Introduction to Effective Field Theories for Hadron Physics

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Outline

References

Useful textbooks/monographs:

J.F. Donoghue, E. Golowich, B.R. Holstein, *Dynamics of the Standard Model* (1992)

H. Georgi, *Weak Interactions and Modern Particle Physics* (2009)

A.V. Manohar, M.B. Wise, *Heavy Quark Physics* (2000)

S. Scherer, M.R. Schindler, *A Primer for Chiral Perturbation Theory* (2012)

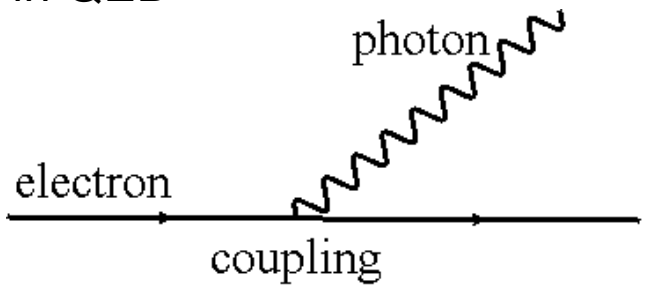
Elementary particles in the Standard Model

QUARKS	up u $2.3^{+0.7}_{-0.5}$ MeV	charm c 1275 ± 25 MeV	top t ≈ 173 GeV	gluon g 0	Higgs boson H ≈ 126 GeV
	down d $4.8^{+0.5}_{-0.3}$ MeV	strange s 95 ± 5 MeV	bottom b 4180 ± 30 MeV	photon γ 0	
LEPTONS	electron e 0.51 MeV	muon μ 105.7 MeV	tau τ 1776.8 ± 0.2 MeV	Z boson Z ≈ 91.2 GeV	GAUGE BOSONS
	electron neutrino ν_e < 2.2 MeV	muon neutrino ν_μ < 0.17 MeV	tau neutrino ν_τ < 15.5 MeV	W boson $W^\pm$ ≈ 80.4 GeV	

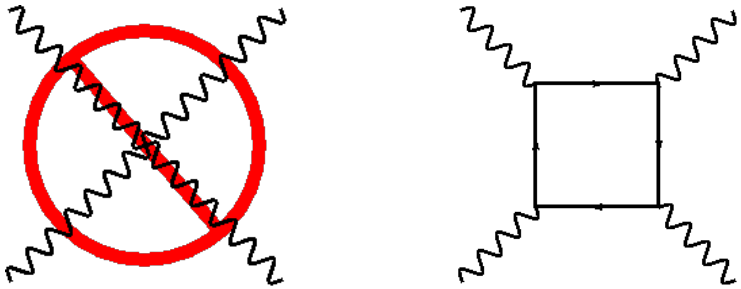
$$M_{\text{proton}} \gg (2m_u + m_d)$$

Quantum Electrodynamics (QED)

- Basic interaction vertex in QED



- Photon does not carry charge, photon-photon interaction only happens at higher orders



- Coupling constant is small, $\alpha \approx 1/137 \Longrightarrow$ great success of perturbation theory

  E.g., the electron magnetic moment

  - ☞ Exp.: $g/2 = 1.001\,159\,652\,180\,73(28)[0.28\ \mathrm{ppt}]$ D. Hanneke et al, PRL100(2008)120801
  - ☞ Th.: $g/2 = 1.001\,159\,652\,181\,78(77)[0.77\ \mathrm{ppt}]$ T. Aoyama et al, PRL109(2008)111807
    (up to $\mathcal{O}\left(\alpha^5\right)$: 12672 Feynman diagrams!)

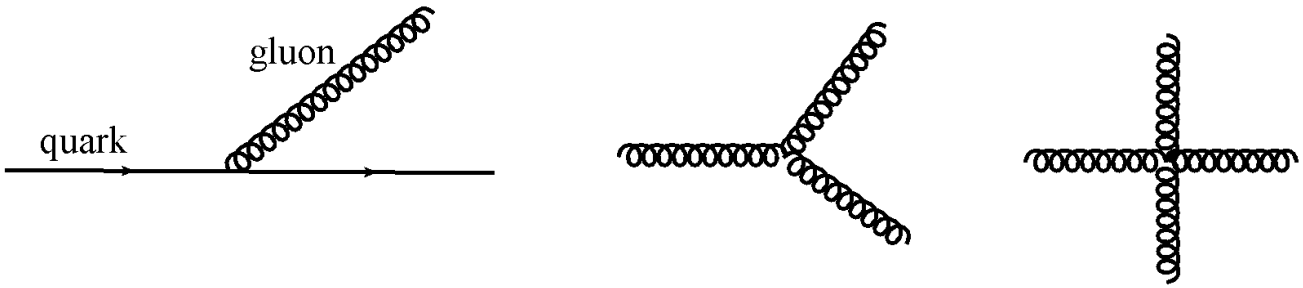
Quantum ChromoDynamics (QCD)

- Quantum ChromoDynamics (QCD) is the theory of the strong interaction

$$\mathcal{L}_{\rm QCD} = \sum_{f=\substack{u,d,s,\\ c,b,t}} \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a} + \frac{g_s^2 \theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

$$D_\mu q_f = \left(\partial_\mu + i g_s A^a_\mu \lambda^a/2\right) q_f, \quad G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$$

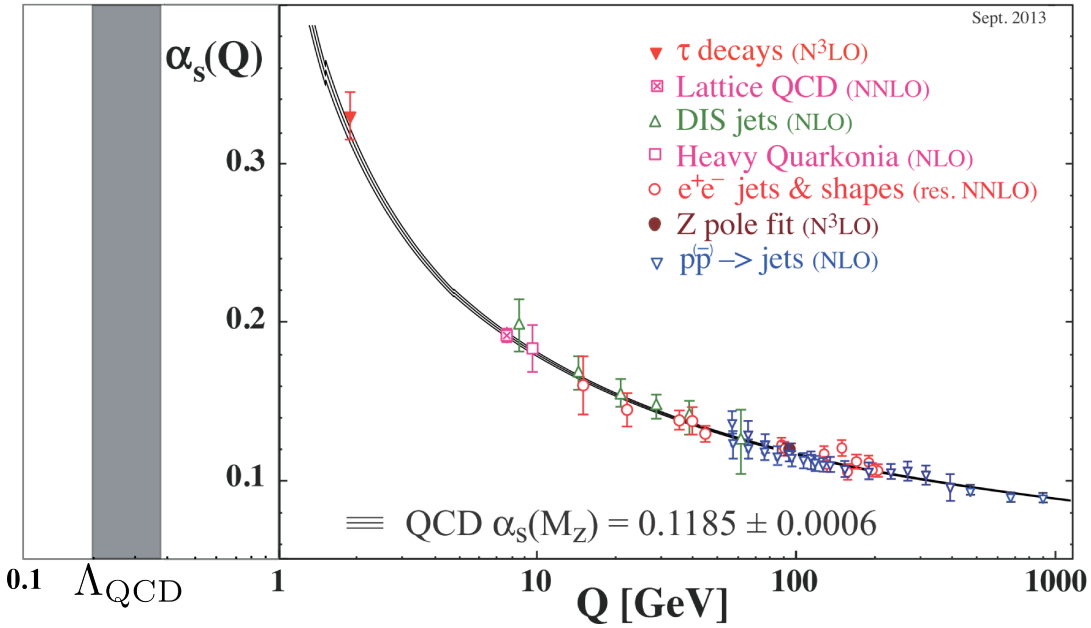
- Color gauge invariance under SU(3)$_c$ (quarks: red, blue and green): non-Abelian
- Gluons: 8 colors, self-interacting



- The θ-term breaks P and CP, θ is tiny: $|\theta| \lesssim 10^{-10}$

Two facets of QCD

- Running of the coupling constant $\alpha_s = g_s^2/(4\pi)$



- High energies
  - ☞ asymptotic freedom, perturbative
  - ☞ degrees of freedom: quarks and gluons
- Low energies
  - ☞ nonperturbative, $\Lambda_{\text{QCD}} \sim 300$ MeV $= \mathcal{O}\left(1\ \text{fm}^{-1}\right)$
  - ☞ color confinement, degrees of freedom: mesons and baryons
  - ☞ theory of quarks and gluons $\overset{?}{\rightarrow}$ low-energy hadron spectrum

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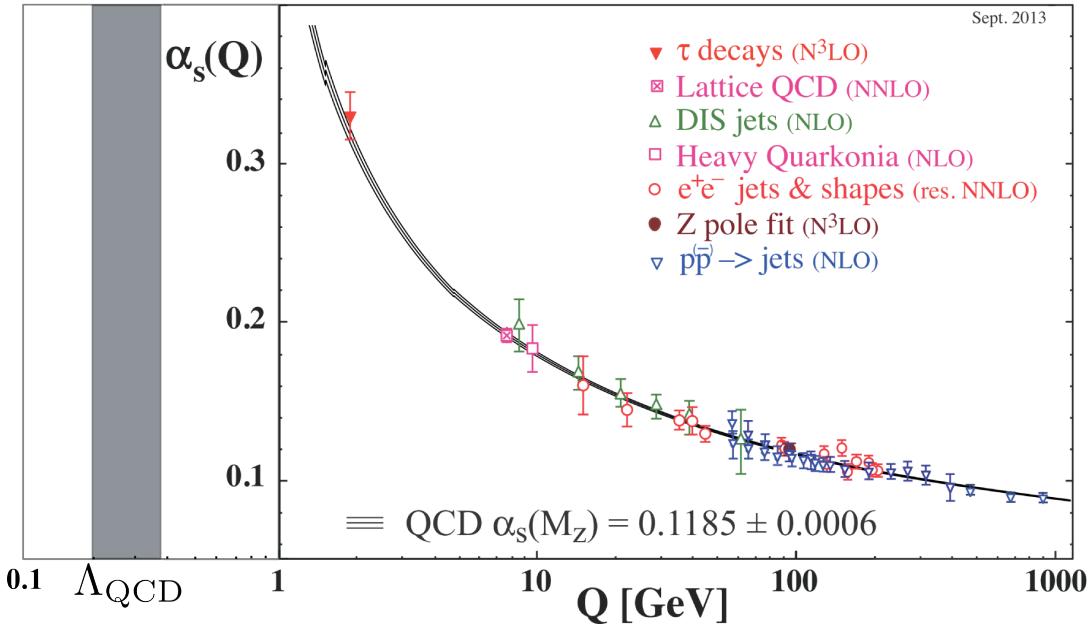

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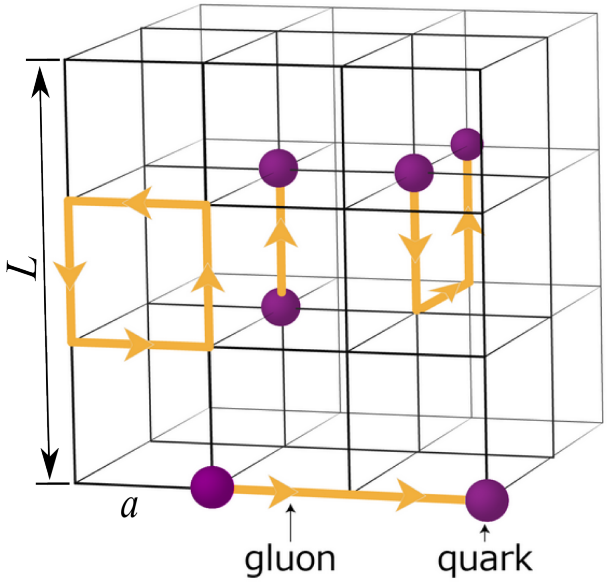
Theoretical tools for studying nonperturbative QCD

- Lattice QCD: numerical simulation in discretized Euclidean space-time
  - ☞ finite volume (L should be large)
  - ☞ finite lattice spacing (a should be small)
  - ☞ normally using $m_{u,d}$ larger than the physical values $\Rightarrow$ chiral extrapolation



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- Low-energy effective field theories of QCD:
  effective degrees of freedom can be mesons and baryons

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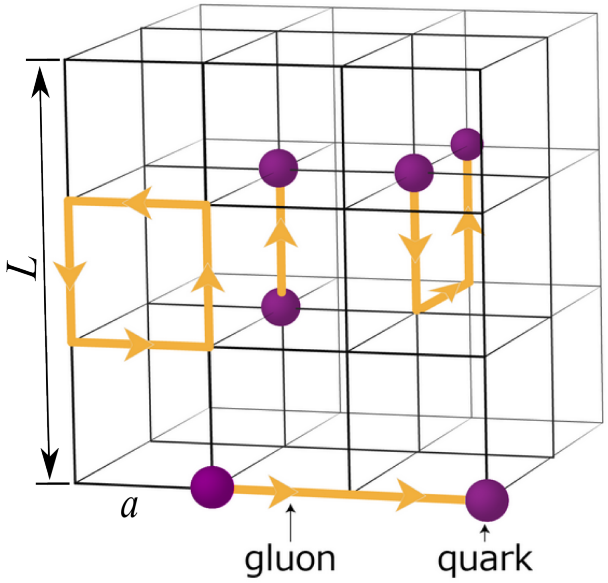

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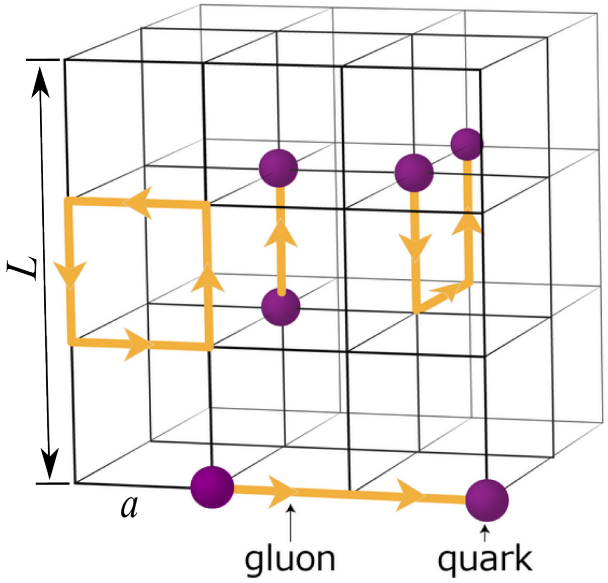

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S. Weinberg, "Phenomenological Lagrangians", Physica **96A** (1979) 327

> been proven, but which I cannot imagine could be wrong. The "theorem" says that although individual quantum field theories have of course a good deal of content, quantum field theory itself has no content beyond analyticity, unitarity, cluster decomposition, and symmetry. This can be put more precisely in the context of perturbation theory: if one writes down the most general possible Lagrangian, including *all* terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. As I said, this has not

Translation:

- ☞ The most general effective Lagrangian, up to a given order, consistent with the symmetries of the underlying theory
  $\Rightarrow$ results consistent with the underlying theory!
- ☞ The degrees of freedom can be different from those of the underlining theory
  $\Rightarrow$ we can work with hadrons directly for low-energy QCD

Low-energy EFT (II)

However, the "most general" means

- an infinite number of parameters $\Rightarrow$ intractable (?)
- nonrenormalizable (in contrast to, e.g., QED and QCD)

Solution: systematic expansion with a power counting

- only a finite number of parameters at a given order, can be determined from
  - ☞ experiments
  - ☞ lattice calculations for QCD
- renormalize order by order
- existence of a small (dimensionless) quantity, e.g.,

  separation of energy scales, $E \ll \Lambda \Rightarrow$ expansion in powers of (E/Λ)

  Neutron decay ($n \to pe^- \bar{\nu}_e$): weak interactions for $|q^2| \ll \Lambda_\chi^2$ (decoupling EFT)

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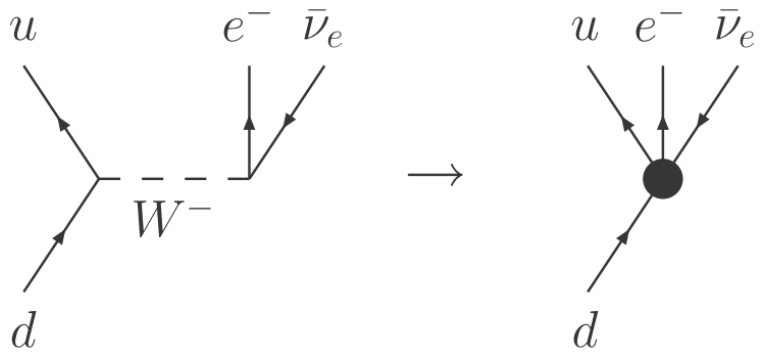
$$\frac{e^2}{8\sin\theta_W}\frac{1}{M_W^2 - q^2}$$
$$= \frac{e^2}{8M_W^2\sin\theta_W}\left(1 + \frac{q^2}{M_W^2} + \ldots\right)$$
$$= \frac{G_F}{\sqrt{2}} + \mathcal{O}\left(\frac{q^2}{M_W^2}\right)$$



Low-energy EFT (III)

- Pro: model-independent, controlled uncertainty
- Con: number of parameters increases fast when going to higher orders

Things need to be remembered for an EFT:

- separation of energy scales:
  systematic expansion with a power counting
- symmetry constraints from the full theory

  For low-energy QCD, we will consider

  ☞ (approximate) chiral symmetry of light quarks
  $\Rightarrow$ CHiral Perturbation Theory (non-decoupling EFT)
  full theory $\Rightarrow$ EFT via spontaneous symmetry breaking (SSB)

  generation of new light degrees of freedom

  ☞ heavy quark symmetry: spin and flavor

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QCD symmetries (I)

$$\mathcal{L}_{\rm QCD} = \sum_{f=\substack{u,d,s,\\ c,b,t}} \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a} + \frac{g_s^2 \theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

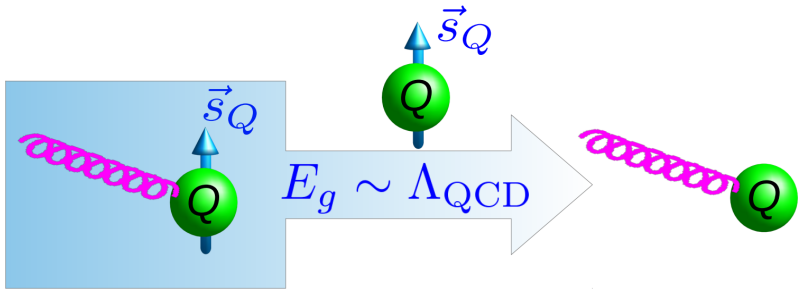
- Exact: Lorentz-invariance, SU$(3)_c$ gauge, C (for $\theta = 0$ with real m_f, P and T as well)
- Hierachy of quark masses: $m_{u,d,s} \ll \Lambda_{\rm QCD}$ ($\sim$ 300 MeV) $\ll m_{c,b,t}$
- For heavy quarks (charm, bottom) in a hadron, typical momentum transfer $\Lambda_{\rm QCD}$
  - ☞ heavy quark flavor symmetry (HQFS) for any hadron containing one heavy quark:

    velocity remains unchanged in the limit $m_Q \to \infty$: $\Delta v = \frac{\Delta p}{m_Q} = \frac{\Lambda_{\rm QCD}}{m_Q}$

    $\Rightarrow$ heavy quark is like a static color triplet source, m_Q is irrelevant
  - ☞ heavy quark spin symmetry (HQSS): see lectures by E.Eichten

    chromomag. interaction $\propto \frac{\vec{\sigma} \cdot \vec{B}}{m_Q}$

    spin of the heavy quark decouples



QCD symmetries (II): Heavy quark symmetry

- In the heavy quark limit $m_Q \to \infty$, consider the quark propagator

$$i\frac{\not{p}+m_Q}{p^2-m_Q^2+i\epsilon} = i\frac{m_Q(\not{v}+1)+\not{k}}{2m_Q v\cdot k+k^2+i\epsilon} \xrightarrow{m_Q\to\infty} \frac{1+\not{v}}{2}\frac{i}{v\cdot k+i\epsilon}$$

here $p = m_Q v + k$, with residual momentum $k \sim \Lambda_{\text{QCD}}$.
Velocity-dependent projector (in the rest frame $\vec{v}=0$) $\Rightarrow$ particle component

$$\frac{1+\not{v}}{2} = \frac{1+\gamma^0}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(in Dirac basis)}$$

- Decompose a heavy quark field into v-dep. fields
$Q(x) = e^{-im_Q v\cdot x}\left[Q_v(x)+q_v(x)\right]$ with:

$$Q_v(x) = e^{im_Q v\cdot x}\frac{1+\not{v}}{2}Q(x), \qquad q_v(x) = e^{im_Q v\cdot x}\frac{1-\not{v}}{2}Q(x)$$

Exercise: Let $P_\pm = \dfrac{1\pm\not{v}}{2}$, show $P_\pm^2 = P_\pm$, $P_+P_- = 0$, and
$\bar{Q}_v\gamma_\mu Q_v = \bar{Q}_v v_\mu Q_v$

QCD symmetries (III): Heavy quark symmetry

- Heavy quark symmetry in the leading order term:

$$\mathcal{L}_Q = \bar{Q}(i\not{D} - m_Q)Q = \bar{Q}_v(iv\cdot D)Q_v + \mathcal{O}\left(m_Q^{-1}\right)$$

- Let total angular momentum $\vec{J} = \vec{s}_Q + \vec{s}_\ell$,
  $\vec{s}_Q$: heavy quark spin,
  $\vec{s}_\ell$: spin of the light degrees of freedom (including orbital angular momentum)
  - ☞ HQSS: s_ℓ is a good quantum number in heavy hadrons
  - ☞ spin multiplets:

    for singly heavy mesons, e.g. $\{D, D^*\}$ with $s_\ell^P = \frac{1}{2}^-$,

    $M_{D^*} - M_D \simeq 140$ MeV, $\quad M_{B^*} - M_B \simeq 46$ MeV

    for heavy quarkonia, e.g. $\{\eta_c, J/\psi\}$ (no flavor symmetry for 2 or more heavy quarks)

Examples of HQSS phenomenology:

- Widths of the two D_1 ($J^P = 1^+$) mesons
  - ☞ $\Gamma[D_1(2420)] = (27.4 \pm 2.5)$ MeV $\ll$ $\Gamma[D_1(2430)] = (384^{+130}_{-110})$ MeV
  - ☞ $\vec{s}_l = \vec{s}_q + \vec{L}$ $\Rightarrow$ for P-wave charmed mesons: $s_\ell^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$
  - ☞ for decays $D_1 \to D^*\pi$:

    $\frac{1}{2}^+ \to \frac{1}{2}^- + 0^-$ in S-wave $\Rightarrow$ large width

    $\frac{3}{2}^+ \to \frac{1}{2}^- + 0^-$ in D-wave $\Rightarrow$ small width
  - ☞ thus, $D_1(2420)$: $s_\ell = \frac{3}{2}$, $\quad D_1(2430)$: $s_\ell = \frac{1}{2}$
- Suppression of the S-wave production of $\frac{3}{2}^+ + \frac{1}{2}^-$ heavy meson pairs in e^+e^- annihilation X. Li, M. Voloshin, Phys. Rev. D **88** (2013) 034012

  Exercise: Try to understand this as a consequence of HQSS.

For more examples and applications, see the lectures by Q. Wang.

QCD symmetries (IV): Heavy quark symmetry

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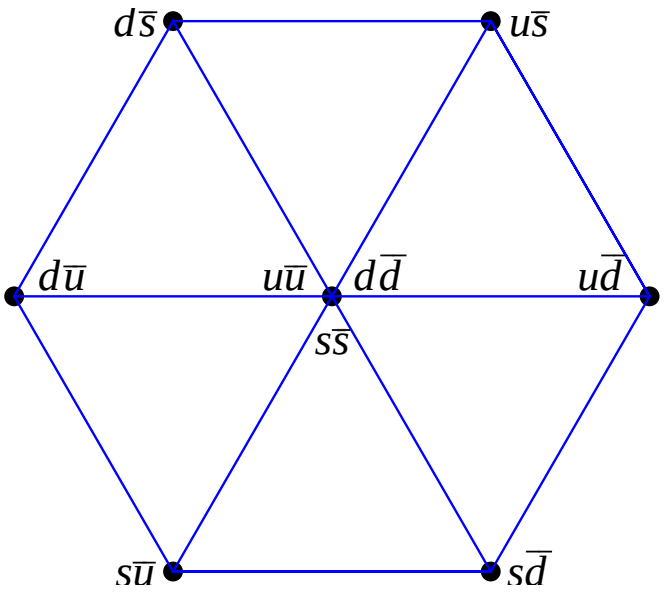
QCD symmetries (V): Light flavor symmetry

Light meson SU(3) $[u, d, s]$ multiplets (octet + singlet):

see lectures by Q.Zhao



- Vector mesons

meson	quark content	mass (MeV)
ρ^+/ρ^-	$u\bar{d}\,/d\bar{u}$	775
ρ^0	$(u\bar{u}-d\bar{d})/\sqrt{2}$	775
K^{*+}/K^{*-}	$u\bar{s}\,/s\bar{u}$	892
$K^{*0}/\bar{K}^{*0}$	$d\bar{s}\,/s\bar{d}$	896
ω	$(u\bar{u}+d\bar{d})/\sqrt{2}$	783
ϕ	$s\bar{s}$	1019

☞ approximate SU(3) symmetry

☞ very good isospin SU(2) symmetry

$$m_{\rho^0} - m_{\rho^\pm} = (-0.7 \pm 0.8)\ \text{MeV}, \quad m_{K^{*0}} - m_{K^{*\pm}} = (6.7 \pm 1.2)\ \text{MeV}$$

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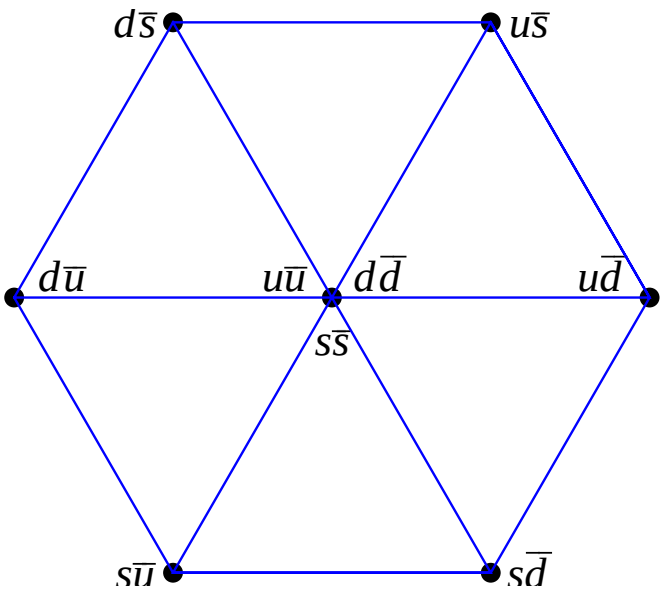

- Pseudoscalar mesons

meson	quark content	mass (MeV)
π^+/π^-	$u\bar{d}\,/d\bar{u}$	140
π^0	$(u\bar{u}-d\bar{d})/\sqrt{2}$	135
K^+/K^-	$u\bar{s}\,/s\bar{u}$	494
$K^0/\bar{K}^0$	$d\bar{s}\,/s\bar{d}$	498
η	$\sim(u\bar{u}+d\bar{d}-2s\bar{s})/\sqrt{6}$	548
η'	$\sim(u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{3}$	958

☞ very good isospin SU(2) symmetry

$$m_{\pi^\pm} - m_{\pi^0} = (4.5936 \pm 0.0005)\ \text{MeV}, \quad m_{K^0} - m_{K^\pm} = (3.937 \pm 0.028)\ \text{MeV}$$

☞ Why are the pions so light?

QCD symmetries (VI): Chiral symmetry

- Masses of the three lightest quarks u, d, s are small
  $\Rightarrow$ approximate chiral symmetry
- Chiral decomposition of fermion fields:

$$\psi = \frac{1}{2}(1-\gamma_5)\psi + \frac{1}{2}(1+\gamma_5)\psi \equiv P_L\psi + P_R\psi \;=\; \psi_L + \psi_R$$

☞ ${P_L}^2 = P_L, \qquad {P_R}^2 = P_R,$
$P_L P_R = P_R P_L = 0, \qquad P_L + P_R = 1$

- For massless fermions, left-/right-handed fields do not interact with each other

$$\mathcal{L}[\psi_L, \psi_R] = i\bar{\psi}_L \not{D}\psi_L + i\bar{\psi}_R \not{D}\psi_R - m\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right)$$

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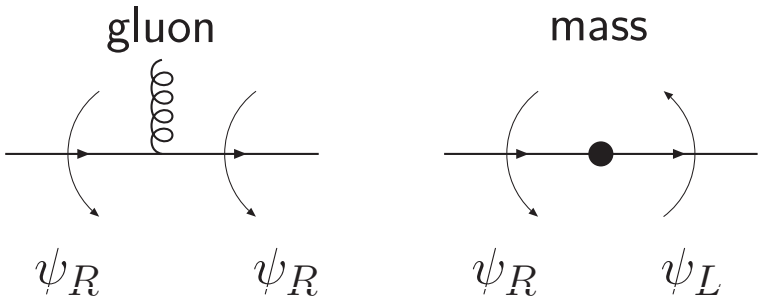

QCD symmetries (VII): Chiral symmetry

- Decompose $\mathcal{L}_{\rm QCD}$ into $\mathcal{L}^0_{\rm QCD}$, the QCD Lagrangian in the 3-flavor chiral limit $m_u = m_d = m_s = 0$, and the light quark mass term:

$$\mathcal{L}_{\rm QCD} = \mathcal{L}^0_{\rm QCD} - \bar{q}\mathcal{M}q\ , \quad q = (u, d, s)^T, \quad \mathcal{M} = {\rm diag}(m_u,\, m_d,\, m_s)$$

- $\mathcal{L}^0_{\rm QCD} = i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R + \ldots$

  is invariant under $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ transformations:

$$\mathcal{L}^0_{\rm QCD}(G_{\mu\nu},\, q',\, D_\mu q') = \mathcal{L}^0_{\rm QCD}(G_{\mu\nu},\, q,\, D_\mu q)$$

$$q' = R\, P_R q + L\, P_L q = R\, q_R + L\, q_L$$

$$R \in \mathrm{U}(3)_R, \quad L \in \mathrm{U}(3)_L$$

- Parity: $\quad q(t, \vec{x}) \xrightarrow{P} \gamma^0 q(t, -\vec{x})$

$$\Rightarrow \quad q_R(t, \vec{x}) \xrightarrow{P} P_R \gamma^0 q(t, -\vec{x}) = \gamma^0 P_L q(t, -\vec{x}) = \gamma^0 q_L(t, -\vec{x})$$

$$q_L(t, \vec{x}) \xrightarrow{P} \gamma^0 q_R(t, -\vec{x})$$

QCD symmetries (VIII): Chiral symmetry

- Decompose $\mathrm{U}(3)_L \times \mathrm{U}(3)_R = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$:

  18 conserved (Noether) currents for massless QCD at the classical level

$$J^{\mu,a}_{L,R} = \bar{q}_{L,R}\gamma^\mu \frac{\lambda^a}{2} q_{L,R}\,, \quad J^{\mu,0}_{L,R} = \bar{q}_{L,R}\gamma^\mu q_{L,R}, \qquad (\lambda_a : \text{ Gell-Mann matrices})$$

- rewritten in terms of vector ($V = L + R$) and axial vector ($A = R - L$) currents

$$V^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q\,, \qquad A^{\mu,a} = \bar{q}\gamma^\mu\gamma_5 \frac{\lambda^a}{2} q$$

$$\partial_\mu V^{\mu,a} = 0\,, \qquad \partial_\mu A^{\mu,a} = 0$$

- $\mathrm{U}(1)_V$: baryon or quark number conservation

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- $\mathrm{U}(1)_A$: broken by quantum effects, anomaly

$$A^{\mu,0} = \bar{q}\gamma_\mu\gamma_5 q, \quad \partial_\mu A^{\mu,0} = \frac{N_f g_s^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}$$

  $\Rightarrow$ the QCD θ-term, $|\theta| \lesssim 10^{-10}$

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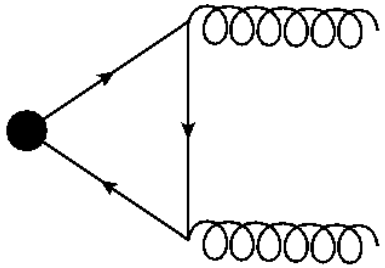

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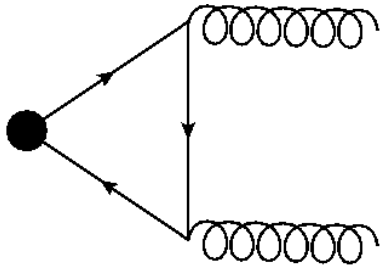

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Wigner–Weyl v.s. Nambu–Goldstone

- Noether's theorem: continuous symmetry $\Rightarrow$ conserved currents

  Let Q^a be symmetry charges: $\quad Q^a = \int d^3\vec{x}\, J^{a,0}(t,\vec{x}), \quad \partial_\mu J^{a,\mu} = 0$
- Q^a is the symmetry generator: $U = e^{i\alpha^a Q^a}$, H: Hamiltonian, thus

$$UHU^{-1} = H \;\Rightarrow\; [Q^a, H] = 0,$$

$$[Q^a, H]|0\rangle = Q^a \underbrace{H|0\rangle}_{=0} - HQ^a|0\rangle = 0$$

- Wigner–Weyl mode: $\quad Q^a|0\rangle = 0 \quad$ or equivalently $\quad U|0\rangle = |0\rangle$

  degeneracy in mass spectrum
- Nambu–Goldstone mode: $\quad U|0\rangle \neq |0\rangle$, spontaneously broken (hidden)

  $Q^a|0\rangle \neq |0\rangle$: new states degenerate with vacuum, massless Goldstone bosons

  - ☞ spontaneously broken continuous global symmetry $\Rightarrow$ massless GBs
  - ☞ the same quantum numbers as $Q^a|0\rangle \Rightarrow$ spinless
  - ☞ $\#$(GBs) $= \#$(broken generators)

Fate of SU(3)$_L\times$SU(3)$_R$ (I)

- Vector and axial charges:

$$Q_V^a = Q_L^a + Q_R^a, \quad Q_A^a = Q_R^a - Q_L^a, \quad [Q_{V,A}^a, H_{\text{QCD}}^0] = 0$$

here H_{QCD}^0: QCD Hamiltonian in the chiral limit.

Under parity transformation: $q_R \to \gamma^0 q_L,\ q_L \to \gamma^0 q_R$

$$J_L^{\mu,a} \to J_{R,\mu}^a, \qquad J_R^{\mu,a} \to J_{L,\mu}^a$$

$$\Rightarrow \quad PQ_V^a P^{-1} = Q_V^a, \qquad PQ_A^a P^{-1} = -Q_A^a$$

For an eigenstate of H_{QCD}^0: $H_{\text{QCD}}^0|\psi\rangle = E|\psi\rangle$ with $P|\psi\rangle = \eta_P|\psi\rangle$, then

$$H_{\text{QCD}}^0 Q_A^a|\psi\rangle = Q_A^a H_{\text{QCD}}^0|\psi\rangle = E\, Q_A^a|\psi\rangle,$$

$$P\, Q_A^a|\psi\rangle = P\, Q_A^a P^{-1} P|\psi\rangle = -\eta_P Q_A^a|\psi\rangle$$

Parity doubling: $Q_A^a|\psi\rangle$ has the same mass but opposite parity

Fate of SU(3)$_L\times$SU(3)$_R$ (II)

- But, in the hadron spectrum:

$$m_{\text{Nucleon},\ P=+} = 939\ \text{MeV} \ll m_{N^*(1535),\ P=-} = 1535\ \text{MeV},$$

$$m_{\pi,\ P=-} = 139\ \text{MeV} \ll m_{a_0(980),\ P=+} = 980\ \text{MeV}$$

- No parity doubling in hadron spectrum $\Rightarrow$

$$Q_A^a|0\rangle \neq 0, \quad \text{or} \quad e^{i\alpha^a Q_A^a}|0\rangle \neq |0\rangle$$

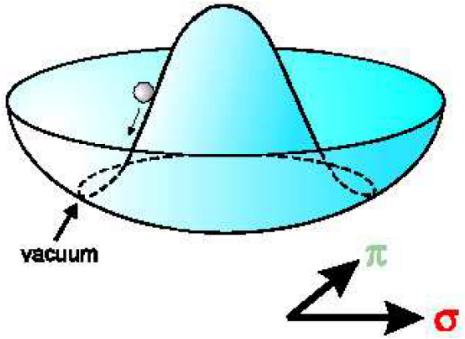
Nambu–Goldstone mode (hidden symmetry):

In QCD, SU(3)$_L\times$SU(3)$_R$ is spontaneously broken down to SU(3)$_V$

- GBs should have $J^P = 0^-$

SSB in linear σ model – 1

- Linear σ model with an $O(4)$ symmetry



$$\mathcal{L}_{\text{LSM}} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi - V(\Phi),$$

$$V(\Phi) = \frac{\lambda}{4}\left(\Phi^T\Phi - v^2\right)^2, \quad \text{with } \Phi^T = (\sigma, \pi_1, \pi_2, \pi_3)$$

- $\sigma = \pi_a = 0$ is not a minimum of $V(\Phi)$

  There is a continuum of degenerate vacua: $\Phi^T_{\text{min}}\Phi_{\text{min}} = v^2$

  Choose $\Phi_{\text{min}} = (v, 0, 0, 0)$, vacuum is only invariant under $O(3)$ rotations

  spontaneous symmetry breaking: $O(4) \to O(3)$

- Perturb around $\Phi_{\text{min}} = (v, \vec{0})$, let $\Phi = \Phi_{\text{min}} + (\sigma', \pi_1, \pi_2, \pi_3)^T$, then

$$\mathcal{L}_{\text{LSM}} = \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi_a + \frac{1}{2}\partial_\mu \sigma' \partial^\mu \sigma' - \frac{\lambda}{4}\left(\sigma'^2 + \pi_a\pi_a + 2v\sigma'\right)^2$$

  ☞ Goldstone bosons (GBs): π_a's are massless

  ☞ σ' is massive: $m^2_{\sigma'} = 2\lambda v^2$

  ☞ $m_{\sigma'}$ and 5 interaction terms described by only 2 parameters

SSB in linear σ model – 1

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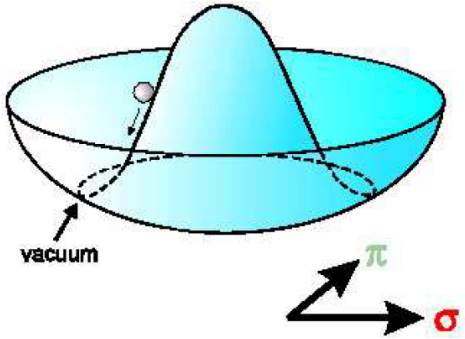

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Only 2 parameters, important for cancellation!

Examples: tree-level scattering amplitudes

- $\pi_3(p_1)\pi_3(p_2) \to \pi_3(p_3)\pi_3(p_4)$ (individually large terms!)

$i\mathcal{A} =$ 

$$= -i6\lambda + \frac{i(-2i\lambda v)^2}{s-(2\lambda v^2)} + \frac{i(-2i\lambda v)^2}{t-(2\lambda v^2)} + \frac{i(-2i\lambda v)^2}{u-(2\lambda v^2)}$$

$$= -i6\lambda + \frac{i}{2\lambda v^2}(2\lambda v)^2\left(3 + \frac{s+t+u}{2\lambda v^2}\right) + \mathcal{O}\left(\frac{p_\pi^4}{m_\sigma^4}\right) = \mathcal{O}\left(\frac{p_\pi^4}{m_\sigma^4}\right)$$

p_π: a generic momentum of GBs

Mandelstam variables: $s = (p_1+p_2)^2$, $t = (p_1-p_3)^2$, $u = (p_1-p_4)^2$,

$s+t+u = \sum_i p_i^2$

SSB in linear σ model – 2

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Examples: tree-level scattering amplitudes

- $\pi_3 \sigma' \to \pi_3 \sigma'$

  Exercise: Show that at tree-level

$$\mathcal{A}(\pi_3 \sigma' \to \pi_3 \sigma') = \mathcal{O}\left(\frac{p_\pi^2}{m_\sigma^2}\right)$$

Lessons from the linear σ model:

- ☞ SSB happens when there is a degeneracy of the vacuum
- ☞ nonvanishing VEV of some Hermitian operator, here $\langle \sigma \rangle = v$
- ☞ SSB $\Rightarrow$ massless GBs (π_a), $\quad \# = \text{dim}(O(4)) - \text{dim}(O(3)) = 3$
- ☞ GBs decouple at vanishing momenta!

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Derivative coupling

Symmetry implies a derivative coupling for GBs, i.e.,

$$\boxed{\text{GBs do not interact at vanishing momenta}}$$

- Consider GB π^a: $\langle \pi^a | Q_A^a | 0 \rangle = \int d^3x \, \langle \pi^a | A_0^a(x) | 0 \rangle \neq 0$

  Lorentz invariance $\Rightarrow$ $\left\langle \pi^a(q) | A_\mu^a(0) | 0 \right\rangle = -i q_\mu F_\pi$

- Consider the matrix element

$$\begin{aligned} \langle \psi_1 | A_\mu^a(0) | \psi_2 \rangle &= \qquad\qquad + \\ &= \quad R_\mu^a \quad + \quad F_\pi q^\mu \frac{1}{q^2} T^a \end{aligned}$$

  Current conservation $\Rightarrow$ $q^\mu A_\mu^a = 0$, thus

$$q^\mu R_\mu^a + F_\pi T^a = 0 \quad \Rightarrow \quad \lim_{q^\mu \to 0} T^a = 0$$

- $\Rightarrow$ GBs couple in a derivative form !!

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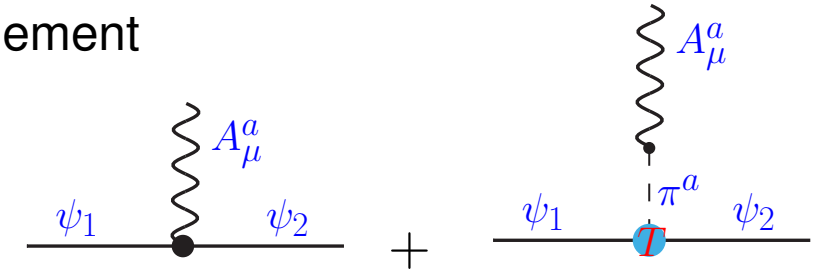

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SSB in QCD

- Hamiltonian invariant under a group $G = \text{SU}(N_f)_L \times \text{SU}(N_f)_R$, vacuum invariant under its vector subgroup $H = \text{SU}(N_f)_V$.

$$Q_V^a|0\rangle = 0, \quad Q_A^a|0\rangle \neq 0$$

- SSB $\Rightarrow$ massless pseudoscalar Goldstone bosons

  $\#(\text{GBs}) = \dim(G) - \dim(H) = N_f^2 - 1$

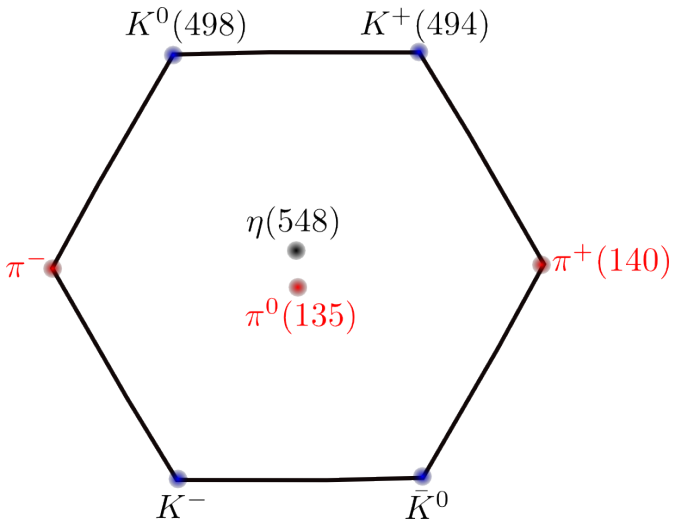
  for $N_f = 3$, 8 GBs: $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

  for $N_f = 2$, 3 GBs: $\pi^\pm, \pi^0$

  Pions get a small mass due to explicit symmetry breaking by tiny $m_{u,d}$ (a few MeV)

  $M_\pi \ll M_{\text{other hadron}}$

  also, $m_s \gg m_{u,d} \quad \Rightarrow \quad M_K \gg M_\pi$



- Mechanism for $\text{SU}(N_f)_L \times \text{SU}(N_f)_R \to \text{SU}(N_f)_V$ in QCD not well understood

Chirl perturbation theory (CHPT) as low-energy EFT for QCD

CHPT: a low energy EFT for QCD:

- an example for a non-decoupling EFT:

  degrees of freedom are different from those of the underlying theory
- a theory for the Goldstone bosons of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V$
- most general Lagrangian with the same global symmetries as QCD

How do the Goldstone bosons transform under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$?

Nonlinear realization of chiral symmetry (I)

Weinberg (1968); Coleman, Wess, Zumino (1969); Callan, Coleman, Wess, Zumino (1969)

To study the transformation properties of the Goldstone bosons (GBs)

- Assume $G \xrightarrow{\text{SSB}} H$, we have GBs

$$\Phi = (\phi_i, \ldots, \phi_n), \quad n = \text{dim}(G) - \text{dim}(H)$$

- Left coset of H with respect to $g \in G$: $gH = \{gh | h \in H\}$

  coset space G/H: set of all left cosets $\{gH | g \in G\}$

  ☞ $\text{dim}(G/H) = \dim(G) - \dim(H)$

- GBs can be parametrized as coordinates of G/H:

  choosing a representative element in each coset, e.g., for $h_{1,2} \in H$,

$$gh_1 H = gh_2 H$$

  we can choose either gh_1 or gh_2 as the parameterization

Nonlinear realization of chiral symmetry (II)

- Representation invariance:

  free to choose the set of representative elements / set of coordinates on G/H

- Transformation properties of the GBs uniquely determined:

  Parameterizing GBs by $u \in G/H$

  transformation under $g \in G$

$$g\,u = u'\,h(g,u)$$

  since any element of the coset $\{u'h(g,u)|h(g,u) \in H\}$ can be used

  $\Rightarrow$ Nonlinear transformation of GBs

$$\boxed{u \overset{g\in G}{\to} u' = g\,u\,h^{-1}(g,u)}$$

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Application to QCD (I)

For QCD, $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \xrightarrow{\text{SSB}} H = \mathrm{SU}(N_f)_V$

- $g = (g_L, g_R) \in \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ for $g_L \in \mathrm{SU}(N_f)_L$, $g_R \in \mathrm{SU}(N_f)_R$

$$g_1 g_2 = (g_{L_1}, g_{R_1})(g_{L_2}, g_{R_2}) = (g_{L_1} g_{L_2}, g_{R_1} g_{R_2})$$

- Choice of a representative element inside each left coset is free

$$(g_L, g_R) H = (g_L g_R^\dagger, \mathbb{1}) \underbrace{(g_R, g_R)}_{\in H = \mathrm{SU}(N_f)_V} H = (g_L g_R^\dagger, \mathbb{1})\, H$$

☞ Goldstone bosons can be parameterized by a unitary matrix

$$\boxed{U = g_L g_R^\dagger = \exp\left(i \frac{\phi}{F'}\right)}$$

here, $\phi = \sum_{a=1}^{8} \lambda_a \phi_a$ for SU(3), and $\vec{\tau} \cdot \vec{\pi}$ for SU(2)

λ_a: Gell-Mann matrices, $\tau_i (i = 1, 2, 3)$: Pauli matrices

ϕ_a (π_i) : Goldstone boson fields

F' : dimensionful constant (to be determined later)

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Application to QCD (II)

- Acting $g = (L,\, R) \in G$ on the coset $(U, \mathbb{1})H$

$$g\,(U, \mathbb{1})H = (L\,U,\, R)H = (L\,U\,R^\dagger\,R,\, R)H = (L\,U\,R^\dagger, \mathbb{1})\ \underbrace{(R,\, R)}_{\in \mathrm{SU}(N_f)_V}\ H$$

$\Rightarrow$ transformation property of U:

$$\boxed{U \xrightarrow{g} L\,U\,R^\dagger}$$

One can also parametrize the GBs such that $U \xrightarrow{g} R\,U\,L^\dagger$. Any one is okay if used consistently.

- For $g \in H = \mathrm{SU}(N_f)_V$, we have $R = L$

$$U \to L\,U\,L^\dagger \quad \Rightarrow \quad \phi \to L\,\phi\,L^\dagger$$

i.e., GB fields transform linearly under $\mathrm{SU}(N_f)_V$

Intermediate summary – 1

- Heavy quark spin and flavor symmetry for hadrons containing a single heavy quark
- QCD in the chiral limit $m_u = m_d = m_s = 0$ has a global symmetry

$$\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$$

  $\mathrm{U}(1)_A$ is anomalously broken by quantum effects

- Strong evidence for chiral symmetry spontaneous breaking

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \xrightarrow{\text{SSB}} \mathrm{SU}(N_f)_V$$

  3 (8) pseudoscalar Goldstone bosons: $\pi^\pm$, π^0 (, $K^\pm$, K^0, $\bar{K}^0$, η)

- For $G \xrightarrow{\text{SSB}} H$, Goldstone bosons can be identified with G/H

$$U = \exp \frac{i\phi}{F'}\,, \quad \phi = \sqrt{2} \begin{pmatrix} \frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\phi_8}{\sqrt{6}} \end{pmatrix} \text{ for } \mathrm{SU}(3)$$

  nonlinear transformation: $U \xrightarrow{g \in G} L U R^\dagger$

Construction of the effective Lagrangian for GBs (I)

Aim: reproduce low-energy structure of QCD

- Effective Lagrangian invariant under $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ (and C, P)

$$U \xrightarrow{G} L\,U\,R^\dagger, \qquad U \xrightarrow{C} U^T, \qquad U \xrightarrow{P} U^\dagger$$

- What does "low-energy" mean here?

  Goldstone boson fields (contained in U) as the only degrees of freedom

  $\Rightarrow$ energy range restricted to well below 1 GeV

  (separation of energy scales, more see later)

- Low energies: expand in powers of momenta (= number of derivatives)
- Lorentz invariance $\Rightarrow$ only even number of derivatives are allowed

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$$

- $\mathcal{L}^{(0)}$? U is unitary, $UU^\dagger = \mathbb{1}$, hence

$$\langle (UU^\dagger)^n \rangle \;=\; \text{const.}\,, \quad \langle \ldots \rangle \equiv \mathrm{Tr}_{\text{flavor}}[\ldots]$$

  $\Rightarrow$ Leading non-trivial term is $\mathcal{L}^{(2)}$

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Construction of the effective Lagrangian for GBs (II)

- Leading term of the effective Lagrangian is $\mathcal{L}^{(2)}$ just one single term (nonlinear σ model):

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle} \quad \text{with} \quad U = \exp \tfrac{i\phi}{F'}$$

$$\phi_{\text{SU(2)}} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \quad \phi_{\text{SU(3)}} = \sqrt{2}\begin{pmatrix} \frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\phi_8}{\sqrt{6}} \end{pmatrix}$$

- $\mathcal{L}^{(2)}$ is invariant under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$:

$$\begin{aligned} \langle \partial_\mu U \partial^\mu U^\dagger \rangle &\rightarrow \langle \partial_\mu U' \partial^\mu U'^\dagger \rangle \\ &= \langle L \partial_\mu U R^\dagger R \partial^\mu U^\dagger L^\dagger \rangle = \langle \partial_\mu U \partial^\mu U^\dagger \rangle \end{aligned}$$

Exercise: Show that $\langle U \partial^\mu U^\dagger \rangle = 0$, therefore $\langle U \partial_\mu U^\dagger \rangle \langle U \partial^\mu U^\dagger \rangle$ is not present.

The low-energy constant F

- Expand U in powers of ϕ, $\quad U = 1 + \frac{i\phi}{F'} - \frac{\phi^2}{2F'^2} + \ldots$

  $\Rightarrow$ canonical kinetic terms $\quad \mathcal{L}^{(2)} = \partial_\mu \pi^+ \partial^\mu \pi^- + \partial_\mu K^+ \partial^\mu K^- + \ldots$

  for $\quad \boxed{F' = F}$

- Calculate the Noether currents L_a^μ, R_a^μ from $\mathcal{L}^{(2)} \quad \Rightarrow$

$$
\begin{aligned}
V_a^\mu &= R_a^\mu + L_a^\mu = i\frac{F^2}{4}\langle \lambda_a \left[\partial^\mu U, U^\dagger\right]\rangle \\
A_a^\mu &= R_a^\mu - L_a^\mu = i\frac{F^2}{4}\langle \lambda_a \left\{\partial^\mu U, U^\dagger\right\}\rangle
\end{aligned}
$$

- Expand the currents in powers of ϕ, $\quad A_a^\mu = -F\partial^\mu \phi_a + \mathcal{O}\left(\phi^3\right)$

$$\langle 0|A_a^\mu(x)|\phi_b(p)\rangle = ip^\mu F\, e^{-ip\cdot x}\, \delta_{ab}$$

$\Rightarrow$ F is the pion decay constant in the chiral limit

$$\boxed{F \approx F_\pi}$$

$F_\pi = 92.2$ MeV measured in the leptonic decay of the pion $\pi^+ \to \ell^+ \nu_\ell$

PCAC

So far, only considered chiral limit $m_u = m_d = m_s = 0$.

For non-zero quark masses,

- the singlet vector current still conserved (baryon number conservation)

$$\partial^\mu V_\mu = 0$$

- vector currents V_μ^a conserved when $m_u = m_d = m_s = m$ (i.e., $\mathcal{M} = m\mathbb{1}$)

Gell-Mann, Ne'eman, *The Eightfold Way*

$$\partial^\mu V_\mu^a = i\,\bar{q}\left[\mathcal{M}, \frac{\lambda^a}{2}\right] q$$

- A_μ^a not conserved any more: Partially Conserved Axial Current (PCAC)

$$\partial^\mu A_\mu^a = i\,\bar{q}\left\{\mathcal{M}, \frac{\lambda^a}{2}\right\} q$$

Explicit symmetry breaking: quark masses

- In the chiral limit $m_u = m_d = m_s = 0$
  ☞ $\mathcal{L}^{(2)} = \frac{F^2}{4}\left\langle \partial_\mu U \partial^\mu U^\dagger \right\rangle$ contains no terms $\propto M^2\phi^2$
  ☞ theory for massless Goldstone bosons, but pions have nonvanishing masses
- In nature, u, d, s quark masses are small ($m_{u,d,s} \ll \Lambda_{\text{QCD}}$), but non-zero
  ☞ chiral symmetry explicitly broken

$$\mathcal{L}_m = -\bar{q}_R\, m\, q_L - \bar{q}_L\, m\, q_R$$

  ☞ if symmetry breaking is weak $\Rightarrow$

  a perturbative expansion in the quark masses

- Effective Lagrangian is still an appropriate tool to systematically derive all symmetry relations

CHiral Perturbation Theory (CHPT):

double expansion in low momenta and quark masses

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Explicit symmetry breaking: the spurion trick (I)

Spurion in 3 steps: very useful trick for explicit symmetry breaking

1. Introduce a spurion field (e.g. quark mass, electric charge, γ_μ, ...) with a transformation property so that the symmetry breaking term in the full theory is invariant
2. Write down invariant operators in EFT including the spurion field
3. Set the spurion field to the value which it should take

Explicit symmetry breaking: the spurion trick (II)

- Apply the spurion trick to quark masses:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^0_{\text{QCD}} - \bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M}^\dagger q_L$$

  - ☞ Treat $\mathcal{M}$ as a complex spurion field

$$\boxed{\mathcal{M} \to \mathcal{M}' = L \mathcal{M} R^\dagger}$$

  - ☞ Then construct Lagrangian invariant under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots, \mathcal{M})$$

  - ☞ This procedure guarantees that chiral symmetry is broken in exactly the same way in the effective theory as it is in QCD

- The spurion trick is very useful to construct EFT operators with a given symmetry transformation property

LO chiral Lagrangian with the mass term (I)

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\left[\langle \partial_\mu U \partial^\mu U^\dagger \rangle + 2B\langle \mathcal{M}U^\dagger + \mathcal{M}^\dagger U \rangle\right]$$

- Read off mass terms for GBs [SU(3)]:

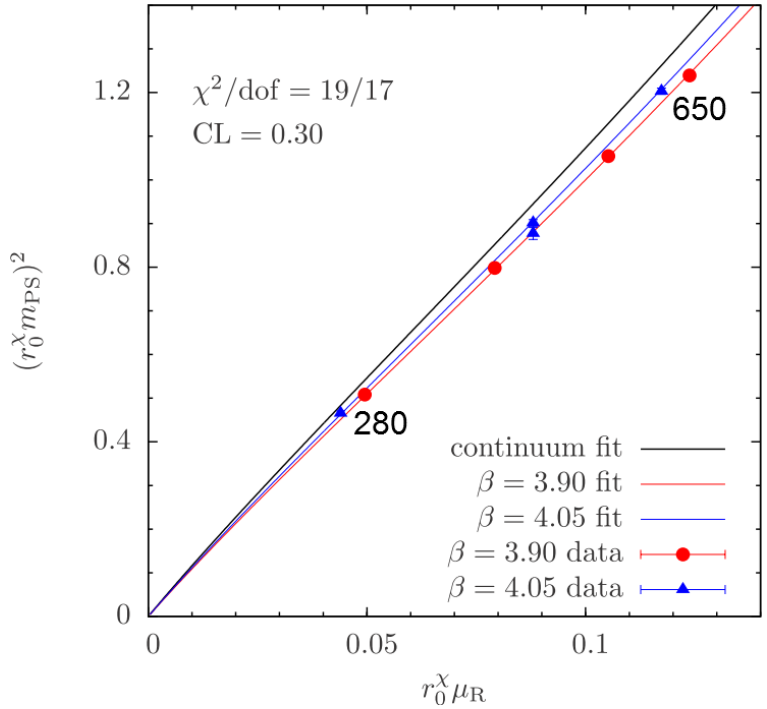
$$\begin{aligned} M^2_{\pi^\pm} &= B(m_u + m_d) \\ M^2_{K^\pm} &= B(m_u + m_s) \\ M^2_{K^0} &= B(m_d + m_s) \end{aligned}$$

- Gell-Mann–Oakes–Renner (GMOR) relation:

$$M^2_{\text{GB}} \propto m_q$$

☞ CHPT can be used to extrapolate lattice results from large to the physical values of $m_{u,d}$ (or equivalently pion masses)

GMOR relation on lattice:



ETM Col., JHEP08(2010)097

- Unified power counting for derivative and quark mass expansions:

$$m_q = \mathcal{O}\left(p^2\right)$$

LO chiral Lagrangian with the mass term (II)

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\left[\langle \partial_\mu U \partial^\mu U^\dagger \rangle + 2B\langle \mathcal{M}U^\dagger + \mathcal{M}^\dagger U\rangle\right]}$$

- Flavor-neutral states ϕ_3, ϕ_8 are mixed:

$$\frac{B}{2}\begin{pmatrix}\phi_3\\ \phi_8\end{pmatrix}^T \begin{pmatrix} m_u + m_d & \frac{1}{\sqrt{3}}(m_u - m_d) \\ \frac{1}{\sqrt{3}}(m_u - m_d) & \frac{1}{3}(m_u + m_d + 4m_s)\end{pmatrix}\begin{pmatrix}\phi_3\\ \phi_8\end{pmatrix}$$

- Diagonalize with

$$\begin{pmatrix}\pi^0\\ \eta\end{pmatrix} = \begin{pmatrix}\cos\epsilon & \sin\epsilon\\ -\sin\epsilon & \cos\epsilon\end{pmatrix}\begin{pmatrix}\phi_3\\ \phi_8\end{pmatrix}$$

- $\pi^0\eta$ mixing angle: $\epsilon = \frac{1}{2}\arctan\left(\frac{\sqrt{3}}{2}\frac{m_d - m_u}{m_s - \hat{m}}\right), \quad \hat{m} = \frac{1}{2}(m_u + m_d)$

LO chiral Lagrangian with the mass term (III)

- Mass eigenvalues:

$$\boxed{\begin{aligned} M_{\pi^0}^2 &= B(m_u + m_d) - \mathcal{O}\left((m_u - m_d)^2\right) \\ M_\eta^2 &= \frac{B}{3}(m_u + m_d + 4m_s) + \mathcal{O}\left((m_u - m_d)^2\right) \end{aligned}}$$

- In the isospin limit $m_u = m_d$:

$$M_{\pi^\pm}^2 = M_{\pi^0}^2 \ , \quad M_{K^\pm}^2 = M_{K^0}^2 \quad \text{(of course!)}$$

- Gell-Mann–Okubo (GMO) mass formula for pseudoscalars:

$$4M_K^2 = 3M_\eta^2 + M_\pi^2$$

$\Rightarrow$ fulfilled in nature at 7% accuracy

Representation invariance

- Freedom to choose coordinates on coset space G/H
- Haag's theorem on field redefinition: Haag (1958); Coleman, Wess, Zumino (1969)

  If fields ϕ and χ are related nonlinearly by a local function as

$$\phi = \chi\, F[\chi] \quad \text{with} \quad F[0] = 1,$$

  then the same physical observables (on-shell S-matrices) can be obtained using either field ϕ with Lagrangian $\mathcal{L}[\phi]$ or χ with $\mathcal{L}[\chi F[\chi]]$.

Representation invariance: $\pi\pi$ scattering (I)

In the chiral limit, $$\mathcal{L}^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle$$

Amplitude for $\pi^+\pi^- \to \pi^0\pi^0$:

- Exponential representation:

$$U = \exp \frac{i\phi}{F}$$

with $\phi = \vec{\tau}\cdot\vec{\pi}$, $\pi^0 = \pi_3$, $\pi^\pm = \frac{1}{\sqrt{2}}(\pi_1 \mp i\pi_2)$ for SU(2).

$$\mathcal{A} = \frac{s}{F^2}$$

☞ parameter-free prediction

☞ in accordance with Goldstone theorem: vanishes at zero-momentum

- Square-root representation:

$$U = \frac{1}{F}\left(\sqrt{F^2 - \vec{\pi}'^{\,2}} + i\vec{\tau}\cdot\vec{\pi}'\right)$$

calculating with $\vec{\pi}'$ as the pion fields gives the same scattering amplitude.

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Representation invariance: $\pi\pi$ scattering (II)

Exercise: Calculate the amplitude and show that the two representations are related by the following field redefinition:

$$\vec{\pi}' = \vec{\pi}\frac{F}{|\vec{\pi}|}\sin\left(\frac{|\vec{\pi}|}{F}\right) = \vec{\pi} + \ldots, \quad \text{here } |\vec{\pi}| \equiv \sqrt{\vec{\pi}^2}$$

Exercise: Show that the amplitude with the quark mass term included reads

$$\boxed{A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}}$$

Weinberg (1966)

Including matter fields

- So far, EFT for (pseudo) Goldstone bosons
- Matter fields (fields which are not Goldstone bosons) can be included as well, e.g.
  - ☞ baryon CHPT:
    nucleons [SU(2)] / baryon ground state octet [SU(3)]
  - ☞ heavy-hadron CHPT:
    heavy-flavor (charm, bottom) mesons and baryons
- At low-energies, 3-momenta remain small $\sim M_\pi$, derivative expansion is feasible

Transformation of matter fields

- Proceed as before
  - ☞ need to know how matter fields transform under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$
  - ☞ construct effective Lagrangians according to increasing number of momenta
- Transformation properties of matter fields:
  - ☞ well-defined transformation rule under the unbroken $\mathrm{SU}(N_f)_V$
  - ☞ not necessarily form representations of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$:
    transformation of matter fields under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$
    not uniquely defined,
    related by field redefinition (again: representation invariance)
- Example: heavy-meson CHPT

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Bispinor fields for heavy mesons

- Recall HQSS for heavy quarks (charm, bottom);
  $\frac{1}{2}(1+\not{v})$ projects onto the particle component of the heavy quark spinor.
- Convenient to introduce heavy mesons as bispinors:

$$H_a = \frac{1+\not{v}}{2}\left[P_a^{*\mu}\gamma_\mu - P_a\gamma_5\right], \qquad \bar{H}_a = \gamma_0 H_a^\dagger \gamma_0$$

  $P = \{Q\bar{u}, Q\bar{d}, Q\bar{s}\}$: pseudoscalar heavy mesons, P^*: vector heavy mesons
- Charge conjugation:
  - ☞ H_a destroys mesons containing a Q, but does not create mesons with a $\bar{Q}$
  - ☞ Free to choose the phase convention for charge conjugation. If we use, e.g.,

$$P_a \xrightarrow{C} +P_a^{(\bar{Q})}, \qquad P^*_{a,\mu} \xrightarrow{C} -P^{*(\bar{Q})}_{a,\mu},$$

  then the fields annihilating mesons containing a $\bar{Q}$ is ($\mathcal{C} = i\gamma^2\gamma^0$)

$$H_a^{(\bar{Q})} = \mathcal{C}\left[\frac{1+\not{v}}{2}\Big(\underbrace{-P^{*(\bar{Q})}_{a,\mu}}_{P^*_{a,\mu}\xrightarrow{C}}\gamma^\mu - \underbrace{P_a^{(\bar{Q})}}_{P_a\xrightarrow{C}}\gamma_5\Big)\right]^T \mathcal{C}^{-1}$$

$$= \left(+P^{*(\bar{Q})}_{a,\mu}\gamma^\mu - P_a^{(\bar{Q})}\gamma_5\right)\frac{1-\not{v}}{2}$$

Transformations of heavy meson fields (I)

Consider the SU(3) case, the charmed meson ground state anti-triplet:

$$H_a = \left(D^0, D^+, D_s^+\right)_a \qquad \left[\left(c\bar{u}, c\bar{d}, c\bar{s}\right)\right]$$

transform under the global unbroken $\mathrm{SU}(3)_V$ as

$$H \xrightarrow{V\in\mathrm{SU}(3)_V} H\,V^\dagger$$

- Representation invariance:
  free to choose how H transforms under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as long as it reduces to the above under $\mathrm{SU}(3)_V$
- Example:
  describe the heavy mesons by H_1 or H_2, under
  $g = (L, R) \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$

  $$H_1 \xrightarrow{g} H_1\,L^\dagger, \quad H_2 \xrightarrow{g} H_2\,R^\dagger$$

  both transform as an anti-triplet under $(V, V) \in \mathrm{SU}(3)_V$

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- Related to each other through field redefinition:

$$H_2 = H_1 U = H_1 + \frac{i}{F} H_1 \phi + \ldots, \qquad U = \exp\left(\frac{i}{F}\phi\right) \xrightarrow{g} L\, U\, R^\dagger$$

- But $H_{1,2}$ are inconvenient: complicated parity transformation (P)
  $L(L^\dagger)$ needs to be replaced by $R(R^\dagger)$ under parity $\Rightarrow$

$$H_{1,a}(t, \vec{x}) \xrightarrow{P} \gamma^0 H_{1,b}(t, -\vec{x}) \gamma^0 U_{ba} \xrightarrow{g} \gamma^0 H_{1,b}(t, -\vec{x}) \gamma^0 U_{ba} R^\dagger$$

  [recall for a spinor: $\psi(t, \vec{x}) \xrightarrow{P} \gamma^0 \psi(t, -\vec{x})$]

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Transformations of heavy meson fields (III)

- An elegant/convenient way:
  introduce
  $$u = \exp\left(\frac{i\phi}{2F}\right) \quad \text{or} \quad u^2 = U$$
  it transforms under $g \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as (recall $U \xrightarrow{g} L\,U\,R^\dagger$)
  $$u \xrightarrow{g} L\,u\,h^\dagger(L,R,\phi) = h(L,R,\phi)\,u\,R^\dagger$$
  $h(L,R,\phi)$: space-time dependent nonlinear function, called compensator field
  For for SU$(3)_V$ transformations ($L = R = V$), reduces to $h(L,R,\phi) = V$
- We can construct $H = H_1 u$ or $H = H_2 u^\dagger$, it transforms as
  $$H \xrightarrow{g} H\,h^\dagger$$
  Exercise: Show that under parity transformation $h(t,\vec{x}) \xrightarrow{P} h(t,-\vec{x})$, and $H(t,\vec{x}) \xrightarrow{P} \gamma^0 H(t,-\vec{x})\gamma^0$.
- Transformation under HQSS: $H_a \to \mathcal{S} H_a$,
  $\mathcal{S}$ does not commute with γ-matrices !

Transformations of heavy meson fields (III)

- An elegant/convenient way:
  introduce
  $$u = \exp\left(\frac{i\phi}{2F}\right) \quad \text{or} \quad u^2 = U$$
  it transforms under $g \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ as (recall $U \xrightarrow{g} L\,U\,R^\dagger$)
  $$u \xrightarrow{g} L\,u\,h^\dagger(L,R,\phi) = h(L,R,\phi)\,u\,R^\dagger$$
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Building blocks of the chiral Lagrangian

- Useful to introduce combinations of u whose transformations only involve h:

$$\Gamma^\mu = \frac{1}{2}\left[u^\dagger(\partial^\mu - i\,l^\mu)u + u(\partial^\mu - i\,r^\mu)u^\dagger\right]$$
$$u^\mu = i\left[u^\dagger(\partial^\mu - i\,l^\mu)u - u(\partial^\mu - i\,r^\mu)u^\dagger\right]$$

Γ^μ: chiral connection, vector; u^μ: chiral vielbein, axial vector

$$\Gamma^\mu \xrightarrow{g} h\,\Gamma^\mu h^\dagger + h\,\partial^\mu h^\dagger, \qquad u^\mu \xrightarrow{g} h\,u^\mu h^\dagger$$

- Introduce a covariant derivative:

$$\mathcal{D}^\mu H = \partial^\mu H - H\,\Gamma^\mu$$

which transform the same way as H under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$

$$\mathcal{D}^\mu H \xrightarrow{g} \mathcal{D}^\mu H\,h^\dagger$$

- Include the quark mass term $\chi = 2B(s + i\,p) = 2B\mathcal{M} + \ldots$ by introducing

$$\chi_+ = u^\dagger\chi u^\dagger + u\chi^\dagger u, \qquad \chi_+ \to h\,\chi_+ h^\dagger$$

All fields transform in terms of h, convenient to construct the effective Lagrangians

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Heavy meson CHPT at LO

- LO Lagrangian [$\mathcal{O}(p)$]:

$$\mathcal{L}_{\text{HM}}^{(1)} = \underbrace{-i\,\mathrm{Tr}\left[\bar{H}_a v_\mu (\mathcal{D}^\mu H)_a\right]}_{\text{kinetic term}+D\pi\text{ scattering}+\ldots} + \underbrace{\frac{g}{2}\mathrm{Tr}\left[\bar{H}_a H_b \gamma_\mu \gamma_5\right] u^\mu_{ba}}_{\text{terms for } D^*\to D\pi+\ldots}$$

  - ☞ invariant under Lorentz transformation, chiral symmetry, parity
  - ☞ Tr: trace in the spinor space, a, b: indices in the light flavor space

- Notice that the mass dimension of H_a is 3/2.
  Nonrelativistic normalization: $H_a \simeq \sqrt{M_H} H_a^{\text{rel.}}$
  D-meson propagator:

$$\frac{i}{2v\cdot k + i\epsilon} \simeq M_H \times \underbrace{\frac{i}{p^2 - M_H^2 + i\epsilon}}_{=\frac{i}{2M_H v\cdot k + i\epsilon}\left[1+\mathcal{O}\left(k^2/M_H^2\right)\right]} \qquad (p = M_H v + k)$$

Simplified two-component notation (I)

The superfield for pseudoscalar and vector heavy mesons:

$$H_a = \frac{1+\not{v}}{2}\left[P_a^{*\mu}\gamma_\mu - P_a\gamma_5\right]$$

In the rest frame of heavy meson, $v^\mu = (1, \vec{0})$. We take the Dirac basis

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Simplifications: $\dfrac{1+\not{v}}{2} = \dfrac{1+\gamma^0}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$

$$H_a = \begin{pmatrix} 0 & -\left(P_a + \vec{P}_a^* \cdot \vec{\sigma}\right) \\ 0 & 0 \end{pmatrix}, \qquad \bar{H}_a = \begin{pmatrix} 0 & 0 \\ \left(P_a^\dagger + \vec{P}_a^{*\dagger} \cdot \vec{\sigma}\right) & 0 \end{pmatrix}$$

Thus, it is convenient to simply use the two-component notation

$$H_a = P_a + \vec{P}_a^* \cdot \vec{\sigma}$$

Simplified two-component notation (II)

- In the simplified two-component notation:

$$-i\,\mathrm{Tr}\left[\bar{H}_a v_\mu (\mathcal{D}^\mu H)_a\right] \Rightarrow i\,\mathrm{Tr}\left[H_a^\dagger\left(\partial^0 H_a - H_b \Gamma^0_{ba}\right)\right]$$

$$= 2i\left(P_a^\dagger \partial^0 P_a + P_a^{*i\,\dagger}\partial^0 P_a^{*i}\right) + \underbrace{\frac{-i}{4F^2}\left(P_a^\dagger P_b + \vec{P}_a^{*\dagger}\cdot\vec{P}_b^*\right)\left[\phi,\partial^0\phi\right]_{ba}}_{\text{scattering between } (D,D^*,\bar{B},\bar{B}^*) \text{ and GBs}(\pi,K,\eta)} + \ldots$$

The LO, $\mathcal{O}(p)$, scattering amplitudes are completely determined by chiral symmetry!

- The axial coupling term ($u^k = -\frac{1}{F}\partial^k\phi + \ldots$):

$$\frac{g}{2}\mathrm{Tr}\left[\bar{H}_a H_b \gamma_\mu\gamma_5\right] u^\mu_{ba} \Rightarrow -\frac{g}{2}\mathrm{Tr}\left[H_a^\dagger H_b \sigma^i\right] u^i_{ba}$$

$$= -\frac{g}{2}\mathrm{Tr}\left[\left(P_a^{*i\,\dagger}\sigma^i + P_a^\dagger\right)\left(P_b^{*j}\sigma^j + P_b^\dagger\right)\sigma^k\right] u^k_{ba}$$

$$= \underbrace{\frac{1}{F}P_a^\dagger P_b^{*i}\partial^i\phi_{ba}}_{\text{term for } D^*\to D\pi} + \frac{1}{F}P_a^{*i\,\dagger}P_b\partial^i\phi_{ba} + \frac{i}{F}\epsilon^{ijk}P_a^{*i\,\dagger}P_b^{*j}\partial^k\phi_{ba} + \ldots$$

Determination of g

- Measured D^* widths: PDG2015

$$\Gamma(D^{*0}) < 2.1 \text{ MeV}, \Gamma(D^{*\pm}) = (83.4 \pm 1.8) \text{ keV}$$
$$\mathcal{B}(D^{*+} \to D^0\pi^+) = (67.7 \pm 0.5)\%,$$
$$\mathcal{B}(D^{*+} \to D^+\pi^0) = (30.7 \pm 0.5)\%$$

- Amplitude from the LO Lagrangian:

$$\mathcal{A}(D^{*+} \to D^0\pi^+) = \frac{i\sqrt{2}g}{F}\vec{\varepsilon}_{D^*} \cdot \vec{q}_\pi \quad \underbrace{\sqrt{M_{D^*}M_D}}_{\text{accounts for NR normalization}}$$

and the two-body decay width

$$\Gamma(D^{*+} \to D^0\pi^+) = \frac{1}{8\pi}\frac{|\vec{q}_\pi|}{M_{D^*}^2}\frac{1}{3}|\mathcal{A}|^2 \quad \Rightarrow \quad g \simeq 0.57$$

- HQFS: g should be approximately the same in bottom sector with a relative uncertainty of $\sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{m_c} - \frac{\Lambda_{\text{QCD}}}{m_b}\right) \sim \mathcal{O}(0.2)$

Lattice QCD results:

$$g_b = 0.492 \pm 0.029$$ ALPHA Collaboration, Phys. Lett. B **740** (2015) 278

$$g_b = 0.56 \pm 0.03 \pm 0.07$$ RBC and UKQCD Collaborations, Phys. Rev. D **93** (2016) 014510

Mass splittings among heavy mesons (I)

- Spin-dependent term $\Rightarrow$ mass difference between vector and pseudoscalar mesons ($\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$)

$$\mathcal{L}_\Delta = \frac{\lambda_2}{m_Q}\mathrm{Tr}\left[\bar{H}_a \sigma_{\mu\nu} H_a \sigma^{\mu\nu}\right] \overset{\text{2-comp.}}{\Rightarrow} -\frac{2\lambda_2}{m_Q}\mathrm{Tr}\left[H_a^\dagger \sigma^i H_a \sigma^i\right]$$
$$= \frac{4\lambda_2}{m_Q}\left(\vec{P}_a^{*\,\dagger} \cdot \vec{P}_a^* - 3P_a^\dagger P_a\right)$$

$$\Rightarrow \qquad M_{P_a^*} - M_{P_a} = -\frac{8\lambda_2}{m_Q}$$

- Thus, we expect

$$\frac{M_{B^*} - M_B}{M_{D^*} - M_D} \simeq \frac{m_c}{m_b} \simeq 0.3$$

measured values:

$$M_{D^*} - M_D \simeq 140 \text{ MeV}, \qquad M_{B^*} - M_B \simeq 46 \text{ MeV}$$

Mass splittings among heavy mesons (II)

- Light quark mass-dependent terms in two-component notation:

$$\mathcal{L}_\chi = -\lambda_1 \mathrm{Tr}\left[H_a^\dagger H_b\right]\chi_{+,ba} - \lambda_1' \mathrm{Tr}\left[H_a^\dagger H_a\right]\chi_{+,bb}$$

here, $$\chi_+ = u^\dagger \chi u^\dagger + u\chi^\dagger u = 4B\mathcal{M} - \frac{B}{2F^2}\{\phi,\{\phi,\mathcal{M}\}\} + \ldots$$

☞ SU(3) mass differences:

$$M_{D_s^+} - M_{D^+} = 4\lambda_1 B(m_s - m_d) = 4\lambda_1\left(M_{K^\pm}^2 - M_{\pi^\pm}^2\right)$$

$\Rightarrow \quad \lambda_1 \simeq 0.11$ GeV^{-1}

☞ Isospin splitting induced by $m_d - m_u$:

$$\begin{aligned}(M_{D^0} - M_{D^+})_{\text{quark mass}} &= 4\lambda_1 B(m_u - m_d)\\ &= 4\lambda_1\left(M_{K^\pm}^2 - M_{K^0}^2 - M_{\pi^\pm}^2 + M_{\pi^0}^2\right)\\ &= -2.3 \text{ MeV}\end{aligned}$$

Exp. value: $$\begin{aligned}M_{D^0} - M_{D^+} &= -(4.77 \pm 0.08) \text{ MeV}\\ &= \left(M_{D^0} - M_{D^+}\right)_{\text{quark mass}} + \left(M_{D^0} - M_{D^+}\right)_{\text{e.m.}}\end{aligned}$$

- $\mathcal{L}_\chi$ also contributes to scattering between a heavy meson and the lightest pseudoscalar mesons (GBs)

Intermediate summary — 2

- Lagrangians need to respect the global symmetries of QCD, explicit symmetry breaking can be included using the spurion technique
- The LO mesonic chiral Lagrangian invariant under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\left[\langle \partial_\mu U \partial^\mu U^\dagger \rangle + 2B\langle \mathcal{M}U^\dagger + \mathcal{M}^\dagger U \rangle\right] \quad \text{with } U = e^{i\phi/F}$$

  GMOR relation: $M_\pi^2 \propto m_q$
- Representation invariance: physical observables do not change under field redefinition

$$\phi = \chi\, F[\chi] \quad \text{with } F[0] = 1$$

- Under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, matter fields do not have a unique transformation law, convenient to introduce the compensator field h
  - ☞ heavy-meson CHPT as an example

Unitarity of S-matrix

- Probability conservation $\Rightarrow$ unitarity of the S-matrix

$$S\,S^\dagger = S^\dagger S = \mathbb{1}$$

T-matrix: $S = \mathbb{1} + i\,T \Rightarrow \qquad T - T^\dagger = i\,T\,T^\dagger$

thus, unitarity dictates a relation for the partial-wave scattering amplitude t_ℓ:

$$\text{Im}\, t_\ell(s) = \sigma(s)|t_\ell(s)|^2$$

here $\sigma(s)$: two-body phase space factor

- From the LO CHPT, the $\pi\pi$ scattering amplitude $A(s,t,u) = \dfrac{s - M_\pi^2}{F^2}$, no imaginary part! $\Rightarrow$ unitarity is broken

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Going to higher orders

- Perturbative unitarity: imaginary part given by loops

$$\mathrm{Im}\, t_\ell^{(2)}(s) = 0, \quad \mathrm{Im}\, t_\ell^{(4)}(s) = \sigma(s)|t_\ell^{(2)}(s)|^2, \ \ldots$$

- Symmetries do not forbid higher order terms in effective Lagrangians: more derivatives, more insertion of quark masses
- More derivatives $\Rightarrow$ non-renormalizable

> ..., if we include in the Lagrangian *all* of the infinite number of interactions allowed by symmetries, then there will be a counterterm available to cancel every ultraviolet divergence. In this sense, ..., non-renormalizable theories are just as renormalizable as renormalizable theories, as long as we include all possible terms in the Lagrangian.
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> S.Weinberg, *The Quantum Theory of Fields*, Vol.1

- Q: How should we deal with the infinite number of terms?
  A: Power counting is needed: organize the infinite number of terms according to power of the expansion parameter, finite number of terms up to a given order

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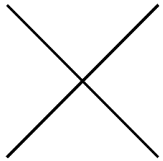
Necessity of higher order Lagrangian from renormalization

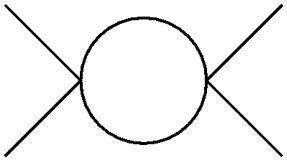
Consider the example of $\pi\pi$ scattering

- At LO, $\mathcal{O}\left(p^2\right)$: two derivatives or one quark mass insertion



$$= \mathcal{O}\left(p^2\right)$$

- One-loop with two $\mathcal{O}\left(p^2\right)$ vertices:



$$= \int d^4q \frac{q_1 q_2 q_3 q_4}{(q^2 - M_\pi^2)[(p-q)^2 - M_\pi^2]}$$

$$= \mathcal{O}\left(p^{4+4-4}\right) = \mathcal{O}\left(p^4\right)$$

- The loop is divergent, divergence absorbed by the counterterms in the $\mathcal{O}\left(p^4\right)$ Lagrangian

  How can we construct the higher order Lagrangian?

Building blocks

- Powerful to include external fields
  $\Rightarrow$ incorporate electroweak interactions, quark masses:

$$\begin{aligned}\mathcal{L}_{\text{QCD}} &= \mathcal{L}^0_{\text{QCD}} + \bar{q}\gamma^\mu\left(v_\mu + \gamma_5 a_\mu\right)q - \bar{q}\left(s + i\gamma_5 p\right)q \\ &= \mathcal{L}^0_{\text{QCD}} + \bar{q}_L\gamma^\mu l_\mu q_L + \bar{q}_R\gamma^\mu r_\mu q_R - \bar{q}_L(s+ip)q_R - \bar{q}_R(s-ip)q_L\end{aligned}$$

$l_\mu = v_\mu - a_\mu, \quad r_\mu = v_\mu + a_\mu$ left-/right-handed sources

$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu], \quad F_L^{\mu\nu} = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu]$

$\chi = 2B(s+ip) = 2B\mathcal{M} + \ldots$ scalar/pseudoscalar sources

- Transformation laws for the building blocks:

$$U \to L\,U\,R^\dagger, \qquad \chi \to L\,\chi\,R^\dagger,$$
$$F_L^{\mu\nu} \to L\,F_L^{\mu\nu}\,L^\dagger, \qquad F_R^{\mu\nu} \to R\,F_R^{\mu\nu}\,R^\dagger$$

- Another set of building blocks: u_μ, $\chi_\pm$ and $f_\pm^{\mu\nu}$. $\quad f_\pm^{\mu\nu} = uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u$
  They all transform as $O \to h\,O\,h^\dagger$

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$$= \mathcal{L}^0_{\text{QCD}} + \bar{q}_L \gamma^\mu l_\mu q_L + \bar{q}_R \gamma^\mu r_\mu q_R - \bar{q}_L (s + ip) q_R - \bar{q}_R (s - ip) q_L$$

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Construction of higher order chiral Lagrangian

- We need a counting scheme (more see later):

U	$\sim$	$\mathcal{O}\left(p^0\right)$	
small momentum / derivative	$\sim$	$\mathcal{O}\left(p\right)$	
light quark masses χ	$\sim$	$\mathcal{O}\left(p^2\right)$	$\Leftarrow M_{\text{GB}}^2 \sim m_q$
external fields $l_\mu,\, r_\mu$	$\sim$	$\mathcal{O}\left(p\right)$	$\Leftarrow D_\mu U = \partial_\mu U - i l_\mu U + i U r_\mu$

- At a given order, write down the most general Lagrangian allowed by symmetries (for $\theta = 0$):

chiral symmetry, P, C and T

most general $\Rightarrow$ be able to absorb all divergences at the same order

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SU(3) chiral Lagrangian at NLO

- SU(3) chiral Lagrangian at $\mathcal{O}\left(p^4\right)$ Gasser, Leutwyler (1985)

$$
\begin{aligned}
\mathcal{L}^{(4)} = & L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle \\
& + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle \\
& + L_5 \langle D_\mu U^\dagger D^\mu U (\chi^\dagger U + \chi U^\dagger) \rangle + L_6 \langle \chi^\dagger U + \chi U^\dagger \rangle^2 \\
& + L_7 \langle \chi^\dagger U - \chi U^\dagger \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
& - i\, L_9 \langle F_L^{\mu\nu} D_\mu U D_\nu U^\dagger + F_R^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle U^\dagger F_L^{\mu\nu} U F_{R\mu\nu} \rangle \\
& + 2 \text{ contact terms}
\end{aligned}
$$

- $L_{1,\ldots,10}$: low-energy constants (LECs)
- NLO SU(2) chiral Lagrangian contains 7 terms: $\ell_{1,\ldots,7}$ Gasser, Leutwyler (1984)

Low-energy constants

- One loop diagrams with vertices from $\mathcal{L}^{(2)}$ are of $\mathcal{O}\left(p^4\right)$, divergences should be absorbed by counterterms in $\mathcal{L}^{(4)}$
- Low-energy constants generally contain two parts:

$$L_i = L_i^r(\mu) + \Gamma_i \times \text{divergence}$$

  renormalized LECs $L_i^r(\mu)$ are finite, scale-dependent
- Scale dependence of LECs cancel the one from loop integrals
  $\Rightarrow$ physical observables are scale-independent !
- L_i^r's are independent of light quark masses by construction, parameterize the short-distance physics

  Values not fixed by chiral symmetry:

  - ☞ extracted using experimental data
  - ☞ estimated with models such as resonance saturation — Ecker et al (1989)
  - ☞ using lattice simulations — Flavour Lattice Averaging Group (FLAG) (2013)

Low-energy constants

- One loop diagrams with vertices from $\mathcal{L}^{(2)}$ are of $\mathcal{O}\left(p^4\right)$, divergences should be absorbed by counterterms in $\mathcal{L}^{(4)}$
- Low-energy constants generally contain two parts:

$$L_i = L_i^r(\mu) + \Gamma_i \times \text{divergence}$$

  renormalized LECs $L_i^r(\mu)$ are finite, scale-dependent
- Scale dependence of LECs cancel the one from loop integrals
  $\Rightarrow$ physical observables are scale-independent !
- L_i^r's are independent of light quark masses by construction, parameterize the short-distance physics

  Values not fixed by chiral symmetry:

  ☞ extracted using experimental data
  ☞ estimated with models such as resonance saturation — Ecker et al (1989)
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Chiral Lagrangian at higher orders

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots$$

- Number of terms allowed by the symmetries increases very fast
- How many LECs are contained in these for SU(N), $N = (2, 3)$?

$\mathcal{L}^{(2)}$	contains	(2, 2)	constants	(F, B)
$\mathcal{L}^{(4)}$	contains	(7, 10)	constants	Gasser, Leutwyler (1984, 1985)
$\mathcal{L}^{(6)}$	contains	(53, 90)	constants	Bijnens, Colangelo, Ecker (1999)

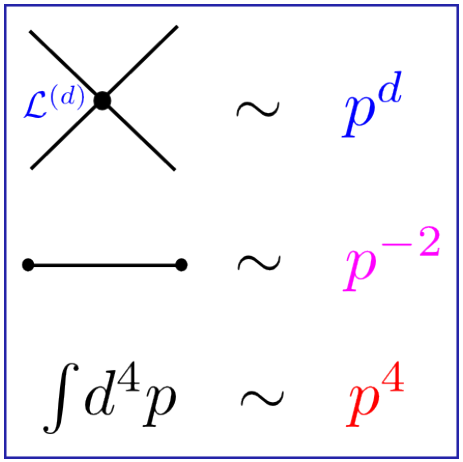
- Why different for SU(2) / SU(3)?
  same most general SU(N) Lagrangian, but
  matrix-trace relations [Cayley–Hamilton] render some of the structures redundant

Weinberg's power counting

Weinberg (1979)

- Consider an arbitrary Feynman diagram with L loops, I internal lines, V_d vertices of order d:

$$\mathcal{A} \propto \int (d^4p)^L \frac{1}{(p^2)^I} \prod_d (p^d)^{V_d}$$



- The chiral dimension of $\mathcal{A}$

$$D = 4L - 2I + \sum_d dV_d$$

- Use topological identity for L to eliminate I, $L = I - \sum_d V_d + 1$

$$D = \sum_d V_d(d-2) + 2L + 2$$

☞ Lowest order is $\mathcal{O}\left(p^2\right)$, i.e. $d \geq 2$ $\Rightarrow$ rhs is a sum of non-negative numbers

☞ For a given order D, there is only a finite number of combinations of L and V_d

☞ Each loop is suppressed by two orders in the momentum expansion

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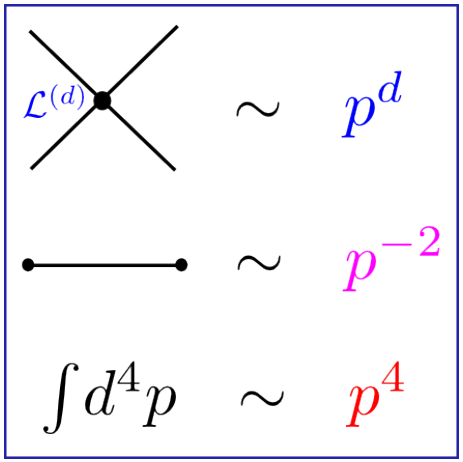

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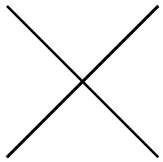
Power counting – example: $\pi\pi$ scattering

$$D = \sum_d V_d (d-2) + 2L + 2$$

- $D = 2$
  - ☞ $L = 0$, lowest order tree-level diagram only



- $D = 4$
  - ☞ $L = 0$, tree-level diagram with one insertion from $\mathcal{L}^{(4)}$

    $V_4 = 1, V_{d>4} = 0$

  - ☞ $L = 1$, one-loop diagram with only $d = 2$ vertices

    $V_{d>2} = 0$

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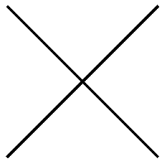

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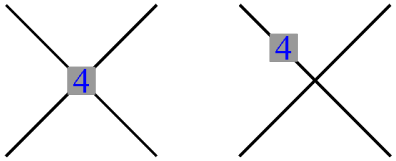

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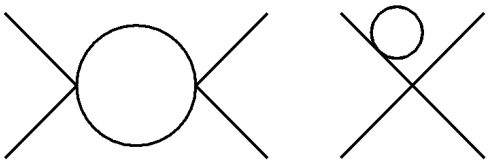

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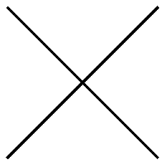
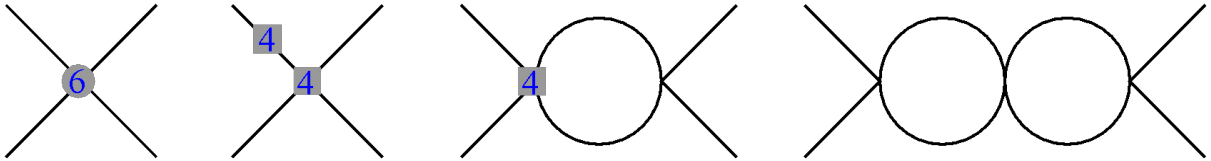


- $D = 6$
  - ☞ $L = 0$, tree-level diagram with one insertion from $\mathcal{L}^{(6)}$
  - ☞ $L = 0$, tree-level diagram with two insertions from $\mathcal{L}^{(4)}$
  - ☞ $L = 1$, one-loop diagram with one insertion from $\mathcal{L}^{(4)}$
  - ☞ $L = 2$, two-loop diagram with $V_d = 2$ vertices



Chiral symmetry breaking scale Λ_χ

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\left(\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle + \frac{1}{\Lambda_\chi^2}\tilde{\mathcal{L}}^{(4)} + \frac{1}{\Lambda_\chi^4}\tilde{\mathcal{L}}^{(6)} + \ldots\right)$$

- What is the scale Λ_χ?
  - ☞ hard energy scale where the momentum expansion definitely fails
  - ☞ uncertainty estimate: higher-order corrections are suppressed by $\sim p^2/\Lambda_\chi^2$
    $\Rightarrow$ how big?
- Estimate with resonance masses:
  The only dynamical degrees of freedom are the GBs, no resonances
  $\Rightarrow$ CHPT must fail once the energy reaches the resonance region: a perturbative momentum expansion cannot generate a pole

$$\Lambda_\chi \approx M_{\text{res}}$$

Lowest narrow resonance $M_\rho \approx 770$ MeV,
typically $M_{\text{res}} \sim 1$ GeV

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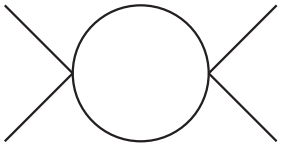
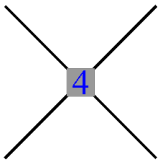
- Naturalness argument: Manohar, Georgi (1984)

  Compare



$$\propto \left(\frac{p^2}{F^2}\right)^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2)^2} \overset{\text{dim.reg}}{\propto} \frac{1}{(4\pi)^2}\frac{p^4}{F^4}\log\mu$$



$$\propto \frac{F^2}{\Lambda_\chi^2}\frac{p^4}{F^4}\tilde{\ell}_i(\mu)$$

- Scale-dependent LEC $\tilde{\ell}_i(\mu)$ compensates for $\log\mu$ dependence of the loop graph
- If no accidental fine-tuning (naturalness), $\tilde{\ell}_i(\mu)$ should not be smaller than shift induced by change in scale μ $\Rightarrow$

$$\Lambda_\chi \approx 4\pi F \approx 1.2 \text{ GeV}$$

Exercise: Using the naturalness assumption, estimate the size of L_i^r in $\mathcal{L}^{(4)}$ (A: 10^{-3})

Meson masses at NLO – 1

Take the pion mass as an example of NLO calculations.

- Mass: pole in the two point correlation function
  At LO, $\quad M_\pi^2 = M^2 \equiv B(m_u + m_d)$
- Higher orders: self-energy

$$i\,\delta^{ab}\Delta_\pi(p) = \int d^4x e^{-ipx} \left\langle 0 \left| T\left[\pi^a(x)\pi^b(0)\right] \right| 0 \right\rangle$$

$=$ 

$$= \frac{i}{p^2 - M^2 + i\epsilon} + \frac{i}{p^2 - M^2 + i\epsilon}\left[-i\Sigma(p^2)\right]\frac{i}{p^2 - M^2 + i\epsilon} + \cdots$$

$$= \frac{i}{p^2 - M^2 - \Sigma(p^2) + i\epsilon} = \frac{Z_\pi}{p^2 - M_\pi^2 + i\epsilon} + \text{non-singular terms}$$

☞ Exercise: show that the wave function renormalization constant is

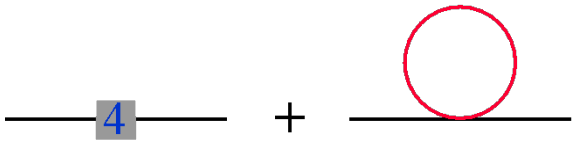
$$Z_\pi = \frac{1}{1 - \Sigma'(M_\pi^2)}$$

☞ M_π: physical pion mass, solution of the equation

$$M_\pi^2 - M^2 - \Sigma(M_\pi^2) = 0$$

Meson masses at NLO – 2

- Self-energy contains two parts:
  π, K, η tadpole loops and counterterms in $\mathcal{L}^{(4)}$



Assuming isospin symmetry ($m_u = m_d = \hat{m}$), the pion mass at NLO:

$$M_\pi^2 = M^2 \left[1 + \frac{I(M^2)}{2F^2} - \frac{I(M_{\eta,2}^2)}{2F^2} + \frac{8M^2}{F^2}(2L_8 - L_5) + \frac{24M_{\eta,2}^2}{F^2}(2L_6 - L_4) \right]$$

here $M^2 = 2B\hat{m}, \;\; M_{\eta,2}^2 = \frac{2}{3}B(2\hat{m} + m_s)$

- Loop is divergent, in dimensional regularization (good for preserving symmetry)

$$I(M^2) = i\mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - M^2 + i\epsilon} = \frac{M^2}{16F^2}\left(\lambda + \log\frac{M^2}{\mu^2}\right)$$

$$\lambda = \frac{2}{d-4} - [\log(4\pi) + \Gamma'(1) + 1], \quad \text{divergent for } d = 4\,!$$

On the other hand, L_i's are divergent either $\quad L_i = L_i^r + \frac{\Gamma_i}{32\pi^2}\lambda$

$$\Gamma_4 = \frac{1}{8}, \quad \Gamma_5 = \frac{3}{8}, \quad \Gamma_6 = \frac{11}{144}, \quad \Gamma_8 = \frac{5}{48}$$

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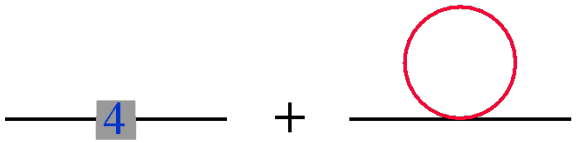

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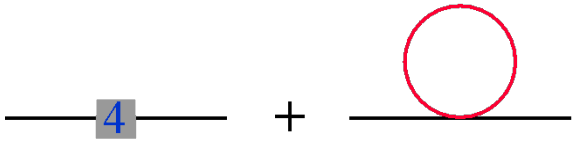

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- Renormalization: divergences from LECs and loops cancel out (as well as the μ-dependence)
  $\Rightarrow$ the pion mass is finite and μ-independent

$$M_\pi^2 = M^2 \left[1 + \frac{1}{32\pi^2 F^2} M^2 \log \frac{M^2}{\mu^2} - \frac{1}{96\pi^2 F^2} M_{\eta,2}^2 \log \frac{M_{\eta,2}^2}{\mu^2} + \frac{8M^2}{F^2}(2L_8^r - L_5^r) + \frac{24M_{\eta,2}^2}{F^2}(2L_6^r - L_4^r) \right]$$

- M_π^2 vanishes in chiral limit, loop correction does not generate a non-zero mass
- Chiral logarithm: non-analytic in the light quark masses

Quark mass ratios revisited – 1

- Calculate pion/kaon masses beyond leading order:

$$M_{\pi^+}^2 = B(m_u + m_d)\Big[1 + \mathcal{O}\left(\hat{m}, m_s\right)\Big]$$
$$M_{K^+}^2 = B(m_u + m_s)\Big[1 + \mathcal{O}\left(\hat{m}, m_s\right)\Big]$$

- Form dimensionless ratios:

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{m_u + m_d}\Big[1 + \Delta_M + \mathcal{O}\left(m_q^2\right)\Big]$$
$$\frac{(M_{K^0}^2 - M_{K^+}^2)_{\text{strong}}}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\Big[1 + \Delta_M + \mathcal{O}\left(m_q^2\right)\Big]$$
$$\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8^r - L_5^r) + \text{ chiral logs}$$

- Double ratio Q^2 particularly stable:

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{(M_{K^0}^2 - M_{K^+}^2)_{\text{strong}}}\Big[1 + \mathcal{O}\left(m_q^2\right)\Big]$$

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Quark mass ratios revisited – 2

- Leutwyler's ellipse: Leutwyler (1996)

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$

- Remember Dashen's theorem:

$$(M^2_{K^+} - M^2_{K^0})_{\text{em}} = (M^2_{\pi^+} - M^2_{\pi^0})_{\text{em}} + \mathcal{O}\left(e^2 m_q\right)$$

$$\boxed{Q_{\text{Dashen}} = 24.2}$$

- Value extracted from lattice simulations FLAG (2013)

$$N_f = 2+1: \quad 22.6 \pm 0.7 \pm 0.6$$
$$N_f = 2: \quad 24.3 \pm 1.4 \pm 0.6$$

$\pi\pi$ scattering beyond LO

- S-wave scattering lengths:

	a_0^0	a_0^2	
LO	0.16	-0.045	Weinberg (1966)
NLO (one-loop)	0.20 ± 0.01	-0.042 ± 0.002	Gasser, Leutwyler (1983)
NNLO (two-loop)	$0.217 \pm \ldots$	$-0.041 \pm \ldots$	Bijnens *et al.* (1996)
NNLO + Roy eq.	0.220 ± 0.005	-0.0444 ± 0.0010	Colangelo *et al.* (2001)

- Compare again with the modern data from NA48/2 (K_{e4} decays & $K^\pm \to \pi^0\pi^0\pi^\pm$) B.Bloch-Devaux, Kaon09

$$a_0^0 = \quad 0.2210 \pm 0.0047_{\text{stat}} \pm 0.0015_{\text{syst}}$$
$$a_0^2 = -0.0429 \pm 0.0044_{\text{stat}} \pm 0.0016_{\text{syst}}$$

Summary – 3

- Symmetries allow terms with more derivatives $\Rightarrow$ non-renormalizable in the traditional sense
- EFT has a power counting to organize terms in the Lagrangian.
  Lagrangian contains the most general terms allowed by symmetries at a given order, thus EFT can be renormalized order by order
- Scale separation: $p, M \ll \Lambda_\chi \sim 1$ GeV
  $\Rightarrow$ allows for an uncertainty estimate
  $\Rightarrow$ chiral Lagrangians should not be naively used for $p \gtrsim \Lambda_\chi$