Introduction to Effective Field Theories for Hadron Physics

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1 Why do we need effective field theories?

2 Effective field theory in a nut shell

3 Symmetries of QCD

4 Spontaneous breaking of chiral symmetry

5 Chiral effective Lagrangians
   - Leading order chiral Lagrangian for Goldstone bosons
   - Combining chiral symmetry and heavy quark symmetry
   - CHPT at the next-to-leading order
Useful textbooks/monographs:


Elementary particles in the Standard Model

**Quarks**
- **Up Quark** ($u$): $2.3^{+0.7}_{-0.5}$ MeV
- **Charm Quark** ($c$): $1275 \pm 25$ MeV
- **Top Quark** ($t$): $\approx 173$ GeV
- **Down Quark** ($d$): $4.8^{+0.5}_{-0.3}$ MeV
- **Strangeness Quark** ($s$): $95 \pm 5$ MeV
- **Bottom Quark** ($b$): $4180 \pm 30$ MeV

**Gauge Bosons**
- **Higgs Boson** ($H$): $\approx 126$ GeV
- **Photon** ($\gamma$): $0$
- **Z Boson** ($Z$): $\approx 91.2$ GeV
- **W Boson** ($W^\pm$): $\approx 80.4$ GeV

**Leptons**
- **Electron** ($e$): $0.51$ MeV
- **Muon** ($\mu$): $105.7$ MeV
- **Tau** ($\tau$): $1776.8 \pm 0.2$ MeV
- **Electron Neutrino** ($\nu_e$): $< 2.2$ MeV
- **Muon Neutrino** ($\nu_\mu$): $< 0.17$ MeV
- **Tau Neutrino** ($\nu_\tau$): $< 15.5$ MeV

**Equation**
$$M_{\text{proton}} \gg (2m_u + m_d)$$
Quantum Electrodynamics (QED)

- Basic interaction vertex in QED

  ![Interaction Vertex](image)

- Photon does not carry charge, photon-photon interaction only happens at higher orders

  ![Interaction Diagrams](image)

- Coupling constant is small, $\alpha \approx 1/137 \implies$ great success of perturbation theory

  E.g., the electron magnetic moment

  $\frac{g}{2} = 1.001\,159\,652\,180\,73(28)$\,[0.28\ ppt] \hspace{1cm} \text{D. Hanneke et al, PRL100(2008)120801}

  $\frac{g}{2} = 1.001\,159\,652\,181\,78(77)$\,[0.77\ ppt] \hspace{1cm} \text{T. Aoyama et al, PRL109(2008)111807}

  (up to $\mathcal{O}(\alpha^5)$: 12672 Feynman diagrams!)
Quantum Chromodynamics (QCD)

- Quantum Chromodynamics (QCD) is the theory of the strong interaction

\[ \mathcal{L}_{\text{QCD}} = \sum_{f = u, d, s, c, b, t} \bar{q}_f (i \not{D} - m_f) q_f - \frac{1}{4} G^{a \mu \nu} G^{\mu \nu, a} + \frac{g_s^2 \theta}{64 \pi^2} \epsilon^{\mu \nu \rho \sigma} G_{\mu \nu}^a G_{\rho \sigma}^a \]

\[ D_\mu q_f = \left( \partial_\mu + ig_s A_\mu^a \lambda^a / 2 \right) q_f, \quad G^{a \mu \nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s \epsilon^{abc} A_\mu^b A_\nu^c \]

- Color gauge invariance under SU(3)$_c$ (quarks: red, blue and green): non-Abelian
- Gluons: 8 colors, self-interacting

- The $\theta$-term breaks P and CP, $\theta$ is tiny: $|\theta| \lesssim 10^{-10}$
Two facets of QCD

- Running of the coupling constant $\alpha_s = g_s^2/(4\pi)$

- High energies
  - asymptotic freedom, perturbative
  - degrees of freedom: quarks and gluons

- Low energies
  - nonperturbative, $\Lambda_{QCD} \sim 300$ MeV = $O\left(1 \text{ fm}^{-1}\right)$
  - color confinement, degrees of freedom: mesons and baryons
  - theory of quarks and gluons $\rightarrow$ low-energy hadron spectrum
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Theoretical tools for studying nonperturbative QCD

- Lattice QCD: numerical simulation in discretized Euclidean space-time
  - finite volume ($L$ should be large)
  - finite lattice spacing ($a$ should be small)
  - normally using $m_{u,d}$ larger than the physical values $\Rightarrow$ chiral extrapolation

- Models (e.g., quark model, see lectures by Q. Zhao)
- Low-energy effective field theories of QCD:
  effective degrees of freedom can be mesons and baryons
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- Low-energy effective field theories of QCD:
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been proven, but which I cannot imagine could be wrong. The “theorem” says that although individual quantum field theories have of course a good deal of content, quantum field theory itself has no content beyond analyticity, unitarity, cluster decomposition, and symmetry. This can be put more precisely in the context of perturbation theory: if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. As I said, this has not

Translation:

- The most general effective Lagrangian, up to a given order, consistent with the symmetries of the underlying theory
  ⇒ results consistent with the underlying theory!

- The degrees of freedom can be different from those of the underlining theory
  ⇒ we can work with hadrons directly for low-energy QCD
Low-energy EFT (II)

However, the “most general” means

- an infinite number of parameters \(\Rightarrow\) intractable (?
- nonrenormalizable (in contrast to, e.g., QED and QCD)

Solution: systematic expansion with a power counting

- only a finite number of parameters at a given order, can be determined from
  - experiments
  - lattice calculations for QCD
- renormalize order by order
- existence of a small (dimensionless) quantity, e.g.,
  - separation of energy scales, \(E \ll \Lambda \Rightarrow\) expansion in powers of \((E/\Lambda)\)

Neutron decay \((n \to pe^-\bar{\nu}_e)\): weak interactions for \(|q^2| \ll M_W^2\) (decoupling EFT)
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Neutron decay ($n \rightarrow p e^- \bar{\nu}_e$): weak interactions for $|q^2| \ll M_W^2$ (decoupling EFT)

\[
\frac{e^2}{8 \sin \theta_W} \frac{1}{M_W^2 - q^2} = \frac{e^2}{8 M_W^2 \sin \theta_W} \left( 1 + \frac{q^2}{M_W^2} + \ldots \right) = \frac{G_F}{\sqrt{2}} + \mathcal{O} \left( \frac{q^2}{M_W^2} \right)
\]
Low-energy EFT (III)

- **Pro:** model-independent, controlled uncertainty
- **Con:** number of parameters increases fast when going to higher orders

Things need to be remembered for an EFT:

- **separation of energy scales:**
  systematic expansion with a power counting
- **symmetry** constraints from the full theory

For low-energy QCD, we will consider

- (approximate) chiral symmetry of light quarks
  ⇒ **CHiral Perturbation Theory** (non-decoupling EFT)
  full theory ⇒ EFT via **spontaneous symmetry breaking** (SSB)
  generation of new light degrees of freedom

- **heavy quark symmetry:** spin and flavor
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- **heavy quark symmetry:** spin and flavor
\[ \mathcal{L}_{\text{QCD}} = \sum_{f=u,d,s,c,b,t} \bar{q}_f (i \slashed{D} - m_f) q_f - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} + \frac{g_s^2 \theta}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} G_{\mu\nu}^a G_{\rho\sigma}^a \]

- Exact: Lorentz-invariance, SU(3)_c gauge, \( C \) (for \( \theta = 0 \) with real \( m_f \), \( P \) and \( T \) as well)

- Hierachy of quark masses: \( m_{u,d,s} \ll \Lambda_{\text{QCD}} (\sim 300 \text{ MeV}) \ll m_{c,b,t} \)

- For heavy quarks (charm, bottom) in a hadron, typical momentum transfer \( \Lambda_{\text{QCD}} \)

\[ \text{heavy quark flavor symmetry (HQFS)} \] for any hadron containing one heavy quark:
velocity remains unchanged in the limit \( m_Q \to \infty \):

\[ \Delta v = \frac{\Delta p}{m_Q} = \frac{\Lambda_{\text{QCD}}}{m_Q} \]

⇒ heavy quark is like a static color triplet source, \( m_Q \) is irrelevant

\[ \text{heavy quark spin symmetry (HQSS)} \]: see lectures by E.Eichten

chromomag. interaction \( \propto \frac{\vec{\sigma} \cdot \vec{B}}{m_Q} \)

spin of the heavy quark decouples
In the heavy quark limit $m_Q \to \infty$, consider the quark propagator

$$i \frac{p + m_Q}{p^2 - m_Q^2 + i\epsilon} = i \frac{m_Q(\gamma + 1) + \not{k}}{2m_Q \not{v} \cdot \not{k} + k^2 + i\epsilon} \xrightarrow{m_Q \to \infty} \frac{1 + \gamma}{2} \frac{i}{v \cdot k + i\epsilon}$$

here $p = m_Q v + k$, with residual momentum $k \sim \Lambda_{\text{QCD}}$.

Velocity-dependent projector (in the rest frame $\vec{v} = 0$) $\Rightarrow$ particle component

$$\frac{1 + \gamma}{2} = \frac{1 + \gamma^0}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(in Dirac basis)}$$

Decompose a heavy quark field into $v$-dep. fields

$$Q(x) = e^{-im_Q \not{v} \cdot x} [Q_v(x) + q_v(x)]$$

with:

$$Q_v(x) = e^{im_Q \not{v} \cdot x} \frac{1 + \gamma}{2} Q(x), \quad q_v(x) = e^{im_Q \not{v} \cdot x} \frac{1 - \gamma}{2} Q(x)$$

Exercise: Let $P_\pm = \frac{1 \pm \gamma}{2}$, show $P_\pm^2 = P_\pm$, $P_+ P_- = 0$, and

$$\bar{Q}_v \gamma_\mu Q_v = \bar{Q}_v v_\mu Q_v$$
• **Heavy quark symmetry** in the leading order term:

\[
\mathcal{L}_Q = \bar{Q}(i\not\!D - m_Q)Q = \bar{Q}_v(i\not\!v \cdot \not\!D)Q_v + \mathcal{O}(m_Q^{-1})
\]

• Let total angular momentum \( \vec{J} = \vec{s}_Q + \vec{s}_\ell \),

\( \vec{s}_Q \): heavy quark spin,
\( \vec{s}_\ell \): spin of the light degrees of freedom (including orbital angular momentum)

⚠️ HQSS: \( s_\ell \) is a good quantum number in heavy hadrons

⚠️ spin multiplets:

for singly heavy mesons, e.g. \( \{D, D^*\} \) with \( s_\ell^P = \frac{1}{2}^- \),

\[ M_{D^*} - M_D \simeq 140 \text{ MeV}, \quad M_{B^*} - M_B \simeq 46 \text{ MeV} \]

for heavy quarkonia, e.g. \( \{\eta_c, J/\psi\} \) (no flavor symmetry for 2 or more heavy quarks)
Examples of HQSS phenomenology:

- Widths of the two $D_1 (J^P = 1^+)$ mesons
  \[ \Gamma[D_1(2420)] = (27.4 \pm 2.5) \text{ MeV} \ll \Gamma[D_1(2430)] = (384^{+130}_{-110}) \text{ MeV} \]

- $s_\ell = s_q + \vec{L} \Rightarrow$ for $P$-wave charmed mesons: $s_\ell^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$

- for decays $D_1 \to D^*\pi$:
  \[ \frac{1}{2}^+ \to \frac{1}{2}^- + 0^- \text{ in } S\text{-wave} \Rightarrow \text{large width} \]
  \[ \frac{3}{2}^+ \to \frac{1}{2}^- + 0^- \text{ in } D\text{-wave} \Rightarrow \text{small width} \]

- thus, $D_1(2420): s_\ell = \frac{3}{2}, \quad D_1(2430): s_\ell = \frac{1}{2}$

- Suppression of the $S$-wave production of $\frac{3}{2}^+ + \frac{1}{2}^-$ heavy meson pairs in $e^+e^-$ annihilation

  X. Li, M. Voloshin, Phys. Rev. D 88 (2013) 034012

**Exercise**: Try to understand this as a consequence of HQSS.

For more examples and applications, see the lectures by Q. Wang.
QCD symmetries (IV): Heavy quark symmetry

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  \[ \text{for decays } D_1 \to D^*\pi: \]
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QCD symmetries (V): Light flavor symmetry

Light meson SU(3) \([u, d, s]\) multiplets (octet + singlet):

See lectures by Q. Zhao

- **Vector mesons**

<table>
<thead>
<tr>
<th>meson</th>
<th>quark content</th>
<th>mass (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho^+ / \rho^-)</td>
<td>(u\bar{d} / d\bar{u})</td>
<td>775</td>
</tr>
<tr>
<td>(\rho^0)</td>
<td>((u\bar{u} - d\bar{d})/\sqrt{2})</td>
<td>775</td>
</tr>
<tr>
<td>(K^{<em>+} / K^{</em>-})</td>
<td>(u\bar{s} / s\bar{u})</td>
<td>892</td>
</tr>
<tr>
<td>(K^{*0} / \bar{K}^{*0})</td>
<td>(d\bar{s} / s\bar{d})</td>
<td>896</td>
</tr>
<tr>
<td>(\omega)</td>
<td>((u\bar{u} + d\bar{d})/\sqrt{2})</td>
<td>783</td>
</tr>
<tr>
<td>(\phi)</td>
<td>(s\bar{s})</td>
<td>1019</td>
</tr>
</tbody>
</table>

- Approximate SU(3) symmetry
- Very good isospin SU(2) symmetry

\[m_{\rho^0} - m_{\rho^\pm} = (-0.7 \pm 0.8) \text{ MeV}, \quad m_{K^{*0}} - m_{K^{*\pm}} = (6.7 \pm 1.2) \text{ MeV}\]
QCD symmetries (V): Light flavor symmetry

Light meson SU(3) $[u, d, s]$ multiplets (octet + singlet):

- Pseudoscalar mesons

<table>
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<tr>
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</tr>
<tr>
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<td>135</td>
</tr>
<tr>
<td>$K^+ / K^-$</td>
<td>$u\bar{s} / s\bar{u}$</td>
<td>494</td>
</tr>
<tr>
<td>$K^0 / \bar{K}^0$</td>
<td>$d\bar{s} / s\bar{d}$</td>
<td>498</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$\sim (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$</td>
<td>548</td>
</tr>
<tr>
<td>$\eta'$</td>
<td>$\sim (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$</td>
<td>958</td>
</tr>
</tbody>
</table>

- Very good isospin SU(2) symmetry

$m_{\pi^\pm} - m_{\pi^0} = (4.5936 \pm 0.0005) \text{ MeV}$, $m_{K^0} - m_{K^\pm} = (3.937 \pm 0.028) \text{ MeV}$

- Why are the pions so light?
QCD symmetries (VI): Chiral symmetry

- Masses of the three lightest quarks $u, d, s$ are small
  $\Rightarrow$ approximate chiral symmetry

- Chiral decomposition of fermion fields:

\[
\psi = \frac{1}{2}(1 - \gamma_5)\psi + \frac{1}{2}(1 + \gamma_5)\psi \equiv P_L \psi + P_R \psi = \psi_L + \psi_R
\]

  \[
P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0, \quad P_L + P_R = 1
\]

- For massless fermions, left-/right-handed fields do not interact with each other

\[
\mathcal{L}[\psi_L, \psi_R] = i \bar{\psi}_L \slashed{D} \psi_L + i \bar{\psi}_R \slashed{D} \psi_R - m (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)
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\]
Decompose $\mathcal{L}_{\text{QCD}}$ into $\mathcal{L}_{\text{QCD}}^0$, the QCD Lagrangian in the 3-flavor chiral limit $m_u = m_d = m_s = 0$, and the light quark mass term:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^0 - \bar{q} M q, \quad q = (u, d, s)^T, \quad M = \text{diag}(m_u, m_d, m_s)$$

$\mathcal{L}_{\text{QCD}}^0 = i \bar{q}_L \not{D} q_L + i \bar{q}_R \not{D} q_R + \ldots$ is invariant under $U(3)_L \times U(3)_R$ transformations:

$$\mathcal{L}_{\text{QCD}}^0(G_{\mu\nu}, q', D_{\mu} q') = \mathcal{L}_{\text{QCD}}^0(G_{\mu\nu}, q, D_{\mu} q)$$

$$q' = R P_R q + L P_L q = R q_R + L q_L$$

$R \in U(3)_R, \quad L \in U(3)_L$

Parity:

$$q(t, \vec{x}) \xrightarrow{P} \gamma^0 q(t, -\vec{x})$$

$$q_R(t, \vec{x}) \xrightarrow{P} P_R \gamma^0 q(t, -\vec{x}) = \gamma^0 P_L q(t, -\vec{x}) = \gamma^0 q_L(t, -\vec{x})$$

$$q_L(t, \vec{x}) \xrightarrow{P} \gamma^0 q_R(t, -\vec{x})$$
QCD symmetries (VIII): Chiral symmetry

- Decompose $U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$:
  18 conserved (Noether) currents for massless QCD at the classical level

$$J_{L,R}^{\mu,a} = \bar{q}_{L,R} \gamma^\mu \frac{\lambda^a}{2} q_{L,R}, \quad J_{L,R}^{\mu,0} = \bar{q}_{L,R} \gamma^\mu q_{L,R}, \quad (\lambda_a : \text{Gell-Mann matrices})$$

- rewritten in terms of vector ($V = L + R$) and axial vector ($A = R - L$) currents

$$V^{\mu,a} = \bar{q} \gamma^\mu \frac{\lambda^a}{2} q, \quad A^{\mu,a} = \bar{q} \gamma^\mu \gamma^5 \frac{\lambda^a}{2} q$$

$$\partial_\mu V^{\mu,a} = 0, \quad \partial_\mu A^{\mu,a} = 0$$

- $U(1)_V$: baryon or quark number conservation

$$V^{\mu,0} = \bar{q} \gamma_\mu q, \quad \partial_\mu V^{\mu,0} = 0$$

- $U(1)_A$: broken by quantum effects, anomaly

$$A^{\mu,0} = \bar{q} \gamma_\mu \gamma^5 q, \quad \partial_\mu A^{\mu,0} = \frac{N_f g_s^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

$\Rightarrow$ the QCD $\theta$-term, $|\theta| \lesssim 10^{-10}$

- Is $SU(3)_L \times SU(3)_R$ realized in hadron spectrum?
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    \[ J^{\mu,a}_{L,R} = \bar{q}_{L,R} \gamma^{\mu} \frac{\lambda^a}{2} q_{L,R}, \quad J^{\mu,0}_{L,R} = \bar{q}_{L,R} \gamma^{\mu} q_{L,R}, \quad (\lambda^a : \text{Gell-Mann matrices}) \]

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    \[ J^{\mu,a}_{L,R} = \bar{q}_{L,R} \gamma^\mu \frac{\lambda_a}{2} q_{L,R}, \quad J^{\mu,0}_{L,R} = \bar{q}_{L,R} \gamma^\mu q_{L,R}, \quad (\lambda_a : \text{Gell-Mann matrices}) \]
- rewritten in terms of vector ($V = L + R$) and axial vector ($A = R - L$) currents
  \[ V^{\mu,a} = \bar{q} \gamma^\mu \frac{\lambda_a}{2} q, \quad A^{\mu,a} = \bar{q} \gamma^\mu \gamma^5 \frac{\lambda_a}{2} q \]
  \[ \partial_\mu V^{\mu,a} = 0, \quad \partial_\mu A^{\mu,a} = 0 \]
- $U(1)_V$: baryon or quark number conservation
  \[ V^{\mu,0} = \bar{q} \gamma_\mu q, \quad \partial_\mu V^{\mu,0} = 0 \]
- $U(1)_A$: broken by quantum effects, anomaly
  \[ A^{\mu,0} = \bar{q} \gamma_\mu \gamma^5 q, \quad \partial_\mu A^{\mu,0} = \frac{N_f g_s^2}{32 \pi^2} \epsilon^{\mu \nu \rho \sigma} G^{\alpha}_{\mu \nu} G^{\alpha}_{\rho \sigma} \]
  \[ \Rightarrow \text{the QCD } \theta\text{-term, } |\theta| \lesssim 10^{-10} \]
- Is $SU(3)_L \times SU(3)_R$ realized in hadron spectrum?
Noether’s theorem: continuous symmetry \( \Rightarrow \) conserved currents

Let \( Q^a \) be symmetry charges:
\[
Q^a = \int d^3 \vec{x} \, J^{a,0}(t, \vec{x}), \quad \partial_\mu J^{a,\mu} = 0
\]

\( Q^a \) is the symmetry generator:
\[
U = e^{i\alpha^a Q^a}, \quad H: \text{Hamiltonian, thus}
\]
\[
U H U^{-1} = H \quad \Rightarrow \quad [Q^a, H] = 0,
\]
\[
[Q^a, H]|0\rangle = Q^a H|0\rangle - HQ^a|0\rangle = 0
\]

Wigner–Weyl mode: \( Q^a|0\rangle = 0 \) or equivalently \( U|0\rangle = |0\rangle \)
degeneracy in mass spectrum

Nambu–Goldstone mode: \( U|0\rangle \neq |0\rangle \), spontaneously broken (hidden)
\( Q^a|0\rangle \neq |0\rangle \): new states degenerate with vacuum, massless Goldstone bosons

- spontaneously broken continuous global symmetry \( \Rightarrow \) massless GBs
- the same quantum numbers as \( Q^a|0\rangle \) \( \Rightarrow \) spinless
- \#(GBs) = \#(broken generators)
Vector and axial charges:

\[ Q^a_V = Q^a_L + Q^a_R, \quad Q^a_A = Q^a_R - Q^a_L, \quad [Q^a_{V,A}, H^0_{\text{QCD}}] = 0 \]

Here \( H^0_{\text{QCD}} \): QCD Hamiltonian in the chiral limit.

Under parity transformation:

\[ q_R \rightarrow \gamma^0 q_L, \quad q_L \rightarrow \gamma^0 q_R \]

\[ J^{\mu,a}_{L} \rightarrow J^{a}_{R,\mu}, \quad J^{\mu,a}_{R} \rightarrow J^{a}_{L,\mu} \]

\[ \Rightarrow \quad PQ^a_V P^{-1} = Q^a_V, \quad PQ^a_A P^{-1} = -Q^a_A \]

For an eigenstate of \( H^0_{\text{QCD}} \): \( H^0_{\text{QCD}} |\psi\rangle = E |\psi\rangle \) with \( P |\psi\rangle = \eta_P |\psi\rangle \), then

\[ H^0_{\text{QCD}} Q^a_A |\psi\rangle = Q^a_A H^0_{\text{QCD}} |\psi\rangle = E Q^a_A |\psi\rangle, \]

\[ P Q^a_A |\psi\rangle = P Q^a_A P^{-1} P |\psi\rangle = -\eta_P Q^a_A |\psi\rangle \]

Parity doubling: \( Q^a_A |\psi\rangle \) has the same mass but opposite parity
• But, in the hadron spectrum:

\[ m_{\text{Nucleon, } P=+} = 939 \text{ MeV} \ll m_{N^* (1535), \, P=-} = 1535 \text{ MeV}, \]

\[ m_{\pi, \, P=-} = 139 \text{ MeV} \ll m_{a_0 (980), \, P=+} = 980 \text{ MeV} \]

• No parity doubling in hadron spectrum \( \Rightarrow \)

\[ Q_a^A |0\rangle \neq 0, \quad \text{or} \quad e^{i \alpha^a Q_A^a} |0\rangle \neq |0\rangle \]

Nambu–Goldstone mode (hidden symmetry):

In QCD, \( SU(3)_L \times SU(3)_R \) is \textit{spontaneously broken down to} \( SU(3)_V \)

• GBs should have \( J^P = 0^- \)
SSB in linear $\sigma$ model – 1

- Linear $\sigma$ model with an $O(4)$ symmetry

$$\mathcal{L}_{\text{LSM}} = \frac{1}{2} \frac{\partial}{\partial \mu} \Phi^T \frac{\partial}{\partial \mu} \Phi - V(\Phi),$$

$$V(\Phi) = \frac{\lambda}{4} (\Phi^T \Phi - v^2)^2, \quad \text{with } \Phi^T = (\sigma, \pi_1, \pi_2, \pi_3)$$

- $\sigma = \pi_a = 0$ is not a minimum of $V(\Phi)$

There is a continuum of degenerate vacua: $\Phi_{\text{min}}^T \Phi_{\text{min}} = v^2$

Choose $\Phi_{\text{min}} = (v, 0, 0, 0)$, vacuum is only invariant under $O(3)$ rotations

spontaneous symmetry breaking: $O(4) \rightarrow O(3)$

- Perturb around $\Phi_{\text{min}} = (v, \vec{0})$, let $\Phi = \Phi_{\text{min}} + (\sigma', \pi_1, \pi_2, \pi_3)^T$, then

$$\mathcal{L}_{\text{LSM}} = \frac{1}{2} \partial_\mu \pi_a \partial^\mu \pi_a + \frac{1}{2} \partial_\mu \sigma' \partial^\mu \sigma' - \frac{\lambda}{4} (\sigma'^2 + \pi_a \pi_a + 2v \sigma')^2$$

- Goldstone bosons (GBs): $\pi_a$’s are massless

- $\sigma'$ is massive: $m_{\sigma'}^2 = 2\lambda v^2$

- $m_{\sigma'}$ and 5 interaction terms described by only 2 parameters
SSB in linear $\sigma$ model – 1

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\[ \sigma' \text{ is massive: } m_{\sigma'}^2 = 2\lambda v^2 \]

\[ m_{\sigma'} \text{ and 5 interaction terms described by only 2 parameters} \]
SSB in linear $\sigma$ model – 2

$$\mathcal{L}_{\text{LSM}} = \frac{1}{2} \partial_\mu \pi_a \partial^\mu \pi_a + \frac{1}{2} \partial_\mu \sigma' \partial^\mu \sigma' - \frac{\lambda}{4} \left( \sigma'^2 + \pi_a \pi_a + 2v \sigma' \right)^2$$

Only 2 parameters, important for cancellation!

Examples: tree-level scattering amplitudes

- $\pi_3(p_1)\pi_3(p_2) \rightarrow \pi_3(p_3)\pi_3(p_4)$ (individually large terms!)

\[iA = \quad + \quad + \quad + \quad +\]

\[= -i6\lambda + \frac{i(-2i\lambda v)^2}{s - (2\lambda v^2)} + \frac{i(-2i\lambda v)^2}{t - (2\lambda v^2)} + \frac{i(-2i\lambda v)^2}{u - (2\lambda v^2)}\]

\[= -i6\lambda + \frac{i}{2\lambda v^2} (2\lambda v)^2 \left( 3 + \frac{s + t + u}{2\lambda v^2} \right) + \mathcal{O} \left( \frac{p_\pi^4}{m_\sigma^4} \right) = \mathcal{O} \left( \frac{p_\pi^4}{m_\sigma^4} \right)\]

$p_\pi$: a generic momentum of GBs

Mandelstam variables: $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$, $s + t + u = \sum_i p_i^2$
SSB in linear $\sigma$ model – 2

$$\mathcal{L}_{\text{LSM}} = \frac{1}{2} \partial_\mu \pi_a \partial^\mu \pi_a + \frac{1}{2} \partial_\mu \sigma' \partial^\mu \sigma' - \frac{\lambda}{4} \left( \sigma'^2 + \pi_a \pi_a + 2v\sigma' \right)^2$$

Only 2 parameters, important for cancellation!

Examples: tree-level scattering amplitudes

- $\pi_3 \sigma' \rightarrow \pi_3 \sigma'$

**Exercise**: Show that at tree-level

$$\mathcal{A}(\pi_3 \sigma' \rightarrow \pi_3 \sigma') = \mathcal{O}\left(\frac{p_{\pi}^2}{m_\sigma^2}\right)$$

Lessons from the linear $\sigma$ model:

- SSB happens when there is a degeneracy of the vacuum
- Nonvanishing VEV of some Hermitian operator, here $\langle \sigma \rangle = v$
- $\text{SSB} \Rightarrow \text{massless GBs } (\pi_a), \quad \# = \dim(O(4)) - \dim(O(3)) = 3$
- GBs decouple at vanishing momenta!
SSB in linear $\sigma$ model – 2

$$\mathcal{L}_{\text{LSM}} = \frac{1}{2} \partial_\mu \pi_a \partial^\mu \pi_a + \frac{1}{2} \partial_\mu \sigma' \partial^\mu \sigma' - \frac{\lambda}{4} \left( \sigma'^2 + \pi_a \pi_a + 2v\sigma' \right)^2$$

Only 2 parameters, important for cancellation!

Examples: tree-level scattering amplitudes

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Lessons from the linear $\sigma$ model:

- SSB happens when there is a degeneracy of the vacuum
- Nonvanishing VEV of some Hermitian operator, here $\langle \sigma \rangle = v$
- SSB $\Rightarrow$ massless GBs ($\pi_a$), $\# = \dim(O(4)) - \dim(O(3)) = 3$
- GBs decouple at vanishing momenta!
Symmetry implies a derivative coupling for GBs, i.e.,

\[
\text{GBs do not interact at vanishing momenta}
\]

- Consider GB \( \pi^a \):
  \[
  \langle \pi^a | Q^a_A | 0 \rangle = \int d^3x \langle \pi^a | A^a_0(x) | 0 \rangle \neq 0
  \]
  Lorentz invariance \( \Rightarrow \)
  \[
  \langle \pi^a(q) | A^a_\mu(0) | 0 \rangle = -iq_\mu F_\pi
  \]

- Consider the matrix element

\[
\langle \psi_1 | A^a_\mu(0) | \psi_2 \rangle = R^a_\mu + F_\pi q^\mu \frac{1}{q^2} T^a
\]

Current conservation \( \Rightarrow \)

\[
q^\mu A^a_\mu = 0, \text{ thus}
\]

\[
q^\mu R^a_\mu + F_\pi T^a = 0 \Rightarrow \lim_{q^\mu \to 0} T^a = 0
\]

\( \Rightarrow \) GBs couple in a derivative form!!
Symmetry implies a derivative coupling for GBs, i.e.,

\[ \langle \pi^a | Q_a^0 | 0 \rangle = \int d^3 x \langle \pi^a | A_0^a(x) | 0 \rangle \neq 0 \]

Lorentz invariance \( \Rightarrow \) \( \langle \pi^a(q) | A_\mu^a(0) | 0 \rangle = -iq_\mu F_\pi \)

Consider the matrix element

\[ \langle \psi_1 | A_\mu^a(0) | \psi_2 \rangle = R_\mu^a + F_\pi q_\mu \frac{1}{q^2} T^a \]

Current conservation \( \Rightarrow q_\mu A_\mu^a = 0 \), thus

\[ q_\mu R_\mu^a + F_\pi T^a = 0 \Rightarrow \lim_{q_\mu \to 0} T^a = 0 \]

\( \Rightarrow \) GBs couple in a derivative form!!
SSB in QCD

- Hamiltonian invariant under a group $G = SU(N_f)_L \times SU(N_f)_R$, vacuum invariant under its vector subgroup $H = SU(N_f)_V$.

$$Q_V^a |0\rangle = 0, \quad Q_A^a |0\rangle \neq 0$$

- SSB $\Rightarrow$ massless pseudoscalar Goldstone bosons

$$\#(\text{GBs}) = \dim(G) - \dim(H) = N_f^2 - 1$$

for $N_f = 3$, 8 GBs: $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

for $N_f = 2$, 3 GBs: $\pi^\pm, \pi^0$

Pions get a small mass due to explicit symmetry breaking by tiny $m_{u,d}$ (a few MeV)

$$M_\pi \ll M_{\text{other hadron}}$$

also, $m_s \gg m_{u,d} \Rightarrow M_K \gg M_\pi$

- Mechanism for $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V$ in QCD not well understood
CHPT: a low energy EFT for QCD:

- an example for a non-decoupling EFT:
  
  degrees of freedom are different from those of the underlying theory

- a theory for the Goldstone bosons of \( \text{SU}(N_f)_L \times \text{SU}(N_f)_R \rightarrow \text{SU}(N_f)_V \)

- most general Lagrangian with the same global symmetries as QCD

How do the Goldstone bosons transform under \( \text{SU}(N_f)_L \times \text{SU}(N_f)_R \)?
Nonlinear realization of chiral symmetry (I)

Weinberg (1968); Coleman, Wess, Zumino (1969); Callan, Coleman, Wess, Zumino (1969)

To study the transformation properties of the Goldstone bosons (GBs)

- Assume \( G \xrightarrow{\text{SSB}} H \), we have GBs
  \[ \Phi = (\phi_i, \ldots, \phi_n), \quad n = \dim(G) - \dim(H) \]

- Left coset of \( H \) with respect to \( g \in G \):
  \[ gH = \{gh \mid h \in H\} \]
  coset space \( G/H \): set of all left cosets \( \{gH \mid g \in G\} \)

  \[ \dim(G/H) = \dim(G) - \dim(H) \]

- GBs can be parametrized as coordinates of \( G/H \):
  choosing a representative element in each coset, e.g., for \( h_{1,2} \in H \),
  \[ gh_1 H = gh_2 H \]

  we can choose either \( gh_1 \) or \( gh_2 \) as the parameterization
Nonlinear realization of chiral symmetry (II)

- **Representation invariance:**
  
  free to choose the set of representative elements / set of coordinates on $G/H$

- Transformation properties of the GBs uniquely determined:
  Parameterizing GBs by $u \in G/H$

  transformation under $g \in G$

  $$g \cdot u = u' \cdot h(g, u)$$

  since any element of the coset $\{ u' \cdot h(g, u) | h(g, u) \in H \}$ can be used

  $\Rightarrow$ **Nonlinear** transformation of GBs

  $$u \xrightarrow{g \in G} u' = g \cdot u \cdot h^{-1}(g, u)$$
• **Representation invariance:**
  free to choose the set of representative elements / set of coordinates on \( G/H \)

• **Transformation properties of the GBs** uniquely determined:
  Parameterizing GBs by \( u \in G/H \)
  transformation under \( g \in G \)

\[
g u = u' h(g, u)
\]

since any element of the coset \( \{u' h(g, u) | h(g, u) \in H\} \) can be used

\[\Rightarrow\] **Nonlinear** transformation of GBs

\[
\begin{array}{c}
\text{\( u \in \rightarrow \quad u' = g u h^{-1}(g, u) \)}
\end{array}
\]
Application to QCD (I)

For QCD, \( G = SU(N_f)_L \times SU(N_f)_R \overset{SSB}{\rightarrow} H = SU(N_f)_V \)

- \( g = (g_L, g_R) \in SU(N_f)_L \times SU(N_f)_R \) for \( g_L \in SU(N_f)_L, g_R \in SU(N_f)_R \)

\[
g_1 g_2 = (g_{L_1}, g_{R_1})(g_{L_2}, g_{R_2}) = (g_{L_1} g_{L_2}, g_{R_1} g_{R_2})
\]

- Choice of a representative element inside each left coset is free

\[
(g_L, g_R) H = (g_L g_R^\dagger, 1) \quad (g_R, g_R) \quad H = (g_L g_R^\dagger, 1) H
\]

\[\in H = SU(N_f)_V\]

Goldstone bosons can be parameterized by a unitary matrix

\[
U = g_L g_R^\dagger = \exp \left( i \frac{\phi}{F'} \right)
\]

here, \( \phi = \sum_{a=1}^{8} \lambda_a \phi_a \) for \( SU(3) \), and \( \vec{\tau} \cdot \vec{\pi} \) for \( SU(2) \)

\( \lambda_a \): Gell-Mann matrices, \( \tau_i (i = 1, 2, 3) \): Pauli matrices

\( \phi_a (= \pi_i) \): Goldstone boson fields

\( F' \): dimensionful constant (to be determined later)
For QCD, $G = SU(N_f)_L \times SU(N_f)_R \xrightarrow{SSB} H = SU(N_f)_V$

- $g = (g_L, g_R) \in SU(N_f)_L \times SU(N_f)_R$ for $g_L \in SU(N_f)_L$, $g_R \in SU(N_f)_R$
  
$$g_1 g_2 = (g_{L_1}, g_{R_1})(g_{L_2}, g_{R_2}) = (g_{L_1} g_{L_2}, g_{R_1} g_{R_2})$$

- Choice of a representative element inside each left coset is free
  
$$(g_L, g_R) H = (g_L g_R^\dagger, 1) \quad (g_R, g_R) \quad H = (g_L g_R^\dagger, 1) \quad H$$

Goldstone bosons can be parameterized by a unitary matrix

$$U = g_L g_R^\dagger = \exp \left( i \frac{\phi}{F'} \right)$$

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- $\lambda_a$: Gell-Mann matrices, $\tau_i (i = 1, 2, 3)$: Pauli matrices
- $\phi_a (\pi_i)$: Goldstone boson fields
- $F'$: dimensionful constant (to be determined later)
Application to QCD (II)

- Acting $g = (L, R) \in G$ on the coset $(U, \mathbb{1})H$

$$g(U, \mathbb{1})H = (LU, R)H = (LU R^\dagger, R)H = (LU R^\dagger, \mathbb{1}) (R, R) H \in SU(N_f)_V$$

⇒ transformation property of $U$:

$$U \xrightarrow{g} LU R^\dagger$$

One can also parametrize the GBs such that $U \xrightarrow{g} RU L^\dagger$. Any one is okay if used consistently.

- For $g \in H = SU(N_f)_V$, we have $R = L$

$$U \rightarrow LU L^\dagger \quad \Rightarrow \quad \phi \rightarrow L \phi L^\dagger$$

i.e., GB fields transform linearly under $SU(N_f)_V$
• Heavy quark spin and flavor symmetry for hadrons containing a single heavy quark

• QCD in the chiral limit $m_u = m_d = m_s = 0$ has a global symmetry

\[ \text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_V \]

$\text{U}(1)_A$ is anomalously broken by quantum effects

• Strong evidence for chiral symmetry spontaneous breaking

\[ \text{SU}(N_f)_L \times \text{SU}(N_f)_R \to \text{SU}(N_f)_V \]

3 (8) pseudoscalar Goldstone bosons: $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

• For $G \to H$, Goldstone bosons can be identified with $G/H$

\[
U = \exp \frac{i\phi}{F'} \quad \phi = \sqrt{2} \begin{pmatrix}
\frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} \\
\pi^- - \frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} \\
K^- - \frac{2\phi_8}{\sqrt{6}}
\end{pmatrix}
\]

for $\text{SU}(3)$

nonlinear transformation: $U \xrightarrow{g \in G} LUR^\dagger$
Aim: reproduce low-energy structure of QCD

- Effective Lagrangian invariant under $G = SU(N_f)_L \times SU(N_f)_R$ (and $C, P$)

$$U \xrightarrow{G} LU R^\dagger, \quad U \xrightarrow{C} U^T, \quad U \xrightarrow{P} U^\dagger$$

- What does "low-energy" mean here?

Goldstone boson fields (contained in $U$) as the only degrees of freedom

$\Rightarrow$ energy range restricted to well below 1 GeV

(separation of energy scales, more see later)

- Low energies: expand in powers of momenta ( = number of derivatives)

- Lorentz invariance $\Rightarrow$ only even number of derivatives are allowed

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$$

- $\mathcal{L}^{(0)}$? $U$ is unitary, $UU^\dagger = 1$, hence

$$\langle (UU^\dagger)^n \rangle = \text{const.}, \quad \langle \ldots \rangle \equiv \text{Tr}_{\text{flavor}}[\ldots]$$

$\Rightarrow$ Leading non-trivial term is $\mathcal{L}^{(2)}$
Construction of the effective Lagrangian for GBs (I)

Aim: reproduce low-energy structure of QCD

- Effective Lagrangian invariant under $G = \text{SU}(N_f)_L \times \text{SU}(N_f)_R$ (and $C$, $P$)

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  $U \xrightarrow{G} L U R^\dagger$, $U \xrightarrow{C} U^T$, $U \xrightarrow{P} U^\dagger$

- What does "low-energy" mean here?
  Goldstone boson fields (contained in $U$) as the only degrees of freedom
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  $\langle (UU^\dagger)^n \rangle = \text{const.}$ , $\langle \ldots \rangle \equiv \text{Tr}_{\text{flavor}}[\ldots]$
  $\Rightarrow$ Leading non-trivial term is $\mathcal{L}^{(2)}$
Construction of the effective Lagrangian for GBs (II)

- Leading term of the effective Lagrangian is $\mathcal{L}^{(2)}$ just one single term (nonlinear $\sigma$ model):

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle$$

with $U = \exp \frac{i\phi}{F'}$

$$\phi_{\text{SU}(2)} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \quad \phi_{\text{SU}(3)} = \sqrt{2} \begin{pmatrix} \frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\phi_8}{\sqrt{6}} \end{pmatrix}$$

- $\mathcal{L}^{(2)}$ is invariant under $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$:

$$\langle \partial_\mu U \partial^\mu U^\dagger \rangle \rightarrow \langle \partial_\mu U' \partial^\mu U'^\dagger \rangle$$

$$= \langle L \partial_\mu U R^\dagger R \partial^\mu U^\dagger L^\dagger \rangle = \langle \partial_\mu U \partial^\mu U^\dagger \rangle$$

**Exercise:** Show that $\langle U \partial^\mu U^\dagger \rangle = 0$, therefore $\langle U \partial_\mu U^\dagger \rangle \langle U \partial^\mu U^\dagger \rangle$ is not present.
The low-energy constant $F$

- Expand $U$ in powers of $\phi$, \[ U = 1 + \frac{i\phi}{F'} - \frac{\phi^2}{2F'^2} + \ldots \]
  \(\Rightarrow\) canonical kinetic terms \[ \mathcal{L}^{(2)} = \partial_\mu \pi^+ \partial^\mu \pi^- + \partial_\mu K^+ \partial^\mu K^- + \ldots \]

- Calculate the Noether currents $L^\mu_a$, $R^\mu_a$ from $\mathcal{L}^{(2)}$ \(\Rightarrow\)

\[ V^\mu_a = R^\mu_a + L^\mu_a = i \frac{F^2}{4} \langle \lambda_a \left[ \partial^\mu U, U^\dagger \right] \rangle \]

\[ A^\mu_a = R^\mu_a - L^\mu_a = i \frac{F^2}{4} \langle \lambda_a \left\{ \partial^\mu U, U^\dagger \right\} \rangle \]

- Expand the currents in powers of $\phi$, \[ A^\mu_a = -F \partial^\mu \phi_a + \mathcal{O} \left( \phi^3 \right) \]

\[ \langle 0 | A^\mu_a (x) | \phi_b (p) \rangle = ip^\mu F e^{-ip \cdot x} \delta_{ab} \]

\(\Rightarrow\) $F$ is the pion decay constant in the chiral limit

\[ F \approx F_\pi \]

$F_\pi = 92.2$ MeV measured in the leptonic decay of the pion $\pi^+ \rightarrow \ell^+ \nu_\ell$
So far, only considered chiral limit $m_u = m_d = m_s = 0$.

For non-zero quark masses,

- the singlet vector current still conserved (baryon number conservation)
  \[ \partial \mu V_\mu = 0 \]

- vector currents $V^a_\mu$ conserved when $m_u = m_d = m_s = m$ (i.e., $\mathcal{M} = m \mathbb{1}$)

\[
\partial \mu V^a_\mu = i \bar{q} \left[ \mathcal{M}, \frac{\lambda^a}{2} \right] q
\]

- $A^a_\mu$ not conserved any more: Partially Conserved Axial Current (PCAC)
  \[ \partial \mu A^a_\mu = i \bar{q} \left\{ \mathcal{M}, \frac{\lambda^a}{2} \right\} q \]
Explicit symmetry breaking: quark masses

- In the **chiral limit** \( m_u = m_d = m_s = 0 \)
  \[ \mathcal{L}^{(2)} = \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle \] contains no terms \( \propto M^2 \phi^2 \)
  
  - theory for **massless** Goldstone bosons, but pions have nonvanishing masses

- In nature, \( u, d, s \) quark masses are small \( (m_u,d,s \ll \Lambda_{QCD}) \), but non-zero
  - chiral symmetry explicitly broken
  \[ \mathcal{L}_m = -\bar{q}_R m q_L - \bar{q}_L m q_R \]
  
  - if symmetry breaking is weak \( \Rightarrow \)
a perturbative expansion in the quark masses

- Effective Lagrangian is still an appropriate tool to systematically derive all symmetry relations

**CHiral Perturbation Theory (CHPT):**

double expansion in low momenta and quark masses
Explicit symmetry breaking: quark masses

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CHiral Perturbation Theory (CHPT):

  double expansion in low momenta and quark masses
### Explicit symmetry breaking: the spurion trick (I)

<table>
<thead>
<tr>
<th>Spurion in 3 steps: very useful trick for explicit symmetry breaking</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1.</strong> Introduce a <strong>spurion field</strong> (e.g. quark mass, electric charge, $\gamma_\mu$, ...) with a transformation property so that the symmetry breaking term in the full theory is invariant</td>
</tr>
<tr>
<td><strong>2.</strong> Write down invariant operators in EFT including the spurion field</td>
</tr>
<tr>
<td><strong>3.</strong> Set the spurion field to the value which it should take</td>
</tr>
</tbody>
</table>
Apply the spurion trick to quark masses:

\[ \mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^0 - \bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M}^\dagger q_L \]

Treat \( \mathcal{M} \) as a complex spurion field

\[ \mathcal{M} \rightarrow \mathcal{M}' = L \mathcal{M} R^\dagger \]

Then construct Lagrangian invariant under \( SU(N_f)_L \times SU(N_f)_R \)

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots, \mathcal{M}) \]

This procedure guarantees that chiral symmetry is broken in exactly the same way in the effective theory as it is in QCD

The spurion trick is very useful to construct EFT operators with a given symmetry transformation property.
LO chiral Lagrangian with the mass term (I)

\[
\mathcal{L}^{(2)} = \frac{F^2}{4} \left[ \langle \partial_\mu U \partial^\mu U^\dagger \rangle + 2B \langle MU^\dagger + M^\dagger U \rangle \right]
\]

- Read off mass terms for GBs [SU(3)]:
  \[
  \begin{align*}
  M^2_{\pi^\pm} &= B(m_u + m_d) \\
  M^2_{K^\pm} &= B(m_u + m_s) \\
  M^2_{K^0} &= B(m_d + m_s)
  \end{align*}
  \]

- Gell-Mann–Oakes–Renner (GMOR) relation:
  \[
  M^2_{\text{GB}} \propto m_q
  \]
  \(^\clubsuit\) CHPT can be used to extrapolate lattice results from large to the physical values of \(m_u,d\) (or equivalently pion masses)

- Unified power counting for derivative and quark mass expansions:
  \[
  m_q = \mathcal{O} \left( p^2 \right)
  \]
$\mathcal{L}^{(2)} = \frac{F^2}{4} \left[ \langle \partial_\mu U \partial^\mu U^\dagger \rangle + 2B \langle MU^\dagger + M^\dagger U \rangle \right]$

- Flavor-neutral states $\phi_3, \phi_8$ are mixed:

$$\frac{B}{2} \begin{pmatrix} \phi_3 \\ \phi_8 \end{pmatrix}^T \begin{pmatrix} m_u + m_d & \frac{1}{\sqrt{3}}(m_u - m_d) \\ \frac{1}{\sqrt{3}}(m_u - m_d) & \frac{1}{3}(m_u + m_d + 4m_s) \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_8 \end{pmatrix}$$

- Diagonalize with

$$\begin{pmatrix} \pi^0 \\ \eta \end{pmatrix} = \begin{pmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_8 \end{pmatrix}$$

- $\pi^0 \eta$ mixing angle:

$$\epsilon = \frac{1}{2} \arctan \left( \frac{\sqrt{3}}{2} \frac{m_d - m_u}{m_s - \hat{m}} \right), \quad \hat{m} = \frac{1}{2}(m_u + m_d)$$
• Mass eigenvalues:

\[
\begin{align*}
M_{\pi_0}^2 &= B(m_u + m_d) - \mathcal{O}((m_u - m_d)^2) \\
M_{\eta}^2 &= \frac{B}{3}(m_u + m_d + 4m_s) + \mathcal{O}((m_u - m_d)^2)
\end{align*}
\]

• In the isospin limit \(m_u = m_d\):

\[
M_{\pi^\pm}^2 = M_{\pi_0}^2, \quad M_{K^\pm}^2 = M_{K^0}^2 \quad \text{(of course!)}
\]

• Gell-Mann–Okubo (GMO) mass formula for pseudoscalars:

\[
4M_K^2 = 3M_\eta^2 + M_\pi^2
\]

⇒ fulfilled in nature at 7% accuracy
Representation invariance

- Freedom to choose coordinates on coset space $G/H$
- **Haag’s theorem** on field redefinition: Haag (1958); Coleman, Wess, Zumino (1969)

If fields $\phi$ and $\chi$ are related nonlinearly by a local function as

$$\phi = \chi F[\chi] \quad \text{with} \quad F[0] = 1,$$

then the same physical observables (on-shell $S$-matrices) can be obtained using either field $\phi$ with Lagrangian $\mathcal{L}[\phi]$ or $\chi$ with $\mathcal{L}[\chi F[\chi]]$. 
Representation invariance: $\pi\pi$ scattering (I)

In the chiral limit,

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \langle \partial_\mu U \partial^{\mu} U^\dagger \rangle$$

Amplitude for $\pi^+\pi^- \rightarrow \pi^0\pi^0$:

- **Exponential representation:**

  $$U = \exp \frac{i\phi}{F}$$

  with $\phi = \vec{r} \cdot \vec{\pi}$, $\pi^0 = \pi_3$, $\pi^\pm = \frac{1}{\sqrt{2}} (\pi_1 \mp i\pi_2)$ for SU(2).

  $$\mathcal{A} = \frac{s}{F^2}$$

* parameter-free prediction
* in accordance with Goldstone theorem: vanishes at zero-momentum

- **Square-root representation:**

  $$U = \frac{1}{F} \left( \sqrt{F^2 - \vec{\pi}'^2} + i\vec{r} \cdot \vec{\pi}' \right)$$

  calculating with $\vec{\pi}'$ as the pion fields gives the same scattering amplitude.
Representation invariance: $\pi \pi$ scattering (I)

In the chiral limit,

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle$$

Amplitude for $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$:

- **Exponential representation:**

$$U = \exp \frac{i\phi}{F}$$

with $\phi = \vec{\tau} \cdot \vec{\pi}$, $\pi^0 = \pi_3$, $\pi^\pm = \frac{1}{\sqrt{2}} \left( \pi_1 \mp i\pi_2 \right)$ for SU(2).

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calculating with $\vec{\pi'}$ as the pion fields gives the same scattering amplitude.
Exercise: Calculate the amplitude and show that the two representations are related by the following field redefinition:

\[ \vec{\pi}' = \vec{\pi} F \frac{\sin}{|\vec{\pi}|} \left( \frac{|\vec{\pi}|}{F} \right) = \vec{\pi} + \ldots, \quad \text{here} \quad |\vec{\pi}| \equiv \sqrt{\vec{\pi}^2} \]

Exercise: Show that the amplitude with the quark mass term included reads

\[
A(s, t, u) = \frac{s - M^2}{F^2_{\pi}}
\]

Weinberg (1966)
Including matter fields

- So far, EFT for (pseudo) Goldstone bosons
- **Matter fields** (fields which are not Goldstone bosons) can be included as well, e.g.
  - **baryon CHPT:**
    - nucleons [SU(2)] / baryon ground state octet [SU(3)]
  - **heavy-hadron CHPT:**
    - heavy-flavor (charm, bottom) mesons and baryons
- At **low-energies**, **3-momenta** remain small $\sim M_\pi$, derivative expansion is feasible
Transformation of matter fields

- Proceed as before
  - need to know how matter fields transform under $SU(N_f)_L \times SU(N_f)_R$
  - construct effective Lagrangians according to increasing number of momenta

- Transformation properties of matter fields:
  - well-defined transformation rule under the unbroken $SU(N_f)_V$
  - not necessarily form representations of $SU(N_f)_L \times SU(N_f)_R$:
    - transformation of matter fields under $SU(N_f)_L \times SU(N_f)_R$
    - not uniquely defined,
      related by field redefinition (again: representation invariance)

- Example: heavy-meson CHPT
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Bispinor fields for heavy mesons

- Recall HQSS for heavy quarks (charm, bottom); $\frac{1}{2} \left( 1 + \gamma^\prime \right)$ projects onto the particle component of the heavy quark spinor.

- Convenient to introduce heavy mesons as bispinors:

$$H_a = \frac{1 + \gamma^\prime}{2} \left[ P_{a \mu}^* \gamma_\mu - P_a \gamma_5 \right], \quad \bar{H}_a = \gamma_0 H_a^\dagger \gamma_0$$

$$P = \{ Q \bar{u}, Q \bar{d}, Q \bar{s} \}: \text{pseudoscalar heavy mesons, } P^*: \text{vector heavy mesons}$$

- Charge conjugation:

  - $H_a$ destroys mesons containing a $Q$, but does not create mesons with a $\bar{Q}$
  - Free to choose the phase convention for charge conjugation. If we use, e.g.,

$$P_a \xrightarrow{C} + P_a(\bar{Q}), \quad P_{a, \mu}^* \xrightarrow{C} - P_{a, \mu}^*(\bar{Q}),$$

then the fields annihilating mesons containing a $\bar{Q}$ is ($C = i \gamma^2 \gamma^0$)

$$H_a(\bar{Q}) = C \left[ \frac{1 + \gamma^\prime}{2} \left( - P_{a, \mu}^*(\bar{Q}) \gamma^\mu - P_a(\bar{Q}) \gamma_5 \right) \right]^T C^{-1}$$

$$= \left( + P_{a, \mu}^*(\bar{Q}) \gamma^\mu - P_a(\bar{Q}) \gamma_5 \right) \frac{1 - \gamma^\prime}{2}$$
Consider the SU(3) case, the charmed meson ground state anti-triplet:

\[ H_a = (D^0, D^+, D_s^+) \]

\[ \left[ (c\bar{u}, c\bar{d}, c\bar{s}) \right] \]

transform under the global unbroken SU(3) as

\[ H \xrightarrow{V \in SU(3)_V} HV^\dagger \]

- Representation invariance:
  free to choose how \( H \) transforms under \( SU(3)_L \times SU(3)_R \) as long as it reduces to the above under \( SU(3)_V \)

- Example:
  describe the heavy mesons by \( H_1 \) or \( H_2 \), under
  \[ g = (L, R) \in SU(3)_L \times SU(3)_R \]

\[ H_1 \xrightarrow{g} H_1 L^\dagger, \quad H_2 \xrightarrow{g} H_2 R^\dagger \]

both transform as an anti-triplet under \( (V, V) \in SU(3)_V \)
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- Related to each other through field redefinition:
  
  \[
  H_2 = H_1 U = H_1 + \frac{i}{F} H_1 \phi + \ldots, \quad U = \exp \left( \frac{i}{F} \phi \right) \xrightarrow{g} L U R^\dagger
  \]

- But $H_{1,2}$ are inconvenient: complicated parity transformation ($P$)
  $L(L^\dagger)$ needs to be replaced by $R(R^\dagger)$ under parity  $\Rightarrow$
  
  \[
  H_{1,a}(t, \vec{x}) \xrightarrow{P} \gamma^0 H_{1,b}(t, -\vec{x}) \gamma^0 U_{ba} \xrightarrow{g} \gamma^0 H_{1,b}(t, -\vec{x}) \gamma^0 U_{ba} R^\dagger
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  [recall for a spinor: $\psi(t, \vec{x}) \xrightarrow{P} \gamma^0 \psi(t, -\vec{x})$]
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Transformations of heavy meson fields (III)

- An elegant/convenient way:
  introduce

  \[ u = \exp \left( \frac{i\phi}{2F} \right) \quad \text{or} \quad u^2 = U \]

  it transforms under \( g \in SU(3)_L \times SU(3)_R \) as  
  (recall \( U \overset{g}{\to} L U R^\dagger \))

  \[ u \overset{g}{\to} L u h^\dagger(L, R, \phi) = h(L, R, \phi) u R^\dagger \]

  \( h(L, R, \phi) \): space-time dependent nonlinear function, called compensator field

  For for \( SU(3)_V \) transformations \( (L = R = V) \), reduces to \( h(L, R, \phi) = V \)

- We can construct \( H = H_1u \) or \( H = H_2u^\dagger \), it transforms as

  \[ H \overset{g}{\to} H h^\dagger \]

  **Exercise**: Show that under parity transformation \( h(t, \vec{x}) \overset{P}{\to} h(t, -\vec{x}) \), and \( H(t, \vec{x}) \overset{P}{\to} \gamma^0 H(t, -\vec{x}) \gamma^0 \).

- Transformation under HQSS: \( H_a \overset{S}{\to} S H_a \), \( S \) does not commute with \( \gamma \)-matrices!
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  \( S \) does not commute with \( \gamma \)-matrices!
Building blocks of the chiral Lagrangian

- Useful to introduce combinations of $u$ whose transformations only involve $h$:

\[ \Gamma^\mu = \frac{1}{2} \left[ u^\dagger (\partial^\mu - i l^\mu)u + u (\partial^\mu - i r^\mu)u^\dagger \right] \]

\[ u^\mu = i \left[ u^\dagger (\partial^\mu - i l^\mu)u - u (\partial^\mu - i r^\mu)u^\dagger \right] \]

$\Gamma^\mu$: chiral connection, vector; $u^\mu$: chiral vielbein, axial vector

- Introduce a covariant derivative:

\[ \mathcal{D}^\mu H = \partial^\mu H - H \Gamma^\mu \]

which transform the same way as $H$ under $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$

\[ \mathcal{D}^\mu H \mapsto \mathcal{D}^\mu H h^\dagger \]

- Include the quark mass term $\chi = 2B(s + ip) = 2BM + \ldots$ by introducing

\[ \chi^+ = u^\dagger \chi u^\dagger + u \chi^\dagger u, \quad \chi^+ \rightarrow h \chi^+ h^\dagger \]

All fields transform in terms of $h$, convenient to construct the effective Lagrangians
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$D^\mu H \to D^\mu H h^\dagger$

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$\chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u$, $\chi_+ \to h \chi_+ h^\dagger$

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$$\chi_+ \rightarrow h \chi_+ h^\dagger$$

All fields transform in terms of $h$, convenient to construct the effective Lagrangians
Heavy meson CHPT at LO

• LO Lagrangian \([\mathcal{O}(p)]\):

\[
\mathcal{L}_{\text{HM}}^{(1)} = -i \, \text{Tr} \left[ \bar{H}_a \nu_\mu (D^\mu H)_a \right] + \frac{g}{2} \, \text{Tr} \left[ \bar{H}_a H_b \gamma_\mu \gamma_5 u^\mu_{ba} \right]
\]

kinetic term + \(D\pi\) scattering + \ldots

terms for \(D^* \rightarrow D\pi + \ldots\)

Invariant under Lorentz transformation, chiral symmetry, parity

\(\text{Tr}\): trace in the spinor space, \(a, b\): indices in the light flavor space

• Notice that the mass dimension of \(H_a\) is 3/2.

Nonrelativistic normalization: \(H_a \simeq \sqrt{M_H} H_a^\text{rel.}\)

\(D\)-meson propagator:

\[
\frac{i}{2 \nu \cdot k + i\epsilon} \simeq \frac{i}{p^2 - M_H^2 + i\epsilon} = \frac{i}{2M_H \nu \cdot k + i\epsilon [1 + \mathcal{O}(k^2/M_H^2)]}
\]

\(p = M_H \nu + k\)
The superfield for pseudoscalar and vector heavy mesons:

\[ H_a = \frac{1 + \phi}{2} \left[ P_a^\mu \gamma_\mu - P_a \gamma_5 \right] \]

In the rest frame of heavy meson, \( v^\mu = (1, \vec{0}) \). We take the Dirac basis

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Simplifications:

\[
\frac{1 + \phi}{2} = \frac{1 + \gamma^0}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
H_a = \begin{pmatrix} 0 & -\left( P_a + \vec{P}_a^* \cdot \vec{\sigma} \right) \\ 0 & 0 \end{pmatrix}, \quad \bar{H}_a = \begin{pmatrix} 0 & 0 \\ \left( P_a^\dagger + \vec{P}_a^{\dagger*} \cdot \vec{\sigma} \right) & 0 \end{pmatrix}
\]

Thus, it is convenient to simply use the two-component notation

\[ H_a = P_a + \vec{P}_a^* \cdot \vec{\sigma} \]
In the simplified two-component notation:

\[ -i \text{Tr} \left[ \tilde{H}_a v_\mu (D^\mu H)_a \right] \Rightarrow i \text{Tr} \left[ H_a^\dagger \left( \partial^0 H_a - H_b \Gamma_{ba}^0 \right) \right] \]

\[ = 2i \left( P_a^\dagger \partial^0 P_a + P_{a*}^i \dagger \partial^0 P_{a*}^i \right) + \left( -\frac{i}{4F^2} \left( P_a^\dagger P_b + \vec{P}_a^* \cdot \vec{P}_b^* \right) \right) [\phi, \partial^0 \phi]_{ba} + \ldots \]

scattering between \((D, D^*, \vec{B}, \vec{B}^*)\) and GBs\((\pi, K, \eta)\)

The LO, \(O(p)\), scattering amplitudes are completely determined by chiral symmetry!

The axial coupling term \((u^k = -\frac{1}{F} \partial^k \phi + \ldots)\):

\[ \frac{g}{2} \text{Tr} \left[ \tilde{H}_a H_b \gamma_\mu \gamma_5 \right] u^\mu_{ba} \Rightarrow -\frac{g}{2} \text{Tr} \left[ H_a^\dagger H_b \sigma^i \right] u^i_{ba} \]

\[ = -\frac{g}{2} \text{Tr} \left[ \left( P_{a*}^i \dagger \sigma^i + P_a^\dagger \right) \left( P_{b*}^j \sigma^j + P_b^\dagger \right) \right] u^k_{ba} \]

\[ = \frac{1}{F} P_a^\dagger P_{b*}^i \partial^i \phi_{ba} + \frac{1}{F} P_{a*}^i \dagger P_b \partial^i \phi_{ba} + \frac{i}{F} \epsilon^{ijk} P_{a*}^i \dagger P_{b*}^j \partial^k \phi_{ba} + \ldots \]

term for \(D^* \rightarrow D\pi\)
Determination of $g$

- Measured $D^*$ widths:
  \[
  \Gamma(D^{*0}) < 2.1 \text{ MeV}, \quad \Gamma(D^{*\pm}) = (83.4 \pm 1.8) \text{ keV}
  \]
  \[
  \mathcal{B}(D^{*+} \rightarrow D^0 \pi^+) = (67.7 \pm 0.5)\%,
  \]
  \[
  \mathcal{B}(D^{*+} \rightarrow D^+ \pi^0) = (30.7 \pm 0.5)\%.
  \]

- Amplitude from the LO Lagrangian:
  \[
  \mathcal{A}(D^{*+} \rightarrow D^0 \pi^+) = \frac{i \sqrt{2} g}{F} \varepsilon_{D^*} \cdot \vec{q}_{\pi} \sqrt{M_{D^*} M_D}
  \]
  and the two-body decay width
  \[
  \Gamma(D^{*+} \rightarrow D^0 \pi^+) = \frac{1}{8\pi} \left| \vec{q}_{\pi} \right| \frac{1}{3} \left| \mathcal{A} \right|^2 \Rightarrow g \simeq 0.57
  \]

- HQFS: $g$ should be approximately the same in bottom sector with a relative uncertainty of $\sim \mathcal{O}\left( \frac{\Lambda_{QCD}}{m_c} - \frac{\Lambda_{QCD}}{m_b} \right) \sim \mathcal{O}(0.2)$

Lattice QCD results:

\[
\begin{align*}
g_b &= 0.492 \pm 0.029 & \text{ALPHA Collaboration, Phys. Lett. B 740 (2015) 278} \\
g_b &= 0.56 \pm 0.03 \pm 0.07 & \text{RBC and UKQCD Collaborations, Phys. Rev. D 93 (2016) 014510}
\end{align*}
\]
Mass splittings among heavy mesons (I)

- Spin-dependent term $\Rightarrow$ mass difference between vector and pseudoscalar mesons $(\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu])$

\[
\mathcal{L}_\Delta = \frac{\lambda_2}{m_Q} \text{Tr} \left[ \bar{H}_a \sigma_{\mu\nu} H_a \sigma^{\mu\nu} \right] \xrightarrow{2\text{-comp.}} -\frac{2\lambda_2}{m_Q} \text{Tr} \left[ H_a^\dagger \sigma^i H_a \sigma^i \right] \\
= \frac{4\lambda_2}{m_Q} \left( \vec{P}_{a}^\dagger \cdot \vec{P}_a^* - 3 P_a^\dagger P_a \right)
\]

$\Rightarrow \quad M_{P_a^*} - M_{P_a} = -\frac{8\lambda_2}{m_Q}$

- Thus, we expect

\[
\frac{M_{B^*} - M_B}{M_{D^*} - M_D} \simeq \frac{m_c}{m_b} \simeq 0.3
\]

measured values:

\[
M_{D^*} - M_D \simeq 140 \text{ MeV}, \quad M_{B^*} - M_B \simeq 46 \text{ MeV}
\]
Mass splittings among heavy mesons (II)

- Light quark mass-dependent terms in two-component notation:
  \[ \mathcal{L}_\chi = -\lambda_1 \text{Tr} \left[ H_a^\dagger H_b \right] \chi_{+,ba} - \lambda'_1 \text{Tr} \left[ H_a^\dagger H_a \right] \chi_{+,bb} \]

  here,
  \[ \chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u = 4BM - \frac{B}{2F^2} \{ \phi, \{ \phi, M \} \} + \ldots \]

- SU(3) mass differences:
  \[ M_{D_s^+} - M_{D^+} = 4\lambda_1 B(m_s - m_d) = 4\lambda_1 (M_{K^\pm}^2 - M_{\pi^\pm}^2) \]

  \[ \Rightarrow \lambda_1 \simeq 0.11 \text{ GeV}^{-1} \]

- Isospin splitting induced by \( m_d - m_u \):
  \[ (M_{D^0} - M_{D^+})_{\text{quark mass}} = 4\lambda_1 B(m_u - m_d) = 4\lambda_1 \left( M_{K^\pm}^2 - M_{K^0}^2 - M_{\pi^\pm}^2 + M_{\pi^0}^2 \right) = -2.3 \text{ MeV} \]

  Exp. value: \( M_{D^0} - M_{D^+} = -(4.77 \pm 0.08) \text{ MeV} \)

- \( \mathcal{L}_\chi \) also contributes to scattering between a heavy meson and the lightest pseudoscalar mesons (GBs)
Lagrangians need to respect the global symmetries of QCD, explicit symmetry breaking can be included using the **spurion** technique

The LO mesonic chiral Lagrangian invariant under $SU(N_f)_L \times SU(N_f)_R$

\[
\mathcal{L}^{(2)} = \frac{F^2}{4} \left[ \langle \partial_\mu U \partial^\mu U^\dagger \rangle + 2B \langle \mathcal{M}U^\dagger + \mathcal{M}^\dagger U \rangle \right] \quad \text{with} \quad U = e^{i\phi/F}
\]

GMOR relation: $M_\pi^2 \propto m_q$

**Representation invariance**: physical observables do not change under field redefinition

\[\phi = \chi F[\chi] \quad \text{with} \quad F[0] = 1\]

Under $SU(N_f)_L \times SU(N_f)_R$, matter fields do not have a unique transformation law, convenient to introduce the **compensator field** $h$

Heavy-meson CHPT as an example
Unitarity of $S$-matrix

- Probability conservation $\Rightarrow$ unitarity of the $S$-matrix

\[ S S^\dagger = S^\dagger S = 1 \]

$T$-matrix: $S = 1 + i T \Rightarrow T - T^\dagger = i TT^\dagger$

thus, unitarity dictates a relation for the partial-wave scattering amplitude $t_\ell$:

\[ \text{Im} \, t_\ell(s) = \sigma(s) |t_\ell(s)|^2 \]

here $\sigma(s)$: two-body phase space factor

- From the LO CHPT, the $\pi\pi$ scattering amplitude $A(s, t, u) = \frac{s - M_\pi^2}{F^2},$

no imaginary part! $\Rightarrow$ unitarity is broken
Unitarity of $S$-matrix

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Going to higher orders

- **Perturbative unitarity**: imaginary part given by loops
  \[ \text{Im} \, t_{\ell}^{(2)}(s) = 0, \quad \text{Im} \, t_{\ell}^{(4)}(s) = \sigma(s) |t_{\ell}^{(2)}(s)|^2, \ldots \]

- Symmetries do not forbid higher order terms in effective Lagrangians: more derivatives, more insertion of quark masses

- More derivatives \(\Rightarrow\) non-renormalizable

... if we include in the Lagrangian *all* of the infinite number of interactions allowed by symmetries, then there will be a counterterm available to cancel every ultraviolet divergence. In this sense, ... non-renormalizable theories are just as renormalizable as renormalizable theories, as long as we include all possible terms in the Lagrangian.


- Q: How should we deal with the infinite number of terms?
  A: *Power counting* is needed: organize the infinite number of terms according to power of the expansion parameter, *finite* number of terms up to a given order
Going to higher orders

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  \[ \text{Im} t^{(2)}_\ell(s) = 0, \quad \text{Im} t^{(4)}_\ell(s) = \sigma(s) |t^{(2)}_\ell(s)|^2, \ldots \]

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- Q: How should we deal with the infinite number of terms?
  A: **Power counting** is needed: organize the infinite number of terms according to power of the expansion parameter, finite number of terms up to a given order
Necessity of higher order Lagrangian from renormalization

Consider the example of $\pi\pi$ scattering

- At LO, $\mathcal{O}(p^2)$: two derivatives or one quark mass insertion

\[
\begin{array}{c}
\times \\
\times \\
\end{array}
= \mathcal{O}(p^2)
\]

- One-loop with two $\mathcal{O}(p^2)$ vertices:

\[
\begin{array}{c}
\times \\
\circ \\
\times \\
\end{array}
= \int d^4q \frac{q_1 q_2 q_3 q_4}{(q^2 - M^2_\pi)[(p - q)^2 - M^2_\pi]}
= \mathcal{O}(p^{4+4-4}) = \mathcal{O}(p^4)
\]

- The loop is divergent, divergence absorbed by the counterterms in the $\mathcal{O}(p^4)$ Lagrangian

How can we construct the higher order Lagrangian?
Building blocks

- Powerful to include external fields
  ⇒ incorporate electroweak interactions, quark masses:

  \[
  \mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^0 + \bar{q} \gamma^\mu (v_\mu + \gamma_5 a_\mu) q - \bar{q} (s + i\gamma_5 p) q
  
  = \mathcal{L}_{\text{QCD}}^0 + \bar{q}_L \gamma^\mu l_\mu q_L + \bar{q}_R \gamma^\mu r_\mu q_R - \bar{q}_L (s + i p) q_R - \bar{q}_R (s - i p) q_L
  \]

  \[l_\mu = v_\mu - a_\mu, \quad r_\mu = v_\mu + a_\mu\]  
  left-/right-handed sources

  \[F_{\mu\nu}^R = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \quad F_{\mu\nu}^L = \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]\]  
  scalar/pseudoscalar sources

- Transformation laws for the building blocks:

  \[U \to L U R^\dagger, \quad \chi \to L \chi R^\dagger, \quad F_{\mu\nu}^L \to L F_{\mu\nu}^L L^\dagger, \quad F_{\mu\nu}^R \to R F_{\mu\nu}^R R^\dagger\]

- Another set of building blocks: \(u_\mu, \chi_\pm\) and \(f_{\mu\nu}^\pm\).
  \[f_{\mu\nu}^\pm = u F_{\mu\nu}^L u^\dagger \pm u^\dagger F_{\mu\nu}^R u\]
  They all transform as \(O \to h O h^\dagger\).
Building blocks

- Powerful to include external fields
  ⇒ incorporate electroweak interactions, quark masses:

\[
\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^0 + \bar{q} \gamma^\mu (v_\mu + \gamma_5 a_\mu) q - \bar{q} (s + i \gamma_5 p) q
\]

\[
= \mathcal{L}_{\text{QCD}}^0 + \bar{q}_L \gamma^\mu l_\mu q_L + \bar{q}_R \gamma^\mu r_\mu q_R - \bar{q}_L (s + i \gamma_5 p) q_R - \bar{q}_R (s - i \gamma_5 p) q_L
\]

\[
l_\mu = v_\mu - a_\mu, \quad r_\mu = v_\mu + a_\mu
\]

left-/right-handed sources

\[
F_{R}^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu], \quad F_{L}^{\mu\nu} = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu]
\]

scalar/pseudoscalar sources

- Transformation laws for the building blocks:

\[
U \rightarrow L U R^\dagger, \quad \chi \rightarrow L \chi R^\dagger, \quad F_{L}^{\mu\nu} \rightarrow L F_{L}^{\mu\nu} L^\dagger, \quad F_{R}^{\mu\nu} \rightarrow R F_{R}^{\mu\nu} R^\dagger
\]

- Another set of building blocks: \( u_\mu, \chi^\pm \) and \( f_{\pm}^{\mu\nu} \).

\[
f_{\pm}^{\mu\nu} = u F_{L}^{\mu\nu} u^\dagger \pm u^\dagger F_{R}^{\mu\nu} u
\]

They all transform as \( O \rightarrow h O h^\dagger \).
Construction of higher order chiral Lagrangian

- We need a counting scheme (more see later):

  \[ U \sim O(p^0) \]
  
  small momentum / derivative \[ \sim O(p) \]
  
  light quark masses \[ \chi \sim O(p^2) \]
  \[ \Leftarrow M_{GB}^2 \sim m_q \]
  
  external fields \[ l_\mu, r_\mu \sim O(p) \]
  \[ \Leftarrow D_\mu U = \partial_\mu U - il_\mu U + iUr_\mu \]

- At a given order, write down the most general Lagrangian allowed by symmetries (for \( \theta = 0 \)):

  chiral symmetry, \( P, C \) and \( T \)

  most general \( \Rightarrow \) be able to absorb all divergences at the same order
Construction of higher order chiral Lagrangian

- We need a counting scheme (more see later):

  \[ U \sim \mathcal{O}(p^0) \]

  \begin{align*}
  \text{small momentum/derivative} & \sim \mathcal{O}(p) \\
  \text{light quark masses } \chi & \sim \mathcal{O}(p^2) \quad \Leftarrow M_{\text{GB}}^2 \sim m_q \\
  \text{external fields } l_\mu, r_\mu & \sim \mathcal{O}(p) \quad \Leftarrow D_\mu U = \partial_\mu U - il_\mu U + iUr_\mu
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  chiral symmetry, \( P, C \) and \( T \)

  most general \( \Rightarrow \) be able to absorb all divergences at the same order
SU(3) chiral Lagrangian at O \( p^4 \)

\[
\mathcal{L}^{(4)} = L_1 \langle D_\mu U^\dagger D_\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D_\mu U^\dagger D_\nu U \rangle + L_3 \langle D_\mu U^\dagger D_\mu U D_\nu U^\dagger D_\nu U \rangle \\
+ L_4 \langle D_\mu U^\dagger D_\mu U \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle \\
+ L_5 \langle D_\mu U^\dagger D_\mu U (\chi^\dagger U + \chi U^\dagger) \rangle + L_6 \langle \chi^\dagger U + \chi U^\dagger \rangle^2 \\
+ L_7 \langle \chi^\dagger U - \chi U^\dagger \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
- i \ L_9 \langle F_L^{\mu\nu} D_\mu U D_\nu U^\dagger + F_R^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle U^\dagger F_L^{\mu\nu} U F_R^{\mu\nu} \rangle \\
+ 2 \text{ contact terms}
\]

- \( L_1, \ldots, 10 \): low-energy constants (LECs)
- NLO SU(2) chiral Lagrangian contains 7 terms: \( \ell_1, \ldots, 7 \)
Low-energy constants

- One loop diagrams with vertices from $\mathcal{L}^{(2)}$ are of $O(p^4)$, divergences should be absorbed by counterterms in $\mathcal{L}^{(4)}$
- Low-energy constants generally contain two parts:
  \[ L_i = L^r_i(\mu) + \Gamma_i \times \text{divergence} \]
  renormalized LECs $L^r_i(\mu)$ are finite, scale-dependent
- Scale dependence of LECs cancel the one from loop integrals
  $\Rightarrow$ physical observables are scale-independent!
- $L^r_i$'s are independent of light quark masses by construction, parameterize the short-distance physics

Values not fixed by chiral symmetry:
- extracted using experimental data
- estimated with models such as resonance saturation
- using lattice simulations

Ecker et al (1989)
Flavour Lattice Averaging Group (FLAG) (2013)
Low-energy constants

- One loop diagrams with vertices from \( \mathcal{L}^{(2)} \) are of \( \mathcal{O}(p^4) \), divergences should be absorbed by counterterms in \( \mathcal{L}^{(4)} \)
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Values not fixed by chiral symmetry:
- extracted using experimental data
- estimated with models such as resonance saturation
  \[ \text{Ecker et al (1989)} \]
- using lattice simulations
  \[ \text{Flavour Lattice Averaging Group (FLAG) (2013)} \]
Chiral Lagrangian at higher orders

$$L_{\text{eff}} = L^{(2)} + L^{(4)} + L^{(6)} + \ldots$$

- Number of terms allowed by the symmetries increases very fast
- How many LECs are contained in these for SU($N$), $N = (2, 3)$?
  - $L^{(2)}$ contains (2, 2) constants ($F, B$)
  - $L^{(4)}$ contains (7, 10) constants
    - Gasser, Leutwyler (1984, 1985)
  - $L^{(6)}$ contains (53, 90) constants
    - Bijnens, Colangelo, Ecker (1999)
- Why different for SU(2) / SU(3)?
  - same most general SU($N$) Lagrangian, but
  - matrix-trace relations [Cayley–Hamilton] render some of the structures redundant
Consider an arbitrary Feynman diagram with $L$ loops, $I$ internal lines, $V_d$ vertices of order $d$:

$$\mathcal{A} \propto \int (d^4 p)^L \frac{1}{(p^2)^I} \prod_d (p^d)^{V_d}$$

The chiral dimension of $\mathcal{A}$

$$D = 4L - 2I + \sum_d dV_d$$

Use topological identity for $L$ to eliminate $I$, $L = I - \sum_d V_d + 1$

$$D = \sum_d V_d (d - 2) + 2L + 2$$

Lowest order is $O(p^2)$, i.e. $d \geq 2 \Rightarrow$ rhs is a sum of non-negative numbers

For a given order $D$, there is only a finite number of combinations of $L$ and $V_d$

Each loop is suppressed by two orders in the momentum expansion
Weinberg's power counting

- Consider an arbitrary Feynman diagram with \( L \) loops, \( I \) internal lines, \( V_d \) vertices of order \( d \):

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\]

- The chiral dimension of \( \mathcal{A} \)

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\]

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- For a given order \( D \), there is only a finite number of combinations of \( L \) and \( V_d \)

- Each loop is suppressed by two orders in the momentum expansion
Power counting – example: $\pi\pi$ scattering

\[ D = \sum_d V_d(d - 2) + 2L + 2 \]

- $D = 2$
  - $L = 0$, lowest order tree-level diagram only

- $D = 4$
  - $L = 0$, tree-level diagram with one insertion from $\mathcal{L}^{(4)}$
    - $V_4 = 1$, $V_{d>4} = 0$

  - $L = 1$, one-loop diagram with only $d = 2$ vertices
    - $V_{d>2} = 0$
Power counting – example: $\pi\pi$ scattering

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D = \sum_d V_d (d - 2) + 2L + 2
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Power counting – example: $\pi\pi$ scattering

$$D = \sum_d V_d (d - 2) + 2L + 2$$

- $D = 2$
  - $L = 0$, lowest order tree-level diagram only

- $D = 6$
  - $L = 0$, tree-level diagram with one insertion from $\mathcal{L}^{(6)}$
  - $L = 0$, tree-level diagram with two insertions from $\mathcal{L}^{(4)}$
  - $L = 1$, one-loop diagram with one insertion from $\mathcal{L}^{(4)}$
  - $L = 2$, two-loop diagram with $V_d = 2$ vertices
Chiral symmetry breaking scale $\Lambda_\chi$

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4} \left( \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \rangle + \frac{1}{\Lambda_\chi^2} \tilde{\mathcal{L}}^{(4)} + \frac{1}{\Lambda_\chi^4} \tilde{\mathcal{L}}^{(6)} + \ldots \right)$$

- What is the scale $\Lambda_\chi$?
  - hard energy scale where the momentum expansion definitely fails
  - uncertainty estimate: higher-order corrections are suppressed by $\sim \frac{p^2}{\Lambda_\chi^2}$
  - how big?

- Estimate with resonance masses:
  The only dynamical degrees of freedom are the GBs, no resonances
  - CHPT must fail once the energy reaches the resonance region: a perturbative momentum expansion cannot generate a pole
  $$\Lambda_\chi \approx M_{\text{res}}$$

  Lowest narrow resonance $M_\rho \approx 770$ MeV,
  typically $M_{\text{res}} \sim 1$ GeV
Chiral symmetry breaking scale $\Lambda_\chi$

\[ \mathcal{L}_{\text{eff}} = \frac{F^2}{4} \left( \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \rangle + \frac{1}{\Lambda^2_\chi} \tilde{\mathcal{L}}^{(4)} + \frac{1}{\Lambda^4_\chi} \tilde{\mathcal{L}}^{(6)} + \ldots \right) \]

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  typically $M_{\text{res}} \sim 1$ GeV
Chiral symmetry breaking scale $\Lambda_\chi$

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4} \left( \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \rangle + \frac{1}{\Lambda_\chi^2} \tilde{\mathcal{L}}^{(4)} + \frac{1}{\Lambda_\chi^4} \tilde{\mathcal{L}}^{(6)} + \ldots \right)$$

- **Naturalness** argument:

  Compare

  $$\propto \left( \frac{p^2}{F^2} \right)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2)^2} \propto \frac{1}{12\pi^2} \frac{p^4}{F^4} \log \mu$$

  $$\propto \frac{F^2}{\Lambda_\chi^2} \frac{p^4}{F^4} \tilde{\ell}_i(\mu)$$

- Scale-dependent LEC $\tilde{\ell}_i(\mu)$ compensates for $\log \mu$ dependence of the loop graph

- If no accidental fine-tuning (naturalness), $\tilde{\ell}_i(\mu)$ should not be smaller than shift induced by change in scale $\mu$  \quad \Rightarrow

  $$\Lambda_\chi \approx 4\pi F \approx 1.2 \text{ GeV}$$

**Exercise**: Using the naturalness assumption, estimate the size of $L^r_i$ in $\mathcal{L}^{(4)}$ (A: $10^{-3}$)
Meson masses at NLO – 1

Take the pion mass as an example of NLO calculations.

- **Mass:** pole in the two point correlation function
  
  At LO, \( M_{\pi}^2 = M^2 \equiv B(m_u + m_d) \)

- **Higher orders:** self-energy

\[
i \delta^{ab} \Delta_{\pi}(p) = \int d^4xe^{-ipx} \langle 0 \left| T \left[ \pi^a(x)\pi^b(0) \right] \right| 0 \rangle
\]

\[
= \frac{i}{p^2 - M^2 + i\epsilon} + \frac{i}{p^2 - M^2 + i\epsilon} \left[ -i \Sigma(p^2) \right] \frac{i}{p^2 - M^2 + i\epsilon} + \cdots
\]

\[
= \frac{i}{p^2 - M^2 - \Sigma(p^2) + i\epsilon} = \frac{Z_\pi}{p^2 - M_{\pi}^2 + i\epsilon} + \text{non-singular terms}
\]

**Exercise:** show that the wave function renormalization constant is

\[
Z_\pi = \frac{1}{1 - \Sigma'(M_{\pi}^2)}
\]

**Exercise:** physical pion mass, solution of the equation

\[
M_{\pi}^2 - M^2 - \Sigma(M_{\pi}^2) = 0
\]
Meson masses at NLO – 2

- Self-energy contains two parts: π, K, η tadpole loops and counterterms in $L^{(4)}$

Assuming isospin symmetry ($m_u = m_d = \hat{m}$), the pion mass at NLO:

$$M_{\pi}^2 = M^2 \left[ 1 + \frac{I(M^2)}{2F^2} - \frac{I(M_{\eta,2}^2)}{2F^2} + \frac{8M^2}{F^2}(2L_8 - L_5) + \frac{24M_{\eta,2}^2}{F^2}(2L_6 - L_4) \right]$$

here $M^2 = 2B\hat{m}$, $M_{\eta,2}^2 = \frac{2}{3}B(2\hat{m} + m_s)$

- Loop is divergent, in dimensional regularization (good for preserving symmetry)

$$I(M^2) = i\mu^{4-d} \int \frac{d^dl}{(2\pi)^d} \frac{1}{l^2 - M^2 + i\epsilon} = \frac{M^2}{16F^2} \left( \lambda + \log \frac{M^2}{\mu^2} \right)$$

$$\lambda = \frac{2}{d-4} - [\log(4\pi) + \Gamma'(1) + 1], \text{ divergent for } d = 4!$$

On the other hand, $L_i$’s are divergent either

$$L_i = L_i^r + \frac{\Gamma_i}{32\pi^2} \lambda$$

$$\Gamma_4 = \frac{1}{8}, \quad \Gamma_5 = \frac{3}{8}, \quad \Gamma_6 = \frac{11}{144}, \quad \Gamma_8 = \frac{5}{48}$$
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\]

here \(M^2 = 2B\hat{m}, \ M_{\eta}^2 = \frac{2}{3}B(2\hat{m} + m_s)\)

- Renormalization: divergences from LECs and loops cancel out (as well as the \(\mu\)-dependence)
  \(\Rightarrow\) the pion mass is finite and \(\mu\)-independent

\[
M_{\pi}^2 = M^2 \left[ 1 + \frac{1}{32\pi^2 F^2} M^2 \log \frac{M^2}{\mu^2} - \frac{1}{96\pi^2 F^2} M_{\eta}^2 \log \frac{M_{\eta}^2}{\mu^2} \right.
\]

\[
+ \frac{8M^2}{F^2} (2L_8^r - L_5^r) + \frac{24M_{\eta}^2}{F^2} (2L_6^r - L_4^r) \right]
\]

- \(M_{\pi}^2\) vanishes in chiral limit, loop correction does not generate a non-zero mass

- Chiral logarithm: non-analytic in the light quark masses
Quark mass ratios revisited – 1

- Calculate pion/kaon masses beyond leading order:

\[ M_{\pi^+}^2 = B(m_u + m_d) \left[ 1 + \mathcal{O}(\hat{m}, m_s) \right] \]

\[ M_{K^+}^2 = B(m_u + m_s) \left[ 1 + \mathcal{O}(\hat{m}, m_s) \right] \]

- Form dimensionless ratios:

\[ \frac{M_K^2}{M_{\pi^+}^2} = \frac{m_s + \hat{m}}{m_u + m_d} \left[ 1 + \Delta M + \mathcal{O}(m_q^2) \right] \]

\[ \frac{(M_{K^0}^2 - M_{K^+}^2)_{\text{strong}}}{M_K^2 - M_{\pi^+}^2} = \frac{m_d - m_u}{m_s - \hat{m}} \left[ 1 + \Delta M + \mathcal{O}(m_q^2) \right] \]

\[ \Delta M = \frac{8(M_{K}^2 - M_{\pi}^2)}{F_{\pi}^2}(2L_8^r - L_5^r) + \text{chiral logs} \]

- Double ratio \( Q^2 \) particularly stable:

\[ Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_{\pi^+}^2} \frac{M_K^2 - M_{\pi^+}^2}{(M_{K^0}^2 - M_{K^+}^2)_{\text{strong}}} \left[ 1 + \mathcal{O}(m_q^2) \right] \]
• Calculate pion/kaon masses beyond leading order:

\[
M_{\pi}^2 = B(m_u + m_d) \left[ 1 + \mathcal{O}(\hat{m}, m_s) \right]
\]

\[
M_K^2 = B(m_u + m_s) \left[ 1 + \mathcal{O}(\hat{m}, m_s) \right]
\]

• Form dimensionless ratios:

\[
\frac{M_K^2}{M_{\pi}^2} = \frac{m_s + \hat{m}}{m_u + m_d} \left[ 1 + \Delta_M + \mathcal{O}(m_q^2) \right]
\]

\[
\frac{(M_{K_0}^2 - M_{K_+}^2)_{\text{strong}}}{M_K^2 - M_{\pi}^2} = \frac{m_d - m_u}{m_s - \hat{m}} \left[ 1 + \Delta_M + \mathcal{O}(m_q^2) \right]
\]

\[
\Delta_M = \frac{8(M_K^2 - M_{\pi}^2)}{F_{\pi}^2}(2L_8^r - L_5^r) + \text{chiral logs}
\]

• Double ratio \(Q^2\) particularly stable:

\[
Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_{\pi}^2} \frac{M_K^2 - M_{\pi}^2}{(M_{K_0}^2 - M_{K_+}^2)_{\text{strong}}} \left[ 1 + \mathcal{O}(m_q^2) \right]
\]
• Leutwyler’s ellipse:

\[
\left( \frac{m_u}{m_d} \right)^2 + \frac{1}{Q^2} \left( \frac{m_s}{m_d} \right)^2 = 1
\]

Leutwyler (1996)

• Remember Dashen’s theorem:

\[
(M^2_{K^+} - M^2_{K^0})_{\text{em}} = (M^2_{\pi^+} - M^2_{\pi^0})_{\text{em}} + \mathcal{O}(e^2 m_q)
\]

\[Q_{\text{Dashen}} = 24.2\]

• Value extracted from lattice simulations

FLAG (2013)

\[
N_f = 2 + 1: \quad 22.6 \pm 0.7 \pm 0.6
\]

\[
N_f = 2: \quad 24.3 \pm 1.4 \pm 0.6
\]
**\( \pi \pi \) scattering beyond LO**

- **\( S \)-wave scattering lengths:**

<table>
<thead>
<tr>
<th></th>
<th>( a_0^0 )</th>
<th>( a_0^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO</td>
<td>0.16</td>
<td>-0.045</td>
</tr>
<tr>
<td>NLO (one-loop)</td>
<td>0.20 ± 0.01</td>
<td>-0.042 ± 0.002</td>
</tr>
<tr>
<td>NNLO (two-loop)</td>
<td>0.217 ± ...</td>
<td>-0.041 ± ...</td>
</tr>
<tr>
<td>NNLO + Roy eq.</td>
<td>0.220 ± 0.005</td>
<td>-0.0444 ± 0.0010</td>
</tr>
</tbody>
</table>

- Compare again with the modern data from NA48/2
  \((K_{e4} \text{ decays} & K^\pm \rightarrow \pi^0\pi^0\pi^\pm)\)

\[
\begin{align*}
a_0^0 &= 0.2210 \pm 0.0047_{\text{stat}} \pm 0.0015_{\text{syst}} \\
a_0^2 &= -0.0429 \pm 0.0044_{\text{stat}} \pm 0.0016_{\text{syst}}
\end{align*}
\]
• Symmetries allow terms with more derivatives ⇒ non-renormalizable in the traditional sense

• EFT has a power counting to organize terms in the Lagrangian. Lagrangian contains the most general terms allowed by symmetries at a given order, thus EFT can be renormalized order by order

• Scale separation: \( p, M \ll \Lambda_\chi \sim 1 \text{ GeV} \)
  ⇒ allows for an uncertainty estimate
  ⇒ chiral Lagrangians should not be naively used for \( p \gtrsim \Lambda_\chi \)