An M-Theory Approach to Topological Strings

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[Alba Grassi, Yasuyuki Hatsuda and M.M., 1410.3382]

[Rinat Kashaev and M.M., 1501.01014]
[M.M. and Szabolcs Zakany, 1502.02958]
[Rinat Kashaev, M.M. and Szabolcs Zakany, 1505.02243]
[Jie Gu, Albrecht Klemm, M.M. and Jonas Reuter, 1506.09716]
[Santiago Codesido, Alba Grassi and M.M., 1507.02096]

[Yasuyuki Hatsuda and M.M., 1511.02860]
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Review: [M.M., 1506.07757]

Related work: Huang-Wang, Huang-Wang-Zhang, Hatsuda,
Okuyama-Zakany

M-theory and localization

In recent years, the combination of AdS4/CFT3 with localization techniques has led to many interesting results on string/M-theory.

The partition function of Chern-Simons-matter theories on the three-sphere provides a *non-perturbative definition* of the string theory partition function. The genus expansion is realized as the asymptotic 't Hooft expansion of a well-defined quantity

ABJM

$$\log Z(N,k) \sim \sum_{g\geq 0} F_g(\lambda) g_{\rm st}^{2g-2}$$

$$N \to \infty \qquad k \to \infty \qquad \lambda = \frac{N}{k} \quad \text{fixed} \qquad g_{\rm st} \sim \frac{1}{k}$$

In ABJM theory, thanks to localization, one can compute all the $F_g(\lambda)$ recursively, on the gauge theory side, and verify some predictions of AdS/CFT

$$F_0(\lambda) \sim \lambda^{3/2}, \quad \lambda \gg 1 \longrightarrow N^{3/2}$$ behavior

The free energies $F_g(\lambda)$ contain *worldsheet instanton corrections*.

In addition, the asymptotic genus expansion receives non-perturbative corrections in the string coupling constant, of the form [Drukker-M.M.-Putrov, Grassi-M.M.-Zakany]

$$\mathrm{e}^{-\sqrt{kN}} \sim \mathrm{e}^{-(L/\ell_p)^3}$$

which in M-theory are due to *membrane instantons*

M-theory expansion

The M-theory regime is

$$N \to \infty \quad \text{and} \quad k \quad \text{fixed}$$

The partition function in this regime has a Rademacher-type *convergent* expansion, which arises from *resumming* the perturbative genus expansion and *including membrane instanton corrections*

$$Z(N,k) = \sum_{p,q,n} a_{p,q,n} \left(-\frac{\partial}{\partial N}\right)^n \mathrm{Ai}\left[C\left(N + 2p + \frac{4q}{k}\right)\right]$$

membrane instanton

worldsheet instanton

$$= \mathrm{Ai}(CN) + \mathcal{O}\left(\mathrm{e}^{-\sqrt{kN}}, \mathrm{e}^{-\sqrt{N/k}}\right)$$

Fermi gas picture

Localization expresses the gauge theory partition function as a matrix integral. To derive the results above, we reformulate it as the canonical partition function for a one-dimensional *ideal* Fermi gas with N particles. For ABJM,

$$Z(N,k) = \int \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{2\pi k} \prod_{i<j} \left(\tanh\left(\frac{x_i - x_j}{2k} \right) \right)^2 \prod_{i=1}^{N} \frac{1}{\cosh\left(\frac{x_i}{2}\right)}$$

$$= \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \int \prod_{i=1}^{N} \mathrm{d}x_i \rho\left(x_i, x_{\sigma(i)}\right)$$

Fermi statistics

density matrix

$$\rho(x,x') = \frac{1}{2\pi k}\frac{1}{\left(2\cosh\left(\frac{x}{2}\right)\right)^{1/2}\left(2\cosh\left(\frac{x'}{2}\right)\right)^{1/2}}\frac{1}{2\cosh\left(\frac{x-x'}{2k}\right)}$$

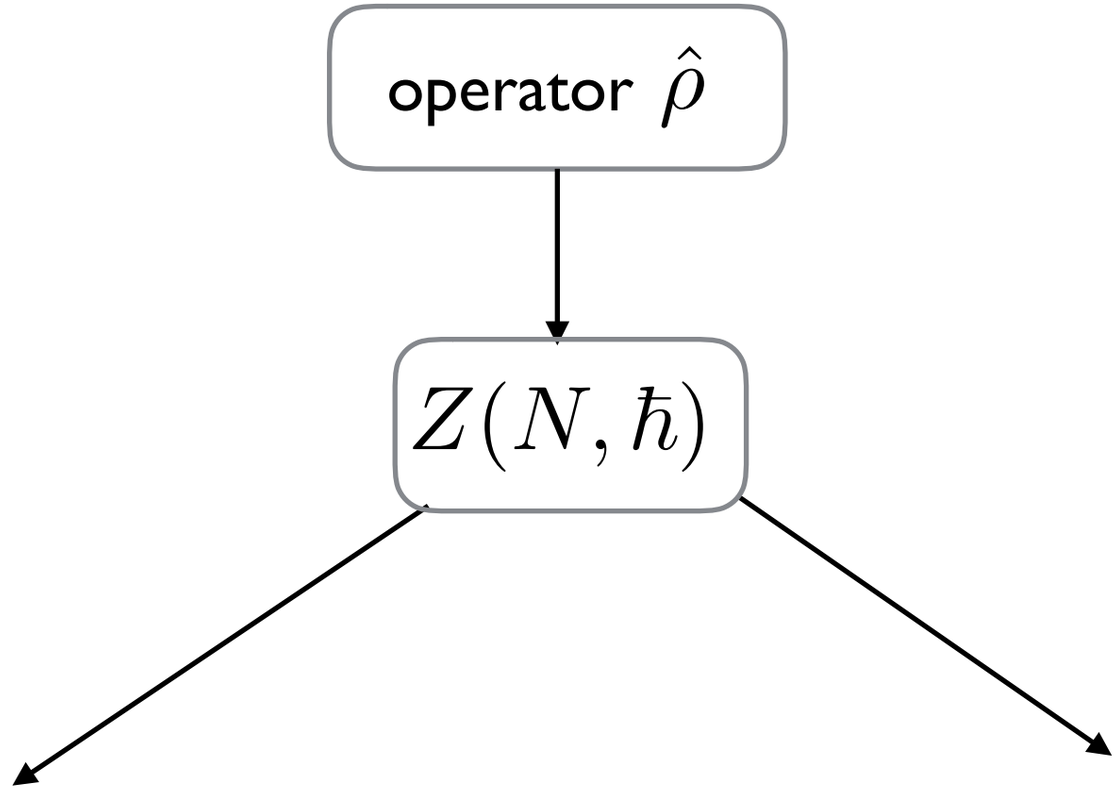
This integral kernel defines a non-trivial Hamiltonian in one dimension

$$\hat{\rho} = \mathrm{e}^{-\hat{H}} \qquad \text{and} \qquad \hbar = 2\pi k \sim \frac{1}{g_s}$$

The semiclassical regime of the Fermi gas is the strong string coupling regime of the superstring. In particular, the membrane instantons can be *computed* in the WKB approximation to the Fermi gas!

Mathematically, $\hat{\rho}$ is a positive-definite, trace class operator on the Hilbert space $L^2(\mathbb{R})$: it has a positive, discrete spectrum and all its traces are well-defined

$$\mathrm{tr}\,\hat{\rho}^n < \infty,\ \ n = 1, 2, \cdots$$



asymptotic 't Hooft expansion: superstring perturbation theory

$$N \to \infty \qquad \hbar \to \infty$$

$$F_g(\lambda)$$

$$\frac{N}{\hbar} = \lambda \qquad \hbar \sim \frac{1}{g_{\rm st}}$$

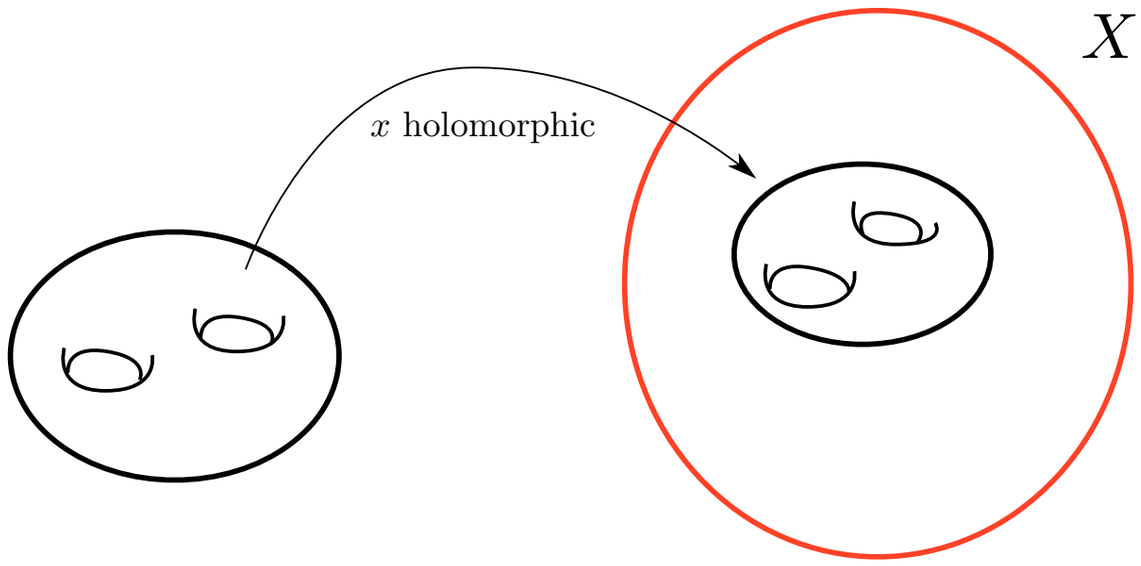
convergent M-theory expansion:
resummed strings
+membrane instantons

$$N \to \infty$$

$\hbar$ fixed

Topological string theory

A simplified version of string theory in which one "counts" worldsheet instantons, i.e. holomorphic maps from a Riemann surface of genus g to a given "target space" X, which will be a Calabi-Yau (CY) threefold



The genus g topological string free energy is given by a sum over worldsheet instantons, and depends on the Kahler moduli, which measure the size of curves in the CY

If X has a single modulus, i.e. a single topological class of worldsheet instantons, one has

$$F_g(t) = \sum_d N_{g,d} \mathrm{e}^{-dt}$$

Gromov-Witten invariants

If X is toric (hence non-compact), one can determine the free energies *at all genera* (e.g. topological vertex).

A powerful way of solving the topological string is *mirror symmetry*, which postulates that the counting of instantons can be solved by using a different, "mirror" manifold $\widehat{X}$. When the CY is toric, its mirror is just an algebraic *mirror curve* or Riemann surface, of the form

$$W_X(\mathrm{e}^x, \mathrm{e}^y) = 0$$

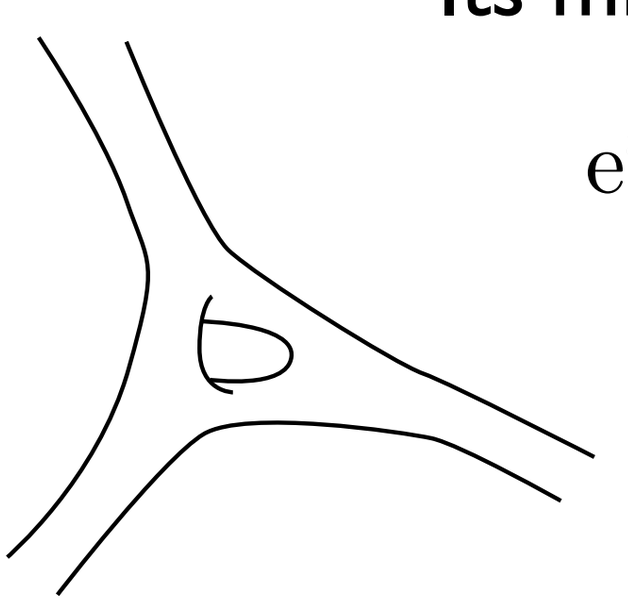
An example

The simplest non-trivial toric CY is a monopole bundle on the complex projective space

$$\mathcal{O}(-3) \to \mathbb{P}^2$$

which is usually called local $\mathbb{P}^2$

Its mirror is described by the curve



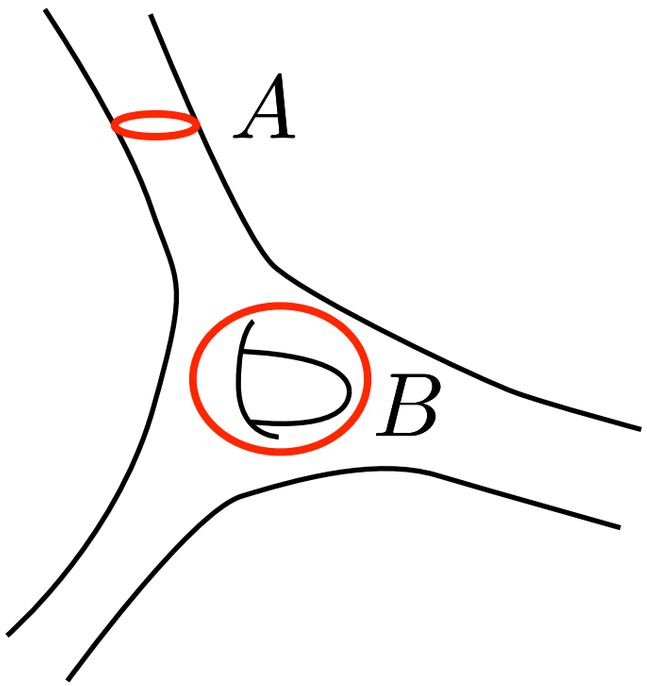
$$\mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{-x-y} + z^{-1/3} = 0$$

a curve of genus one!

parametrizes the complex structure of the mirror

For simplicity I restrict to curves of genus one, but everything I will say has a higher genus extension [Codesido-Grassi-M.M.]

The main statement of mirror symmetry in this case is that the genus zero free energy can be computed by just doing integrals on the curve!



$$t = \oint_A y(x)\mathrm{d}x$$

mirror map

$$\frac{\partial F_0}{\partial t} = \oint_B y(x)\mathrm{d}x$$

$$F_0(t) = Ct^3 + 3\mathrm{e}^{-t} + \cdots$$

As in the case of strings with CFT duals, we would like to realize the perturbative series of topological string theory on *X* as the asymptotic expansion of a *well-defined* partition function $Z_X(N,\hbar)$, in the 't Hooft regime

$$N \to \infty \qquad \frac{N}{\hbar} = \text{fixed} \qquad \hbar \sim \frac{1}{g_{\text{st}}}$$
$$\hbar \to \infty$$

This is suggested by the close relationship between the *1/N* expansion of the partition function of ABJM theory, and the topological string free energies on a particular toric CY [M.M.-Putrov]

operator ρ_X

?

$Z_X(N,\hbar)$

?

asymptotic 't Hooft expansion: topological string free energies

$$N \to \infty \qquad \hbar \to \infty$$

$$F_g(\lambda)$$

$$\frac{N}{\hbar} = \lambda \qquad \hbar \sim \frac{1}{g_{\rm st}}$$

convergent M-theory expansion

?

$$N \to \infty$$

$$\hbar \quad \text{fixed}$$

Operators from mirror curves

It turns out that one can find the operator ρ_X that corresponds to a given toric CY *X* by *quantizing its mirror curve*!

Main idea: promote the variables *x*, *y* appearing in the mirror curve to canonically conjugate Heisenberg operators

$$[\mathrm{x}, \mathrm{y}] = \mathrm{i}\hbar \qquad\qquad \hbar > 0$$

Each mirror curve gives then an operator

$$W_X(\mathrm{e}^x, \mathrm{e}^y) \to \mathsf{O}_X$$

For example, for local $\mathbb{P}^2$ we find

$$\mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{-x-y} + z^{-1/3} \to \quad \mathsf{O} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{x}-\mathsf{y}}$$

Theorem
[Kashaev-M.M.]

The operator $\rho_X = \mathsf{O}_X^{-1}$ on $L^2(\mathbb{R})$ is positive definite and of trace class

One can then regard this operator as a QM density matrix and consider the corresponding Fermi gas partition function

$$Z_X(N,\hbar)$$

Conjecture [Grassi-Hatsuda-M.M.]: in the 't Hooft regime

$$\log\, Z_X(N,\hbar) \sim \sum_{g=0}^{\infty} F_g(\lambda)\hbar^{2-2g}$$

cognoscendi: conifold frame

Notice that this is the *strongly coupled regime* $\hbar \gg 1$ of the Fermi gas. The expansion in the weakly coupled regime $\hbar \ll 1$ involves a different topological string theory [NS, Mironov-Morozov, ACDKV]: the NS limit of the refined topological string

Evidence for the conjecture, I

The canonical partition functions of the Fermi gas are *well-defined* and *calculable* directly in operator theory. For example, one has

local $\mathbb{P}^2$

$$Z(1, \hbar = 2\pi) = \frac{1}{9}$$

$$Z(2, \hbar = 2\pi) = \frac{1}{12\sqrt{3}\pi} - \frac{1}{81}$$

In fact, in some cases one can compute them explicitly in terms of *matrix integrals*! These are the analogues of the localized partition functions on the sphere in Chern-Simons-matter theories

For local $\mathbb{P}^2$ and its generalizations, the matrix model is an *O(2)* matrix model

$$Z_X(N,\hbar) = \frac{1}{N!}\int_{\mathbb{R}^N} \frac{\mathrm{d}^N u}{(2\pi)^N} \prod_{i=1}^{N} \mathrm{e}^{-\frac{1}{g_s}V(u_i,g_s)} \frac{\prod_{i<j} 4\sinh\left(\frac{u_i-u_j}{2}\right)^2}{\prod_{i,j} 2\cosh\left(\frac{u_i-u_j}{2}+\mathrm{i}\pi C\right)}.$$

$g_s \sim \hbar^{-1}$

known potential

Its 't Hooft expansion can be computed with large *N* techniques and one can test the conjecture order by order in $\hbar^{-1}$ and λ [M.M.-Zakany, Kashaev-M.M.-Zakany].

For example, one finds, directly from the matrix model

$$F_0(\lambda) = \frac{\lambda^2}{2}\left(\log\left(\frac{4\pi^2\lambda}{9\sqrt{3}}\right) - \frac{3}{2}\right) - \frac{3V}{4\pi^2}\lambda - \frac{\pi^2}{9\sqrt{3}}\lambda^3 + \frac{\pi^4}{486}\lambda^4 + \frac{56\pi^6}{10935\sqrt{3}}\lambda^5 + \cdots$$

$$V = 2\,\mathrm{Im}\left(\mathrm{Li}_2\left(\mathrm{e}^{\pi \mathrm{i}/3}\right)\right)$$

This agrees with known results from topological strings
[Haghighat-Klemm-Rauch]

In particular, we get a matrix model realization of the topological string on *any* toric CY geometry!

The M-theory expansion

Can we find the *exact* expansion of the partition function at large *N* but fixed string coupling constant? This requires the *total grand potential of the CY X* [Hatsuda-M.M.-Moriyama-Okuyama]

$$J_X(\mu,\hbar) = J^{\mathrm{WKB}}(\mu,\hbar) + J^{\mathrm{WS}}(\mu,\hbar)$$

Grand potential of the Fermi gas in the all-orders WKB approximation. It can be expressed in terms of the NS limit of the topological string.

Series in $\mathrm{e}^{-r\mu}$

Expressed in terms of the Gopakumar-Vafa resummation of the worldsheet instantons.

Series in $\mathrm{e}^{-2\pi r\mu/\hbar}$

μ fugacity of the Fermi gas, related to the modulus of the CY

The total grand potential can be computed from the BPS invariants of X and is a well defined function if μ is sufficiently large (large radius)

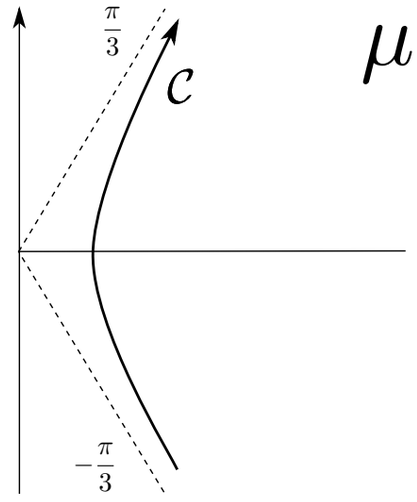
From the point of view of standard TS theory, the WKB grand potential provides a non-perturbative correction in the string coupling constant. However, from the point of view of the QM problem, the WS contribution is a non-perturbative effect in $\hbar$.

Note that the GV resummation (which is equivalent to topological vertex/instanton counting in 5d) has been presented very often as the M-theory version of the topological string. However, it does *not* give by itself a well-defined function. It has poles which have to be cancelled by the WKB part (similar to the HMO mechanism in ABJM theory).

Our exact M-theoretic formula for the partition function is

$$Z_X(N,\hbar) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^{J_X(\mu,\hbar) - N\mu} \mathrm{d}\mu$$

Airy contour



This formula implies our previous claim for the asymptotic expansion, since the 't Hooft limit keeps only the standard topological string. It also leads to a convergent expansion in terms of Airy functions, as in ABJM theory (and to a $N^{3/2}$ scaling)

local $\mathbb{P}^2$

$$Z(N,\hbar) = \sum_{p,q,n} a_{p,q,n} \left(-\frac{\partial}{\partial N}\right)^n \mathrm{Ai}\left[C\left(N + 3p + \frac{6\pi q}{\hbar}\right)\right]$$

Evidence for the conjecture, 2

To test the exact expression for the partition function as an convergent series of instanton corrections, we compare partial sums of the series to known exact values

local $\mathbb{P}^2$ $$Z(1,6\pi) = -\frac{2}{27} - \frac{1}{9\sqrt{3}}\sin\left(\frac{\pi}{9}\right) + \frac{1}{9}\cos\left(\frac{\pi}{9}\right)$$

up to $e^{-d\mu}$ with	$Z(1,6\pi)\cdot 10^3$
$d=0$	**8.** 403...
$d=2$	**8.39561470** 120...
$d=4$	**8.39561470021027651**1 486...
$d=6$	**8.39561470021027651154265956252** 243...
$d=8$	**8.39561470021027651154265956252194562493213418137** 290...
exact value	**8.39561470021027651154265956252194562493213418137**

from [Okuyama-Zakany]

Comments

We have found a reformulation (and non-perturbative completion) of topological string theory on *any* toric CY, in terms of a QM problem in one dimension. Conversely, we can conjecturally *solve* this QM problem by using topological string data. Therefore, we have a new, precise and testable connection between spectral theory and topological strings.

There might be other, potentially different, non-perturbative completions of the topological string. For example, for some geometries there is a large N duality with Chern-Simons theory [GV, AKMV].

An application to integrable systems

Nekrasov and Shatashvili proposed in 2009 a powerful approach to quantum integrable systems based on instanton counting in susy gauge theories. They proposed in particular exact quantization conditions for a variety of integrable systems, based on the NS free energy.

Some of these integrable systems have a realization in terms of toric CY manifolds, e.g. the *relativistic* Toda lattice. It is clear from our analysis that the NS free energy is *not* enough to solve them: one should take into account non-perturbative corrections encoded in the *standard* TS free energy.

Recently, Goncharov and Kenyon (GK) constructed an integrable system for *any* toric CY. When the mirror curve has genus one, the GK system has one single operator, which agrees with our O_X .

Using our results and their reformulation in [Wang-Zhang-Huang] one can find a truly exact quantization condition for the general GK system (including relativistic Toda) [Hatsuda-M.M., Franco-Hatsuda-M.M.], which shows an S-duality structure in the Planck constant:

$$\frac{\partial F^{\rm NS}}{\partial t}(t(\hbar),\hbar)+\frac{\hbar}{2\pi}\frac{\partial F^{\rm NS}}{\partial t}\left(\frac{2\pi}{\hbar}t(\hbar),\frac{4\pi^2}{\hbar}\right)=2\pi\hbar\left(n+\frac{1}{2}\right)$$

standard NS

non-perturbative correction

Open questions

1) We have found a matrix model for each toric CY. Is this the localized partition function of a full gauge theory? Do we have a theory of M2 branes for every toric CY? So far, only in some very special cases (cf [Hatsuda, ABJM on ellipsoid]).

2) What is the spacetime physics of our non-perturbative M-theoretic completion? What is the geometric interpretation of the non-perturbative corrections we have found? Are they membranes?

3) We have conjectured a precise relation between the enumerative geometry of topological strings, and the spectral theory of trace class operators. Can one prove this conjecture?