On the metric theory of gravity
or
A conformal model for gravitons

I) What I will not talk (extensively) about - past work

Phenomenology of emergent symmetry
- power and log symmetry violations
- gauge symmetry violations

II) What I am most interested in discussing (incomplete)

Origin of metric theory of Einstein gravity
- treat spin connection as an independent field
- spin connection asymptotically free → confined?

A conformal model of gravity
- Einstein action from dimensional transmutation

John Donoghue
CERN workshop
June 20, 2016

I. Phenomenology of emergent symmetry

Emergent phenomena are ubiquitous is science

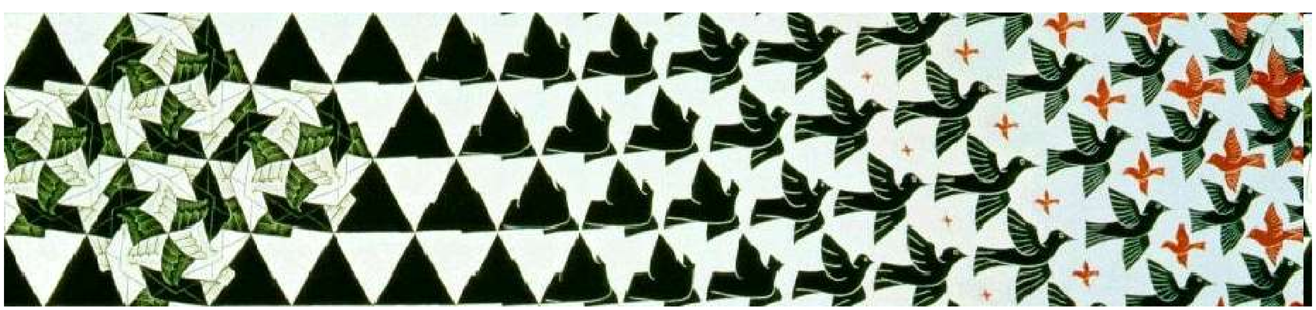
But less so in particle physics…. Ever increasing symmetry

Perhaps we should look for fundamental theories with <u>less</u> symmetry

Can our fields and symmetries (i.e. those of SM and GR) be emergent?

What are the potential consequences?

Work with Mohamed Aber, Ufuk Aydemir and Basem El-Menoufi



Emergent fields: Waves from interacting masses

Take a series of masses interacting with neighbors:

$$S = \int dt L[y_i, \dot{y}_i] = \int dt \sum_i [\frac{1}{2} m \dot{y}_i^2 - V(y_i - y_{i-1})] \approx \int dt \sum_i [\frac{1}{2} m \dot{y}_i^2 - \frac{1}{2} k (y_i - y_{i-1})^2]$$

Go to the continuum limit:

$$y_j(t) \equiv \sqrt{\frac{1}{ka}} \phi(x,t) \qquad x = aj$$

Get a field satisfying the wave eq. (= massless 1D Klein Gordon equation)

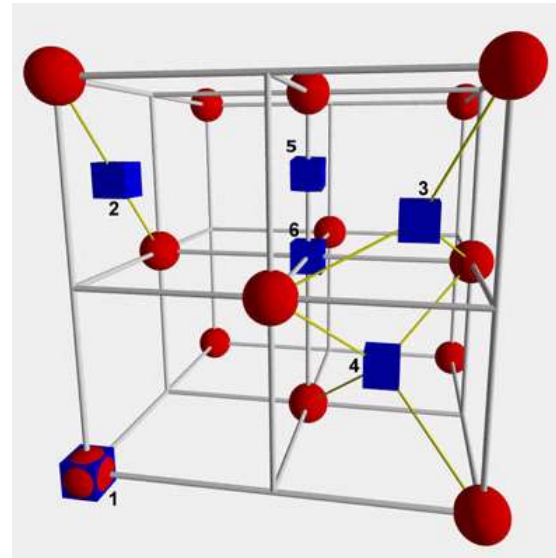
$$S = \int dx dt \frac{1}{2} [\frac{1}{v^2} (\frac{\partial \phi}{\partial t})^2 - (\frac{\partial \phi}{\partial x})^2] = \int dx dt \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$$

with

$$\partial_\mu = (\frac{1}{v_s} \frac{\partial}{\partial t}, -\frac{\partial}{\partial x}) \qquad v = \sqrt{\frac{ka^2}{m}}$$

Emergent symmetry

Three emergent symmetries in phonon/string examples:



1) **Translation symmetry**

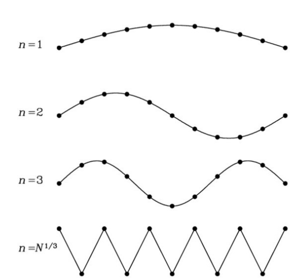
$$x \rightarrow x + c$$

2) **Lorentz-like symmetry**

$$L = \int d^3x [\frac{1}{2}\partial_\mu \phi \partial^\mu \phi]$$

leads to extra invariance

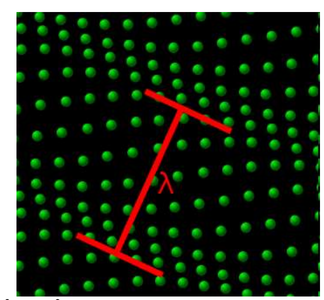
$$x'^\mu = \Lambda^\mu_\nu x^\nu$$



3) **Shift symmetry:**

- why the massless wave equation?

shift symmetry $\phi \rightarrow \phi + c$



corresponds to translating the overall system

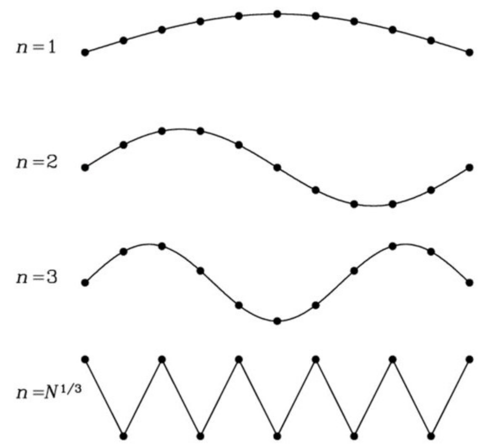
-no cost in energy

These are not symmetries of the original system but emerge in continuum limit

Key to phenomeonlogy: violation of emergent symmetry

In examples of emergence: strings and phonons

1) **Translation invariance violated at small scales**



2) **Waves do not exist at small wavelength**
Emergent DOF no longer exist

3) **Next order in L is not Lorentz invariant:**

$$V(y_i - y_{i-1}) = \frac{1}{2}k(y_i - y_{i-1})^2 + \frac{1}{4}\lambda(y_i - y_{i-1})^4 + \ldots.$$

Then there is a new term in the action without Lorentz-like symmetry

$$S = \int dxdt \frac{1}{2}[\frac{1}{v^2}(\frac{\partial \phi}{\partial t})^2 - (\frac{\partial \phi}{\partial x})^2 + \bar{\lambda}(\frac{\partial \phi}{\partial x})^4]$$

These are generic features of an emergent symmetry

Symmetry is not forever

But the symmetry violation can be logarithmic also

With Mohamed Anber

Eg. Emergence of a common "speed of light"?

- without Lorentz invariance, different limiting velocities are the norm
- in coupled system, do speeds evolve towards each other at low energy? RGE

$$\mathcal{L}_0 = ic_f\bar{\psi}_r\gamma^0\partial_0\psi_r - i\bar{\psi}_r\vec{\gamma}\cdot\vec{\partial}\psi_r + \frac{1}{2}c_b^2\partial_0\phi_r\partial^0\phi_r - \frac{1}{2}\vec{\partial}\phi_r\cdot\vec{\partial}\phi_r - g\phi_r\bar{\psi}_r\psi_r\,,$$

Calculate self energies and renormalize:

$$\beta(g) = \frac{g^3\left(14c_fc_b + 13c_f^2 + 3c_b^2\right)}{24\pi^2 c_f(c_f+c_b)^2}\,,$$

$$\beta(c_b) = \frac{g^2\left(c_b^2 - c_f^2\right)}{8\pi^2 c_f c_b}\,,$$

$$\beta(c_f) = \frac{g^2 c_f(c_f - c_b)}{6\pi^2(c_f+c_b)^2}\,.$$

For the ratio $r=c_f/c_b$:

$$\beta(r) = \frac{\beta(c_f)}{c_b} - \frac{1}{c_b r}\beta(c_b) = \frac{g^2}{24\pi^2}\frac{(r-1)\left[4r+3(1+r)^3\right]}{c_b(1+r)^2}$$

IR fixed point at $c_f = c_b$

Comments:

The running is weak – emergence scale distant

$$\left(\frac{c_f}{c_b}-1\right)_m \sim \left(\frac{c_f}{c_b}-1\right)_\Lambda \left[\frac{m}{\Lambda}\right]^{\alpha/\pi}$$

Should gravity share the same speed as light? Gravity waves as test(C. Will)

Violations of general covariance:

Anber, Aydemir, JFD

Gravity is BEST for testing gauge violations
- gravity is already suppressed by M_P^2
- violations also suppressed, but can stand out more

Non-covariant terms in Lagrangian
- treat $g_{\mu\nu}$ as basic field

Bjorken

At zero derivatives, can only have $V(\sqrt{g})$

Ground state condition $V'(\sqrt{g}) = 0$

Equivalent to unimodular gravity
- Einstein eq with Λ as an integration constant

Henneaux and Teitelboim

Proceed to higher orders in the derivative expansion

(Pauli-Fierz mass also violates covariance, but not with $g_{\mu\nu}$)

Terms with two derivatives

-adding non-covariant terms to the actions
-the tightest constraints come from nonlinear analysis - use full metric

$$\mathcal{L} = \mathcal{L}_{EH} + \sum_{i=1}^{7} a_i \mathcal{L}_i + \mathcal{L}_m \, .$$

with

$$\begin{aligned}
\mathcal{L}_1 &= -\, g^{\mu\nu}\Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha} \, , \quad \mathcal{L}_2 = -\, g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\lambda\alpha} \\
\mathcal{L}_3 &= -\, g^{\alpha\gamma} g^{\beta\rho} g_{\mu\nu}\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\gamma\rho} \, , \quad \mathcal{L}_4 = -\, g^{\alpha\gamma} g_{\beta\lambda} g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}\Gamma^{\beta}_{\gamma\alpha} \\
\mathcal{L}_5 &= -g^{\alpha\beta}\Gamma^{\lambda}_{\lambda\alpha}\Gamma^{\mu}_{\mu\beta} \, , \quad \mathcal{L}_6 = -g^{\mu\nu}\partial_\nu \Gamma^{\lambda}_{\mu\lambda} \\
\mathcal{L}_7 &= -g^{\mu\nu}\partial_\lambda \Gamma^{\lambda}_{\mu\nu} \, ,
\end{aligned} \qquad (3)$$

Parameterized Post-Newtonian (PPN) expansion:

- general expansion of metric theories around Newtonian limit

$$U \sim v^2 \sim p/\rho \sim \Pi \sim \mathcal{O}(2)$$

Expansion of equations of motion:

$$\begin{aligned} g_{00} &= -1 + \overset{(2)}{g}_{00} + \overset{(4)}{g}_{00} + \ldots, \quad g_{ij} = \delta_{ij} + \overset{(2)}{g}_{ij} + \overset{(2)}{g}_{ij} + \ldots \\ g_{0i} &= \overset{(3)}{g}_{0i} + \overset{(5)}{g}_{0i} + \ldots, \end{aligned} \tag{29}$$

Parameterized form used to test alternative theories of gravity Will

$$\begin{aligned} g_{00} = &-1 + 2U - 2\beta U^2 - 2\xi\Phi_W + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 + 2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 \\ &+2(1+\zeta_3)\Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4(\zeta_1 - 2\xi)A - (\alpha_1 - \alpha_2 - \alpha_3)w^i w^i U \\ &-\alpha_2 w^i w^j U_{ij} + (2\alpha_3 - \alpha_1)w^i V_i \\ g_{0i} = &-\tfrac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\eta)V_i - \tfrac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_i - \tfrac{1}{2}(\alpha_1 - 2\alpha_2)w^i U - \alpha_2 w^j U_{ij} \\ g_{ij} = &(1 + 2\gamma U)\delta_{ij} \end{aligned}$$

$$\begin{aligned} U &= \int d^3x' \frac{\rho'}{|\vec{x} - \vec{x}'|} \\ U_{ij} &= \int d^3x' \frac{\rho' (x - x')_i (x - x')_j}{|\vec{x} - \vec{x}'|^3} \\ V_i &= \int d^3x' \frac{\rho' v'_i}{|\vec{x} - \vec{x}'|} \\ W_i &= \int d^3x' \frac{\rho' \vec{v}' \cdot (\vec{x} - \vec{x}')(x - x')_i}{|\vec{x} - \vec{x}'|^3} \\ \Phi_1 &= \int d^3x' \frac{\rho' v'^2}{|\vec{x} - \vec{x}'|}, \Phi_2 = \int d^3x' \frac{\rho' U'}{|\vec{x} - \vec{x}'|} \\ \Phi_3 &= \int d^3x' \frac{\rho' \Pi'}{|\vec{x} - \vec{x}'|}, \Phi_4 = \int d^3x' \frac{p'}{|\vec{x} - \vec{x}'|} \\ \mathcal{A} &= \int d^3x' \frac{\rho' [\vec{v}' \cdot (\vec{x} - \vec{x}')]^2}{|\vec{x} - \vec{x}'|^3} \\ \mathcal{B} &= \int d^3x' \frac{\rho' d\vec{v}'/dt \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|} \\ \Phi_W &= \int d^3x' \rho' \rho'' \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \cdot \left(\frac{\vec{x}' - \vec{x}''}{|\vec{x} - \vec{x}''|} - \frac{\vec{x} - \vec{x}''}{|\vec{x}' - \vec{x}''|} \right). \end{aligned}$$

All theories map onto a set of coefficients:

-phenomenology provides bounds on parameters

Very strong constraint emerges

parameter	value	effect	limit
$\gamma - 1$	-3a	time delay	2.3×10^{-5}
		light deflection	4×10^{-4}
$\beta - 1$	$-\frac{85}{32}a$	perihelion shift	3×10^{-3}
		Nordtvedt effect	2.3×10^{-4}
ξ	$\frac{3}{8}a$	earth tides	10^{-3}
α_1	0	orbital polarization	10^{-4}
α_2	0	orbital polarization	4×10^{-7}
α_3	$\frac{13}{8}a$	orbital polarization	4×10^{-20}
ζ_1	$\frac{39}{8}a$	—	2×10^{-2}
ζ_2	$-\frac{179}{16}a$	binary acceleration	4×10^{-5}
ζ_3	$-a$	Newtons 3rd law	10×10^{-8}
ζ_4	$\frac{5}{8}a$	—	—

TABLE I: The values and limits on the PPN parameters [14].

Strongest constraint from rotating binary pulsars (Damour)

constraint $a \sim 10^{-20}$

If interpreted as a mass scale:

$M_E \sim 10^{10}\, M_{Pl}$

Also QED gauge invariance test

With
Basem El-Menoufi

Gauge non-invariant interaction:

$$\mathcal{L}_{gv} = -\frac{1}{8}\kappa A_\mu A^\mu A_\nu A^\nu$$

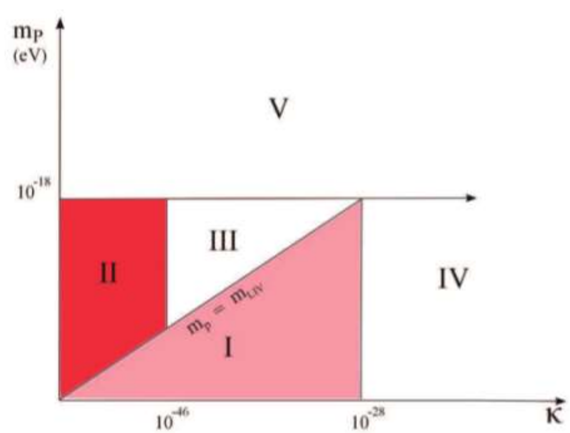
Generates cross-section which blows up in IR

$$\sigma = \frac{29\kappa^2}{120\pi s}$$

Also generates a mass from moving through the CMB

$$\omega^2 - \vec{k}^2 = \frac{5}{3}\kappa I \quad \text{with} \quad I = \int \frac{d^3p}{(2\pi)^3 E_p} n_B(E_p) = \frac{\zeta(2)}{2\pi^2}\beta^{-2}$$

Very tight constraints:



Summary of what I am not talking about

Symmetries can be emergent

Signal can be found in small violations of symmetries

Violations can be higher dimension, or due to log running

Gauge symmetry violation has some strong constraints

II. On the metric theory of Einstein gravity

Setting: In construction of GR with fermions, naturally have two fields

-vierbein (tetrad) e_μ^a and spin connection ω_μ^{ab}

- ω_μ^{ab} appears naturally as a gauge field

Recover GR only by extra assumption – metricity for vierbein

$$\nabla_\mu e_\nu^a = 0 = \partial_\mu e_\nu^a + \omega^a{}_{b\mu} e_\nu^b - \Gamma_{\mu\nu}{}^\lambda e_\lambda^a$$

Removes ω_μ^{ab} as independent field

$$\omega_\mu^{ab}(x) = e^{a\nu}(\partial_\mu e_\nu^b - \Gamma_{\mu\nu}{}^\lambda e_\lambda^b)$$

What if we do not assume metricity?

Explorations:

1) With usual gauge action, spin connection is asymptotically free

2) Is the spin connection confined (or condensed, gapped)
   - would yield metric theory without extra assumption

3) In scale invariant theory for ω_μ^{ab} , dimensional transmutation
   will give Einstein-Hilbert action

4) With conformally invariant theory for ω_μ^{ab}, richer set of invariants
   → conformal model for gravitons

Whenever you move in some direction in GR, there are always others ahead of you

Important work done by:

Schwinger
Utiyama
Kibble
Mansouri MacDowell
DeWitt
Stelle
‘t Hooft
Fradkin Tseytlin
Smilga
Holdom and Ren
Mannhiem
Salvio and Strumia
Lu , Perkins, Pope, Stelle
And many others

Quick review: Vierbein and spin connection

From Equivalence Principle one can write the metric in terms of vierbein variables

$$g_{\mu\nu}(x) = \eta_{ab}\ e^a_\mu(x)\ e^b_\nu(x)$$

In addition to general covariance

$$e'^a_\mu = \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu$$

there is an extra **local Lorentz** symmetry

$$e'^a(x) = \Lambda^a{}_c(x)\ e^c(x) \quad \text{with} \quad \eta_{ab}\ \Lambda^a{}_c(x)\ \Lambda^b{}_d(x) = \eta_{cd}$$

For scalars, this feature is irrelevant. But for fermions, it is important

$$\mathcal{L} = \bar{\psi}[i\gamma^a e^\mu_a(x)\partial_\mu + ...]\psi$$

To include the local Lorentz symmetry

$$\psi \to \psi'(x') = S(x)\psi(x)$$

where

$$S(x) = \exp\left(\frac{-i}{2} J_{ab} \alpha^{ab}(x)\right) \quad , \quad J_{ab} = \frac{\sigma_{ab}}{2} \quad \text{with} \quad \sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b] \ .$$

To include this, need spin connection and gauge covariant derivative

$$\mathcal{L} = \bar{\psi}[i\gamma^a e_a^\mu(x) D_\mu]\psi$$

$$D_\mu = \partial_\mu - ig\frac{J_{ab}}{2}\omega_\mu^{ab} \equiv \partial_\mu - ig\boldsymbol{\omega}_\mu$$

with gauge transformation

$$\boldsymbol{\omega}'_\mu = S\boldsymbol{\omega}_\mu S^{-1} - \frac{2i}{g}(\partial_\mu S)S^{-1}$$

$$S^{-1}(x)\gamma^a S(x)\Lambda_a^b(x) = \gamma^b$$

$$e_a^{\mu'} = \Lambda_a^b(x) e_b^\mu$$

Relation to GR:

- at this stage we have two fields

- field strength tensor

$$[D_\mu, D_\nu] = -ig\frac{J_{ab}}{2}R^{ab}_{\mu\nu}$$

$$R^{ab}_{\mu\nu} = \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + g(\omega^{ac}_\mu\ \omega_{\nu c}{}^b - \omega^{ac}_\nu\ \omega_{\mu c}{}^b)$$

Impose metricity (or first order formalism) (g absorbed here)

$$\nabla_\mu e^a_\nu = 0 = \partial_\mu e^a_\nu + \omega^a{}_{b\mu}e^b_\nu - \Gamma_{\mu\nu}{}^\lambda e^a_\lambda$$

Obtain GR with Riemann tensor

$$R_{\mu\nu\alpha\beta} = e_{a\alpha}e_{b\beta}R^{ab}_{\mu\nu}$$

Notation:

It is often useful to define different notations for various compbinations of derivatives and connections. First let us define the simple partial derivatives:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \partial_a \equiv e_a^\mu \partial_\mu \tag{52}$$

Next, it is often useful to define the derivative which includes only the spin connection:

$$d_\mu \equiv \partial_\mu - i\frac{g}{2} J_{ab}\omega_\mu^{ab} \qquad d_a \equiv e_a^\mu d_\mu \qquad (5$$

This has various forms depending on the object that is being acted on. For a scalar

$$d_\mu \phi = \partial_\mu \phi \qquad (5$$

while for a spinor

$$d_\mu \psi = \left[\partial_\mu - i\frac{g}{2} J_{ab}\omega_\mu^{ab}\right]\psi \qquad \text{with} \quad J_{ab} = \frac{1}{2}\sigma_{ab} \qquad (5$$

and for a Lorentz vector

$$d_\mu A^a = \partial_\mu A^a + g\omega_\mu^{a}{}_{b} A^b \qquad (5$$

We also define the fully covariant derivative, which involves both ω_μ^{ab} and $\Gamma_{\mu\nu}{}^{\lambda}$ in the usual ways. In particular tl
metricity condition displays this covariant derivative

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a + g\omega_\mu^{a}{}_{b} e_\nu^b - \Gamma_{\mu\nu}{}^{\lambda} e_\lambda^a \qquad (5$$

Asymptotic Freedom:

Consider usual gauge Lagrangian

$$\mathcal{L} = -\frac{1}{4} R^{ab}_{\mu\nu} R^{\mu\nu}_{ab}$$

This has SO(3,1) gauge symmetry (**non-compact**)

$$[J_{ab}, J_{cd}] = i\left(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}\right)$$

which can be repackaged in more usual gauge notation

$$[J_{ab}, J_{cd}] = 2i f_{[ab][cd][ef]} J^{ef}$$

$$\begin{aligned} f_{[ab][cd][ef]} &= -\frac{1}{4}\left[\eta_{bc}\eta_{de}\eta_{fa} - \eta_{bd}\eta_{ce}\eta_{fa} - \eta_{bc}\eta_{df}\eta_{ea} + \eta_{bd}\eta_{cf}\eta_{ea}\right. \\ &\qquad \left. - \eta_{bca}\eta_{de}\eta_{fb} + \eta_{ad}\eta_{ce}\eta_{fb} + \eta_{ac}\eta_{df}\eta_{eb} - \eta_{ad}\eta_{cf}\eta_{eb}\right] \\ &\equiv 2\eta_{b][c}\eta_{d][e}\eta_{f][a} \end{aligned}$$

and

$$R^{[ab]}_{\mu\nu} = \partial_\mu \omega^{[ab]}_\nu - \partial_\nu \omega^{[ab]}_\mu + g f^{[ab]}{}_{[cd][ef]} \omega^{[cd]}_\mu \omega^{[ef]}_\nu$$

Gauge loops then proceed in the usual way, with substitution

$$f_{imn} f_{jmn} = C_2 \delta_{ij} \qquad \longrightarrow \qquad f_{[ab][cd][ef]} f^{[gh][cd][ef]} = C_2 \delta^{[gh]}_{[ab]}$$

with

$$C_2 = 2.$$

This then yields the beta function

$$\beta(g) \; = \; -\frac{11 C_2}{3} \frac{g^3}{16\pi^2} \; = \; -\frac{22}{3} \frac{g^3}{16\pi^2}$$

Note: Fermion loops do not contribute to this coupling. **Return to this later**

Confined, condensed, gapped?

Spin connection weakly coupled in UV

Strongly coupled in IR

Running defines a scale – perhaps M_P

Analogies would suggest confinement, but non-compact group?

Singlet channel is attractive, then perhaps condensation

Assume spin connection is not propagating at low energy
- **then symmetry must be realized with metric only**
- explains metric theory without need to assume metricity of vierbein

Should be able to be answered by lattice work

Note: Smilga and Holdom + Ren have suggested confinement for the metric field

What happens at low energy?

Lattice version:

Lattice link variable:

$$U_\mu = \exp\left[\frac{a}{2} J_{ab} \omega_\mu^{ab}\right]$$

And action defined on a plaquette

$$S_W = -\sum_p \frac{2}{g_0^2} \mathbf{Re}(\mathbf{Tr}(U(p))).$$

Confinement test would be area law for Wilson loop

Analogy – Two flavor massless QCD

Theory is weakly coupled in both UV and IR

Massless QCD is classically scale invariant, yet running coupling defines QCD scale

UV story is well known – asymptotic freedom

As we come down in energy – strong coupling region 2 GeV to 0.5 GeV

But at low energy, the chiral symmetry requires massless degrees of freedom
- organized as an effective field theory

$$\mathcal{L} = \frac{F^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) \qquad \text{with} \qquad U = exp\left[\frac{i\tau \cdot \pi}{F}\right]$$

This is weakly coupled in the IR
- explicitly depends on QCD scale
- going up in energy enters the strong coupling region

If we had uncovered pionic theory first, we would think that there was an impassable barrier at 1 GeV.

Can we do the same thing with gravity and the spin connection?

Start with scale/conformal invariant action
Running coupling defines the Planck scale
Confine/gap the spin connection
Low energy theory is EFT for the metric - using dimensional transmutation for the scale

$$S = \int d^4x \sqrt{-g} \left[-\Lambda - \frac{2}{\kappa^2} R + ... \right]$$

Preliminary version: scale invariant

$$\mathcal{L} = \frac{1}{4g^2} R^{ab}_{\mu\nu} R^{\mu\nu}_{ab} = \frac{1}{4g^2} g^{\mu\alpha} g_{\nu\beta} R^{ab}_{\mu\nu} R^{\alpha\beta}_{ab}$$

Induced effects:

The following is not a real calculation in strongly coupled theory
- but can illustrate nature of effects

Consider heat kernel evaluation of functional determinant:

$$\det \mathcal{D} = e^{\mathrm{tr}\, \ln \mathcal{D}} = e^{\int d^4x \;\; \mathrm{Tr}\, \langle x | \ln \mathcal{D} | x \rangle}$$

$$\langle x | \ln \mathcal{D} | x \rangle = - \int_0^\infty \frac{d\tau}{\tau} \left\langle x \left| e^{-\tau \mathcal{D}} \right| x \right\rangle + C$$

$$H(x,\tau) \equiv \left\langle x \left| e^{-\tau \mathcal{D}} \right| x \right\rangle$$

$$H(x,\tau) = \frac{i}{(4\pi)^{d/2}} \frac{e^{-\tau m^2}}{\tau^{d/2}} \left[a_0(x) + a_1(x)\tau + a_2(x)\tau^2 + \ldots \right]$$

Induced effects encoded in heat kernel coefficients:

$a_0 = 1$, $a_1 = \frac{2R}{3}$, …….

With proper time cutoff

$$\Delta\mathcal{L} = \int d^4x \frac{1}{16\pi^2}\left[-\frac{24}{\tau_0^4} - \frac{4}{\tau_0^2}R + ...\right]$$

Again, this is only free vector loop – not a real calculation
- proper time regularization not appropriate for scale invariant theory

But still, the nature of the corrections is clear.

With dimensional transmutation, Einstein action will appear

But, but, but…..

This example has flaws:

The a_2 coefficient describes real divergences involving the metric:

$$\Delta\mathcal{L} = \frac{1}{d-4} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}$$

where the Weyl tensor is

$$\begin{aligned} C_{\mu\nu\alpha\beta} &= R_{\mu\nu\alpha\beta} - \frac{1}{2}\left(R_{\mu\alpha} g_{\nu\beta} - R_{\nu\alpha} g_{\mu\beta} - R_{\mu\beta} g_{\mu\alpha} + R_{\nu\beta} g_{\mu\alpha}\right) \\ &+ \frac{R(g)}{6}\left(g_{\mu\alpha} g_{\nu\beta} - e_{\nu\alpha} e_{\mu\beta}\right) \end{aligned}$$

Therefore one needs to include scale (conformal) invariant action for metric also

In addition, fermion loop leads to new divergences (below)

Need a more extensive action for consistency

- but many, many possible new terms

Improved strategy

Weyl term is conformally invariant:

$$\Delta\mathcal{L} = \frac{1}{d-4} C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$$

So is the result of the fermion loop.

Suggests starting with a **conformally invariant** theory

Extra freedom with both metric and spin connection

Basic conformal symmetry

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \qquad g^{\mu\nu} \to \Omega^{-2}(x) g^{\mu\nu} \qquad \text{or} \qquad e^a_\mu \to \Omega^{(}x) e^a_\mu \qquad e^\mu_a \to \Omega^{-1}(x) e^\mu_a \ .$$

Notation:

$$\Omega^2(x) = e^{2\sigma(x)} \qquad \text{such that} \qquad \Omega^{-1}(x)\partial_\mu \Omega(x) = \partial_\mu \sigma$$

Fermions and the spin connection

The fermion action

$$S_D = \int d^4x \sqrt{-g}\bar{\psi}\left[i\gamma^a e_a^\mu(\partial_\mu - i\frac{J_{ab}}{2}\omega_\mu^{ab})\right]\psi$$

is conformally invariant under

$$\psi \to \Omega^{-3/2}\psi \qquad\qquad e_a^\mu \to \Omega^{-1}(x)e_a^\mu$$

if the spin connection transforms as

$$\omega_\mu^{ab} \to \omega_\mu^{ab} + (e_\mu^a\partial^b\sigma - e_\mu^b\partial^a\sigma)$$

New spin Weyl tensor:

While the Weyl tensor transforms covariantly

$$C_{\mu\nu\alpha\beta} \to \Omega^2 C_{\mu\nu\alpha\beta}$$

the equivalent formed from the spin connection does not

$$\begin{aligned} \delta R^{ab}_{\mu\nu} &= (d_\mu \partial^b \sigma) e^a_\nu - (d_\nu \partial^b \sigma) e^a_\mu \\ &- (d_\mu \partial^a \sigma) e^b_\nu + (d_\nu \partial^a \sigma) e^b_\mu \\ &+ \partial^b \sigma E^a_{\mu\nu} - \partial^a \sigma E^b_{\mu\nu} \end{aligned}$$

where

$$\begin{aligned} E^a_{\mu\nu} &= \nabla_\mu e^a_\nu - \nabla_\nu e^a_\mu = d_\mu e^a_\nu - d_\nu e^a_\mu \\ &= \partial_\mu e^a_\nu + g\omega_\mu^a{}_b e^b_\nu - \partial_\nu e^a_\mu - g\omega_\nu^a{}_b e^b_\mu \end{aligned}$$

The last term causes a lack of conformal invariance (vanishes if metricity assumed)

To compensate define

$$[d_a, d_b] = F_{ab}{}^c\, d_c - i\frac{g}{2} e^\mu_a e^\nu_b J_{cd}\, R^{cd}_{\mu\nu}$$

with

$$d_\mu \equiv \partial_\mu - i\frac{g}{2} J_{ab} \omega^{ab}_\mu \qquad d_a \equiv e^\mu_a d_\mu$$

such that

$$\begin{aligned} F_{ab}{}^c &= (d_a e^\mu_b - d_b e^\mu_a)\, e^c_\mu \\ &= e^\lambda_a \left(\partial_\lambda e^\mu_b + g\omega_{\lambda b}{}^d e^\mu_d\right) e^c_\mu - e^\lambda_b \left(\partial_\lambda e^\mu_a + g\omega_{\lambda a}{}^d e^\mu_d\right) e^c_\mu \end{aligned}$$

Then since:

$$E^a_{\mu\nu} \to \Omega E^a_{\mu\nu}$$

$$F_{ab}{}^c \to \Omega^{-1}\left[F_{ab}{}^c + 2(\partial_a\sigma\delta^c_b - \partial_b\sigma\delta^c_a)\right]$$

the new Weyl tensor can be formed using

$$\bar{R}^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + \frac{1}{2}F^{ab}{}_c E^c_{\mu\nu}$$

and has the form

$$\begin{aligned} D^{ab}_{\mu\nu} &= \bar{R}^{ab}_{\mu\nu} - \frac{1}{2}\left(\bar{R}^a_\mu e^b_\nu - \bar{R}^a_\nu e^b_\mu - \bar{R}^b_\mu e^a_\mu + \bar{R}^b_\nu e^a_\mu\right) \\ &+ \frac{\bar{R}}{6}\left(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu\right) \end{aligned}$$

This is conformally invariant $D^{ab}_{\mu\nu} \to D^{ab}_{\mu\nu}$

and its action is also:

$$S_D = \int d^4x\sqrt{-g}\; D^{ab}_{\mu\nu}D^{\mu\nu}_{ab}$$

Now look at fermion loop effect

Must be conformal and fully covariant

Define $W_d = \frac{1}{2}\epsilon_{abcd} e^{a\mu} w_\mu^{bc}$

Direct calculation:

$$\Delta\mathcal{L} = -\frac{1}{96\pi^2\epsilon}\partial_a W_b \partial_{a'} W_{b'} \left[\eta^{aa'}\eta^{bb'} - \eta^{ab'}\eta^{ba'}\right]$$

The full covariant completion:

$$W_{ab} = d_a W_b - d_b W_a = \partial_a W_b - \partial_b W_a + e_a^\mu \omega_\mu^{bc} W_c - e_b^\mu \omega_\mu^{ac} W_c$$

$$\Delta\mathcal{L} = -\frac{1}{48\pi^2\epsilon} W_{ab} W^{ab}$$

Vanishes if metricity is imposed

Other conformal invariants:

The metricity condition is itself conformally covariant

$$\nabla_\mu e^a_\nu \to \Omega \nabla_\mu e^a_\nu \quad .$$

which implies also

$$E^a_{\mu\nu} \to \Omega E^a_{\mu\nu}$$

$$\begin{aligned} E^a_{\mu\nu} &= \nabla_\mu e^a_\nu - \nabla_\nu e^a_\mu = d_\mu e^a_\nu - d_\nu e^a_\mu \\ &= \partial_\mu e^a_\nu + g\omega_\mu{}^a{}_b e^b_\nu - \partial_\nu e^a_\mu - g\omega_\nu{}^a{}_b e^b_\mu \end{aligned}$$

This allows two new conformal invariants

The full conformally invariant model

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4g_1^2} D^{ab}_{\mu\nu} D^{\mu\nu}_{ab} - \frac{1}{4g_2^2} W_{ab} W^{ab} \\ &+ \alpha_1 C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + \alpha_2 C_{\mu\nu\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} D^{ab\mu\nu} \\ &- \lambda_1 \left[g^{\alpha\beta} \eta_{ab} \nabla_\mu e^a_\alpha \nabla^\mu e^b_\beta \right]^2 - \lambda_2 \left[E^a_{\mu\nu} E^{\mu\nu}_a \right]^2\end{aligned}$$

I have not yet verified AF for this action – still much to do

Side comment: Alternate possibility

Use metricity condition in reverse to express vierbein in terms of spin connection

$$e_b^\nu \left[\partial_\mu e_\nu^a - \Gamma_{\mu\nu}^{\ \ \lambda} e_\lambda^a\right] = -\omega^a_{\ b\mu}$$

with boundary condition

$$e_\nu^a = \delta_\nu^a \qquad \text{when} \qquad \omega^a_{\ b\mu} = 0$$

Then the Lagrangian collapses down to a single term

$$\mathcal{L} = \frac{1}{4g^2} D_{\mu\nu}^{ab} D_{ab}^{\mu\nu} .$$

Of course, it could be tricky dealing with such a constraint

Wilson line representation?

The λ_1 term can serve as a lagrange multiplier

Much to be done:

Next steps:

Gauge fixing for conformal model without explicit conformal breaking

Beta function calculations

……

Comments:

The Planck scale may not be the ultimate barrier

- certainly EFT indicates strong coupling
- but can emerge as weak coupling in the UV

If gravity can be a conventional field theory, it probably should look like this

- scale/conformal invariant actions are most promising
- extra conformal symmetry attractive for fundamental gravity

The spin connection can live as an independent field

- most natural as a gauge field

The spin connection (with usual gauge interaction) is asymptotically free

- Confined or condensed?
- weak coupling beyond Planck scale

Dimensional transmutation can yield Einstein action

- weak coupling in the IR

Conformal model should be closed under renormalization