{q}$ component of $\Psi^{sf}_{[31]_{FS}[22]_F[31]_S}$, then we have from $M(q^3) = 1283\ \mathrm{MeV}$,$M(q^3 s\bar{s})_{s=1/2} = 2331\ \mathrm{MeV}$ and $M(q^4\bar{q})_{s=3/2} = 2586\ \mathrm{MeV}$,

States	J^P	$q^3\%$	$q^4\bar{q}\%$
$N(1535)$	$\frac{1}{2}^-$	76.0	24.0
$N(1520)$	$\frac{3}{2}^-$	81.8	18.2

- The work gives masses of $4310 \sim 4660$ MeV for ground state pentaquarks $q^3 c\bar{c}$. It is consistent with the LHCb observation of $P_c^+(4380)$ and $P_c^+(4450)$.
- The work predicts that ground state pentaquarks $q^3 b\bar{b}$ may have masses around 11 GeV.

Thank you for your attention!

Error of the theoretical results

Baryon	PDG Data (MeV)	Calculated mass (MeV)	Error (MeV)
N	938	934.7	-3.3
Δ	1232	1238.2	6.2
Λ	1115	1131.6	16.6
Σ	1193	1163.7	-29.1
Σ^*	1385	1386.6	1.6
Θ	1315	1331.8	16.8
Θ^*	1530	1527.4	-2.6
Ω	1672	1666.0	-6.0

- The quark mass in one-gluon-exchange contribution should be the dressed mass which include bare quark mass and harmonic oscillator energy.
- The goldstone-boson-exchange contribution can be ignored.

Error of the theoretical results

$$m_u^3 \approx 126.3 \text{ MeV}, \quad m_s^3 \approx 399.4 \text{ MeV}, \quad m_u^2 \approx 192.7 \text{ MeV}, \quad m_s^2 \approx 382.0 \text{ MeV},$$
$$C_m \approx 18.96 \text{ MeV}, \quad w_0 \approx 136.2 \text{ MeV}, \tag{111}$$

Baryon	PDG Data (MeV)	Calculated mass (MeV)	Error (MeV)
Proton	938	935.0	-3.0
Δ	1232	1238.3	6.3
Λ	1115	1131.4	16.4
Σ	1193	1163.6	-29.4
Σ^*	1385	1386.3	1.3
Θ	1315	1331.7	16.7
Θ^*	1530	1527.3	-2.7
Ω	1672	1666.4	-5.6
ρ	770	775.6	5.6
ω	783	775.6	-7.4
k^*	892	895.7	3.7
ϕ	1020	1018.2	-1.8

Pentaquark Spatial Wave Function, $NLM = 322$

- Symmetric:

$$\Psi^S = \frac{1}{3}\left[-\Psi_{021}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{011}(\xi) + \sqrt{2}\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{010}(\xi) \right.$$
$$-\Psi_{000}(\lambda)\Psi_{021}(\rho)\Psi_{000}(\eta)\Psi_{011}(\xi) + \sqrt{2}\Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{010}(\xi)$$
$$\left. -\Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{021}(\eta)\Psi_{011}(\xi) + \sqrt{2}\Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{022}(\eta)\Psi_{010}(\xi)\right] \quad (112)$$

- Antisymmetric: Non
- λ and ρ types of [22]: Non
- λ, ρ and η types of [211]: Non

Pentaquark Spatial Wave Function, $NLM = 322$

First Set of λ, ρ and η types of [31]:

$$\Psi^{\lambda[31]} = \frac{1}{\sqrt{6}}[\sqrt{2}\Psi_{010}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) - \Psi_{011}(\lambda)\Psi_{021}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi)$$
$$+\sqrt{2}\Psi_{010}(\lambda)\Psi_{000}(\rho)\Psi_{022}(\eta)\Psi_{000}(\xi) - \Psi_{011}(\lambda)\Psi_{000}(\rho)\Psi_{021}(\eta)\Psi_{000}(\xi)]$$

$$\Psi^{\rho[31]} = \frac{1}{\sqrt{6}}[\sqrt{2}\Psi_{022}(\lambda)\Psi_{010}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) - \Psi_{021}(\lambda)\Psi_{011}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi)$$
$$+\sqrt{2}\Psi_{000}(\lambda)\Psi_{010}(\rho)\Psi_{022}(\eta)\Psi_{000}(\xi) - \Psi_{000}(\lambda)\Psi_{011}(\rho)\Psi_{021}(\eta)\Psi_{000}(\xi)]$$

$$\Psi^{\eta[31]} = \frac{1}{\sqrt{6}}[\sqrt{2}\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{010}(\eta)\Psi_{000}(\xi) - \Psi_{021}(\lambda)\Psi_{000}(\rho)\Psi_{011}(\eta)\Psi_{000}(\xi)$$
$$+\sqrt{2}\Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{010}(\eta)\Psi_{000}(\Xi) - \Psi_{000}(\lambda)\Psi_{021}(\rho)\Psi_{011}(\eta)\Psi_{000}(\Xi)]$$

Pentaquark Spatial Wave Function, $NLM = 322$

Second Set of λ, ρ and η types of [31]:

$$\Psi^{\lambda[31]} = \frac{1}{\sqrt{3}}[\sqrt{2}\Psi_{010}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{022}(\xi) - \Psi_{011}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{021}(\xi)]$$

$$\Psi^{\rho[31]} = \frac{1}{\sqrt{3}}[\sqrt{2}\Psi_{000}(\lambda)\Psi_{010}(\rho)\Psi_{000}(\eta)\Psi_{022}(\xi) - \Psi_{000}(\lambda)\Psi_{011}(\rho)\Psi_{000}(\eta)\Psi_{021}(\xi)]$$

$$\Psi^{\eta[31]} = \frac{1}{\sqrt{3}}[\sqrt{2}\Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{010}(\eta)\Psi_{022}(\xi) - \Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{011}(\eta)\Psi_{021}(\xi)]$$

Pentaquark Spatial Wave Function, Symmetric $NLM = 444$

$$\Psi^{S_1} = \Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{044}(\xi)$$

$$\Psi^{S_2} = \frac{1}{\sqrt{3}}[\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{022}(\xi) + \Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{022}(\xi) + \Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{022}(\eta)\Psi_{022}(\xi)]$$

$$\Psi^{S_3} = \sqrt{\frac{1}{17}}[\Psi_{033}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{011}(\xi) - \sqrt{7}\Psi_{011}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{011}(\xi) + \sqrt{2}\Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{033}(\eta)\Psi_{011}(\xi) - \sqrt{\frac{7}{2}}\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{011}(\eta)\Psi_{011}(\xi) - \sqrt{\frac{7}{2}}\Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{011}(\eta)\Psi_{011}(\xi)]$$

$$\Psi^{S_4} = \sqrt{\frac{5}{57}}[\Psi_{044}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) + \Psi_{000}(\lambda)\Psi_{044}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) + \Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{044}(\eta)\Psi_{000}(\xi) + \sqrt{\frac{14}{5}}\Psi_{022}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) + \sqrt{\frac{14}{5}}\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{022}(\eta)\Psi_{000}(\xi) + \sqrt{\frac{14}{5}}\Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{022}(\eta)\Psi_{000}(\xi)]$$

Matrix Representations of S_4

- S4 $[211]$

$$
\begin{aligned}
D^{[211]}(12) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; D^{[211]}(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
D^{[211]}(34) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix} \qquad (114)
\end{aligned}
$$

- S4 $[31]$

$$
\begin{aligned}
D^{[31]}(12) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; D^{[31]}(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
D^{[31]}(34) &= \begin{pmatrix} 1/3 & 0 & 2\sqrt{2}/3 \\ 0 & 1 & 0 \\ 2\sqrt{2}/3 & 0 & -1/3 \end{pmatrix} \qquad (115)
\end{aligned}
$$

q^3 color WF ρ type	$q\bar{q}$	$q^3\ \lambda$
$\frac{1}{\sqrt{2}}(RGR - GRR)$	$B\bar{R}$	$\frac{1}{\sqrt{6}}($
$\frac{1}{\sqrt{2}}(RGG - GRG)$	$B\bar{G}$	$\frac{1}{\sqrt{6}}($
$\frac{1}{\sqrt{2}}(RBR - BRR)$	$-G\bar{R}$	$\frac{1}{\sqrt{6}}($
$\frac{1}{2}(RBG + GBR - BRG - BGR)$	$\frac{1}{\sqrt{2}}(R\bar{R} - G\bar{G})$	$\frac{1}{\sqrt{12}}$
$\frac{1}{\sqrt{2}}(GBG - BGG)$	$R\bar{G}$	$\frac{1}{\sqrt{6}}($
$\frac{1}{\sqrt{12}}(2RGB - 2GRB - GBR + BGR - BRG + RBG)$	$\frac{1}{\sqrt{6}}(2B\bar{B} - R\bar{R} - G\bar{G})$	$\frac{1}{2}(R$
$\frac{1}{\sqrt{2}}(RBB - BRB)$	$-G\bar{B}$	$\frac{1}{\sqrt{6}}($
$\frac{1}{\sqrt{2}}(GBB - BGB)$	$R\bar{B}$	$\frac{1}{\sqrt{6}}($

q^3 Baryon States:
(1) $N, L = 1, 1$:

$$
\begin{aligned}
\Psi^{(q^3)}_{Singlet} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\Phi_A(\phi^1_{1m\lambda}\chi_\rho - \phi^1_{1m\rho}\chi_\lambda), \\
\Psi^{(q^3)}_{Octet1} &= \frac{1}{2}\psi^c_{[111]}[\phi^1_{1m\rho}(\Phi_\lambda\chi_\rho + \Phi_\rho\chi_\lambda) \\
&\quad +\phi^1_{1m\lambda}(\Phi_\rho\chi_\rho - \Phi_\lambda\chi_\lambda)], \\
\Psi^{(q^3)}_{Octet2} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\chi_S(\phi^1_{1m\lambda}\Phi_\lambda + \phi^1_{1m\rho}\Phi_\rho), \\
\Psi^{(q^3)}_{Decuplet} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\Phi_S(\phi^1_{1m\lambda}\chi_\lambda + \phi^1_{1m\rho}\chi_\rho)
\end{aligned}
\tag{116}
$$

(2) $N, L = 2, 0$ (spatial part symmetric):

$$\begin{aligned} \Psi^{(q^3)}_{Octet} &= \frac{1}{\sqrt{2}} \psi^c_{[111]} \phi^2_{00S} (\Phi_\rho \chi_\rho + \Phi_\lambda \chi_\lambda), \\ \Psi^{(q^3)}_{Decuplet} &= \psi^c_{[111]} \phi^2_{00S} \Phi_S \chi_S \end{aligned} \tag{117}$$

$N, L = 2, 0$ (spatial part mixed symmetric):

$$\begin{aligned} \Psi^{(q^3)}_{Singlet} &= \frac{1}{\sqrt{2}} \psi^c_{[111]} \Phi_A (\phi^2_{00\lambda} \chi_\rho - \phi^2_{00\rho} \chi_\lambda), \\ \Psi^{(q^3)}_{Octet1} &= \frac{1}{2} \psi^c_{[111]} [\phi^2_{00\rho} (\Phi_\lambda \chi_\rho + \Phi_\rho \chi_\lambda) \\ &\quad + \phi^2_{00\lambda} (\Phi_\rho \chi_\rho - \Phi_\lambda \chi_\lambda)], \\ \Psi^{(q^3)}_{Octet2} &= \frac{1}{\sqrt{2}} \psi^c_{[111]} \chi_S (\phi^2_{00\lambda} \Phi_\lambda + \phi^2_{00\rho} \Phi_\rho), \\ \Psi^{(q^3)}_{Decuplet} &= \frac{1}{\sqrt{2}} \psi^c_{[111]} \Phi_S (\phi^2_{00\lambda} \chi_\lambda + \phi^2_{00\rho} \chi_\rho) \end{aligned} \tag{118}$$