# Entanglement Renormalization and Two Dimensional String Theory

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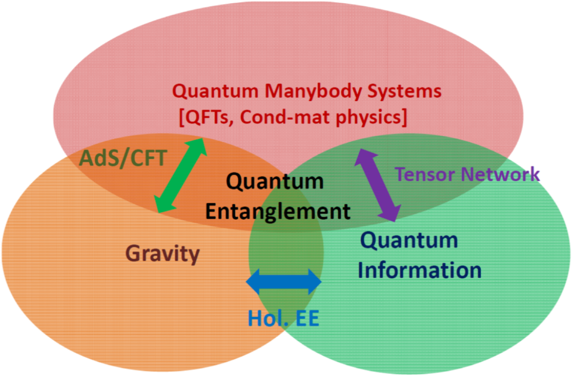
# Talk partially based on works:

- Entanglement Renormalization and two dimensional String Theory, PhysLett B755 (2016) 421-425. [arXiv: 1510.09020] (JMV)
- Information Geometry of Entanglement Renormalization for free Quantum Fields,JHEP09(2015)002. [arXiv:1503.07699] (JMV)

# Introduction: “The unreasobale connectivity of physics”



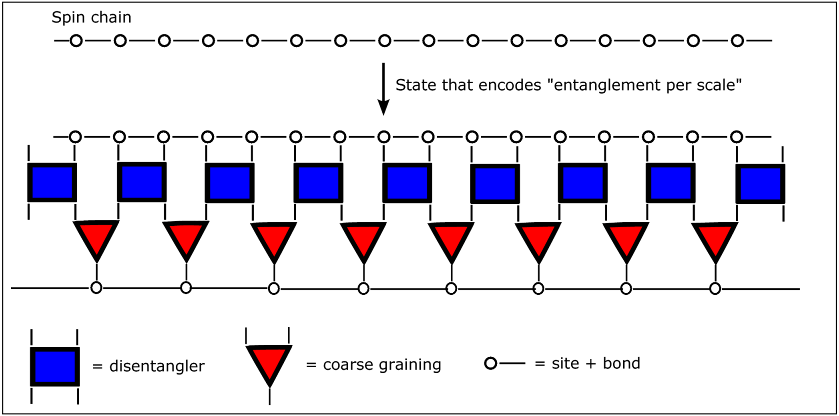
# Introduction

The career of a young *(?)* theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction"

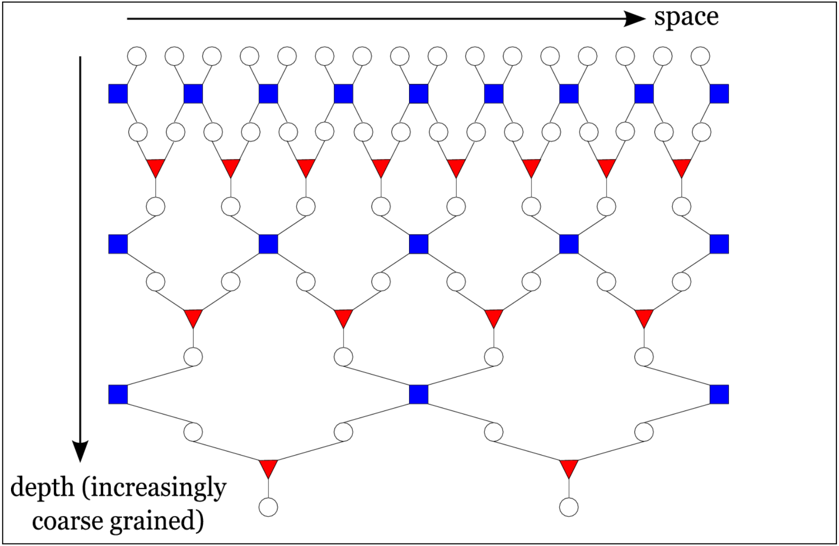
Sidney Coleman

# Entanglement Renormalization. MERA.

(Vidal 2007)



# Entanglement Renormalization. MERA



# Entanglement Renormalization and Holography.

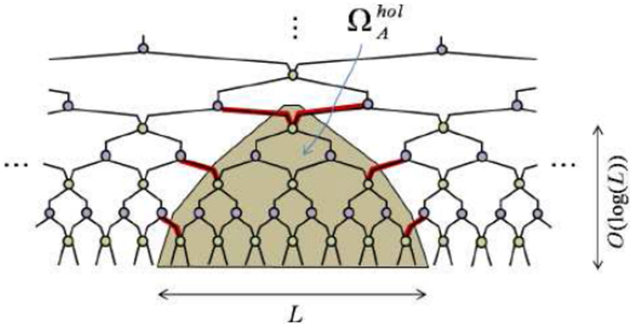
Proposal (Swingle, 2009) connecting the ideas of:

- AdS/CFT (Maldacena, 1998)
- Holographic Entanglement Entropy (Ryu-Takayanagi, 2006)
- Quantum Renormalization Group (Vidal, 2007)

Conceptual difficulties

- The network is fixed so it is difficult to see “how spacetime emerges”.
- Indepently of the Large N limit, we always attain a discrete version of a classical spacetime

# Entanglement Renormalization and Holography.




$$S_A \leq \partial\Omega_A^{hol} \log \mathcal{W} = \text{Length}(\gamma_A) \log \mathcal{W}$$

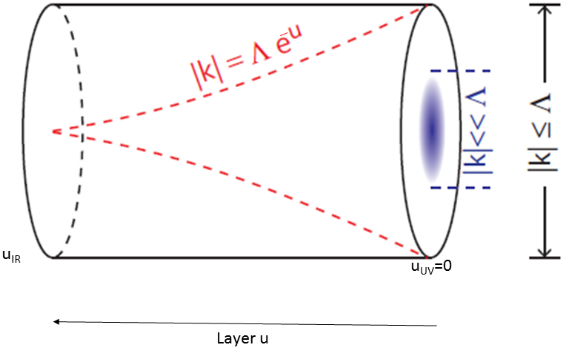
$$S_A = k \log L$$

# Continuous Entanglement Renormalization. cMERA.

- cMERA (Haegeman et al 2011) is a real-space RG in a hamiltonian framework at the level of wavefunctions.
- cMERA represents the wavefunction of the system at each length scale u.
- Rather than integrating out high-frequency modes around a cutoff $\Lambda$, an operator $\mathcal{K}(u)$ first disentangles these modes in such a way that they can be isometrically projected onto a reference and totally disentangled vacuum $|\Psi_{IR}\rangle$.

# Continuous Entanglement Renormalization. cMERA

- Scale dependent representations of the state are obtained by adding left-right moving modes with $|k| \leq \Lambda e^{-u}$ to $|\Psi_{IR}\rangle$.



# cMERA Coherent State Formulation

cMERA is described by the operation

$$|\Psi(u)\rangle = \mathcal{P}\exp\left(-i\int_{u_{IR}}^{u} d\hat{u}\,(\mathcal{K}(\hat{u})+\mathcal{L})\right)|\Psi_{IR}\rangle$$

where $\mathcal{K}(\hat{u})$ is the entangler operator, $\mathcal{L}$ is the coarse-graining operation and $|\Psi_{IR}\rangle$ is a boundary state in the sense that is a completely factorized state in real space.

We will make some analysis in the interaction picture where we only attend to the disentangling operation and

$$|\Phi(u)\rangle = e^{iu\mathcal{L}}|\Psi(u)\rangle = \mathcal{P}\exp\left(-i\int_{u_{IR}}^{u} d\hat{u}\,\widetilde{K}(\hat{u})\right)|\Psi_{IR}\rangle$$

# cMERA Interaction Picture

while the entangler reads

$$\tilde{\mathcal{K}}(u) = e^{-iu\mathcal{L}}\mathcal{K}(u)\, e^{iu\mathcal{L}}$$

- The merit of the 'interaction' picture is that at each layer $u$ of the cMERA, we have the same Hilbert space defined in $0 < |k| < \Lambda$ in momentum space. This allows us to define and calculate the overlaps,

$$\langle \Phi(u) | \Phi(u+\epsilon) \rangle$$

very useful for us later.

# cMERA Coherent State Formulation. Free Boson Theory

For the free boson theory

$$S = \int dtdx \left( (\partial_t \phi)^2 + (\partial_x \phi)^2 - m^2 \phi^2 \right)$$

The entangler operator is given by

$$\widetilde{\mathcal{K}}(u) = -\frac{i}{2} \int_{|k| \leq \Lambda e^{-u}} dk \left( g_k(u)\, a_k^\dagger a_{-k}^\dagger - \bar{g}_k(u) a_k a_{-k} \right)$$

with $a_k |0\rangle = 0$, $a_{-k} |0\rangle = 0$

# cMERA Coherent State Formulation

The state at scale u can be written as a $SU(1,1)$ gaussian squeezed state

$$|\Phi(u)\rangle = \mathcal{N} \exp\left(-\frac{1}{2}\int dk\, \Phi_k(u)\; R_k^\dagger L_k^\dagger\right) |0\rangle_R |0\rangle_L$$

with $R_k = a_k$, $L_k = a_{-k}$ and $R_k|0\rangle_R = 0$, $L_k|0\rangle_L = 0$

or

$$|\Phi(u)\rangle = \mathcal{N} \exp\left(-\frac{1}{2}\int dk\, \Phi_k(u)\; a_k^\dagger a_{-k}^\dagger\right) |0\rangle_R |0\rangle_L$$

with $\Phi_k(u) = \langle\Phi(u)|\, R_k L_k\, |\Phi(u)\rangle = \int_0^u d\hat{u}\, g_k(\hat{u})$

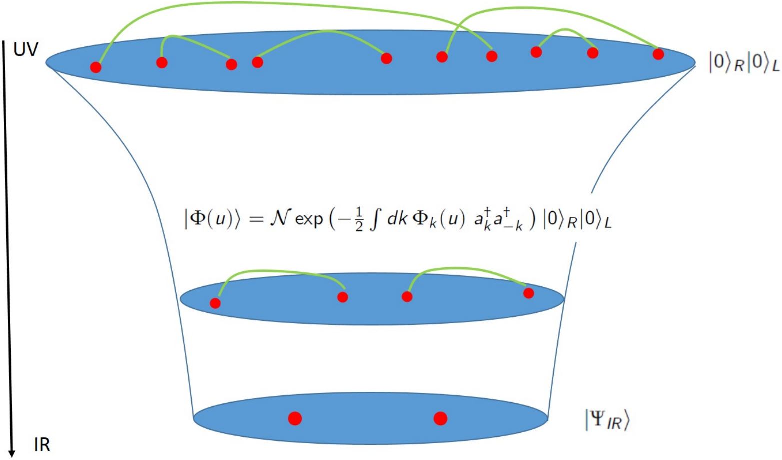
# cMERA Coherent State Formulation

$$|\Phi(u)\rangle = \propto \prod_{|k|\leq\Lambda e^{-u}} \exp\left(\Phi_k(u)\ a_k^\dagger a_{-k}^\dagger\right) |0\rangle_R |0\rangle_L$$

$$\times \prod_{|k|>\Lambda e^{-u}} |\Psi_{IR}\rangle$$

and $\Phi_k(u) = \langle\Phi(u)|\, R_k L_k \,|\Phi(u)\rangle \sim$ 'condensate'.

# cMERA Coherent State Formulation



# cMERA Coherent State Formulation. Final Remark

- Proposal for $|\Psi_{IR}\rangle$ as the Dirichlet boundary state (Miyaji et al 2015)

$$|\Psi_{IR}\rangle \equiv |B\rangle = \exp\left(-\frac{1}{2}\int dk\, a_k^\dagger\, a_{-k}^\dagger\right)|0\rangle_R|0\rangle_L$$

- In these states the entanglement between the left and right moving modes LREE (see Digression below) is maximal so there is no real space entanglement.
- They appear in the worldsheet description of propagating strings in spacetimes with boundaries (D-branes).
- Interesting LREE computations on D-branes (Pando-Zayas et al, 2016)

# cMERA & Emergent space-time

Is it possible to ‘see’ an emergent spacetime from the Entanglement Renormalization of the wavefunction?

# cMERA Path Integral

Write the cMERA amplitude

$$G(u_F, u_{IR}) = \langle \Phi(u_F) | \mathcal{P} \exp \left( -i \int_{u_{IR}}^{u_F} \tilde{\mathcal{K}}(\hat{u}) \, d\hat{u} \right) | \Phi(u_{IR}) \rangle$$

as a SU(1,1) coherent state path integral.

By standard methods,

$$\mathcal{Z} = G(0, u_{IR}) = \int \mathcal{D}(\Phi, \bar{\Phi})(u) \, \exp \left( i \, S_{eff} \left[ \Phi, \bar{\Phi} \right] \right)$$

$$\mathcal{Z} = \int \mathcal{D}(\Phi, \bar{\Phi})(u) \, \exp \left( -i \int dxdu \, \left[ \mathrm{B}(\Phi, \bar{\Phi}) + \tilde{\mathcal{K}}(\Phi, \bar{\Phi}) \right] e^{-u} \right)$$

# cMERA Path Integral

$$\mathcal{Z} = \int \mathcal{D}(\Phi,\bar{\Phi})(u)\ \exp\left(-i\int dxdu\ \left[\mathrm{B}(\Phi,\bar{\Phi}) + \tilde{\mathcal{K}}(\Phi,\bar{\Phi})\right] e^{-u}\right)$$

with

$$\mathrm{B}(\Phi,\bar{\Phi}) = \frac{1}{2i}\left(\bar{\Phi}\,\partial_u\Phi - \Phi\,\partial_u\bar{\Phi}\right)$$

$$\tilde{\mathcal{K}}(\Phi,\bar{\Phi}) = \langle\Phi(u)|\tilde{\mathcal{K}}(u)|\Phi(u)\rangle$$

$$S_{eff}\left[\Phi,\bar{\Phi}\right] = -2\int dxdu\,\bar{\Phi}(u)\ e^{-u}\,\partial_u\Phi(u)$$

# cMERA Path Integral

The coherent state description of cMERA for a free boson yields a natural geometric representation of the RG flow by means of a two dimensional metric on a manifold $(u, x)$ given by

$$ds^2 = g_{uu} du^2 + e^{-2u} dx^2 \qquad g_{ab} = diag\left\{g_{uu},\ e^{-2u}\right\}$$

with

$$g_{uu} = g_k(u)^2$$

**THE VARIATIONAL PARAMETER** $\Phi_k(u)$ of CMERA

$$\Phi_k(u) = \left[-\frac{1}{4}\ \log\ \frac{k^2 + m^2}{\Lambda^2 + m^2}\right]_{k=\Lambda\, e^{-u}} = -\frac{1}{4}\ \log\ \frac{e^{-2u} + \bar{m}^2}{1 + \bar{m}^2}$$

# cMERA Path Integral

is obtained by

$$\frac{\delta\,\mathcal{E}}{\delta\Phi_k} = \frac{\delta}{\delta\Phi_k}\,\langle\Psi_{IR}|\mathcal{H}(u_{IR})|\Psi_{IR}\rangle = 0$$

where $\mathcal{H}(u_{IR})$ is the hamiltonian density of the boson theory at the length scale $u_{IR}$.

Then it is straightforward to obtain

$$g_k(u) = \partial_u\Phi_k(u) = \frac{1}{2}\,\frac{e^{-2u}}{(e^{-2u} + \bar{m}^2)}$$

with $\bar{m} = {}^{m}/\Lambda \ll 1$

# cMERA Path Integral

$$e^{-u}\partial_u \Phi_k(u) = \sqrt{g_{uu}} e^{-u} = \sqrt{g} \quad \text{with} \quad \sqrt{g} = \det\, g_{ab}$$

Thus one may formally write

$$S_{eff}[\Phi, g] = \frac{1}{4} \int d^2\sigma \, \sqrt{g}\, \mathcal{R}^{(2)} \Phi(u)$$

$\mathcal{R}^{(2)} = -8$ scalar curvature of the metric tensor $g_{ab}$

In JHEP09(2015)002 it is shown how $\Phi(u)$ is related with the **EE** of the left and right moving modes at scale u needed to create $|\Phi(u)\rangle$.

# Digression: Left-Right Entanglement Entropy

## CMERA as a scale-dependent Bogoliubov transformation

The state $|\Phi(u)\rangle$ is anihilated by:

$$b_k(u) = A_k(u)\, a_k + B_k(u)\ a^\dagger_{-k} \qquad \text{i.e,}\ b_k(u)|\Phi(u)\rangle = 0$$

with

$$A_k(u) = \cosh \Phi_k(u) \qquad B_k(u) = -\sinh \Phi_k(u)$$

# Digression: Left-Right Entanglement Entropy

Now, we trace-out the left-moving modes of $|\Phi(u)\rangle$,

$$\rho_k^R(u) = Tr_{[L]}\left[|\Phi(u)\rangle\langle|\Phi(u)|\right] = \sum_n \gamma_k(u)^n(1-\gamma_k(u))\,|n_R\rangle\langle n_R|$$

with

$$\gamma_k(u) = \left[\frac{B_k(u)}{A_k(u)}\right]^2 = (\tanh\Phi_k(u))^2 \quad |n_R\rangle = \frac{1}{\sqrt{n_R!}}\left(a_k^\dagger\right)^{n_R}|0\rangle_R$$

# Digression: Left-Right Entanglement Entropy

The reduced density matrix is diagonal of the form

$$\rho_k^R(u) = (1-\gamma_k(u))\, diag\left(1, \gamma_k(u), \gamma_k(u)^2, \gamma_k(u)^3 \cdots\right)$$

and can be written as

$$\rho_k^R(u) = e^{-\beta\, \mathcal{H}_{ent}(u)}$$

with $\beta = 2\pi$ and $\mathcal{H}_{ent}(u) = \varepsilon_k(u)\, a_k^\dagger a_k$ known as the entanglement hamiltonian, where

$$\varepsilon_k(u) = -\frac{1}{\beta}\log\, \gamma_k(u)$$

# Digression: Left-Right Entanglement Entropy

The **LREE,** which amounts to the Von Neumann entropy

$$S_k(u) = -Tr[\,\rho_k^R(u)\,\log\rho_k^R(u)\,],$$

can be written as

$$S_k(u) = -\frac{\gamma_k(u)}{1-\gamma_k(u)}\log\gamma_k(u) - \log(1-\gamma_k(u))$$

and

$$\partial_u S_k(u) \approx -2\,g_k(u)$$

in the limit of $\gamma \approx 1$ . Thus,

$$g_{uu} \propto (\partial_u S_k(u))^2$$

# Closed Bosonic String Action

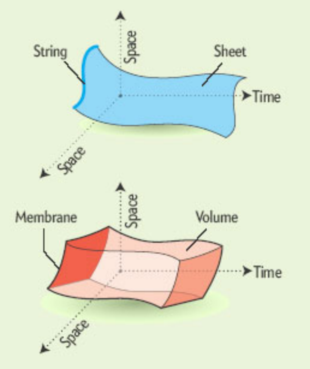
The worldsheet action for a closed bosonic string in a curved background

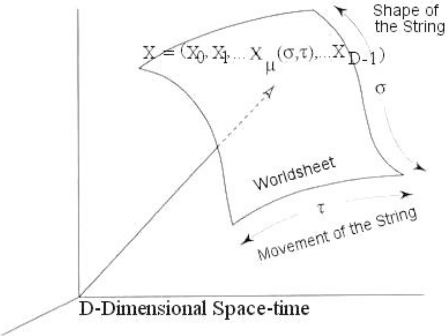
$$S_{ws} = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{g}\left[g^{ab}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu\right] +$$

$$\frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{g}\mathcal{R}^{(2)}\Phi(X)$$

where $\sigma$ , $g_{ab}$ and $\mathcal{R}^{(2)}$ are coordinates, metric and scalar curvature on the world-sheet respectively; $X^\mu(\sigma)$ denote target space coordinates with $\mu = 0 \cdots D-1$ and $D$ the dimension of the target spacetime, $G_{\mu\nu}(X)$ is the target spacetime metric and $\Phi(X)$ is the dilaton field.

# Closed Bosonic String Action



# Closed Bosonic String Action

$$S_{ws} = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{g} \left[ g^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \alpha' \mathcal{R}^{(2)} \Phi(X) \right]$$

Can be understood as a theory of 2D gravity ($g_{ab}$) coupled to conformal matter ($X^\mu(\sigma)$ ) with coupling constants $G_{\mu\nu}(X)$ and $\Phi(X)$

To impose this theory to be a 2D conformal theory on the 2D world-sheet, the trace of Energy-Momentum tensor $T_{ab}$ on the world-sheet must vanish.

# Background Fields E.O.M's

This amounts to impose that the Weyl anomaly $\beta$-functions,

$$\frac{\beta^{\Phi}}{\alpha'} = \frac{D-26}{6\alpha'} + \frac{1}{2}\left[4(\nabla_{\mu}\Phi)^2 - 2\nabla^2\Phi - \mathcal{R}\right] = 0$$

$$\beta^{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} + 2\nabla_{\mu}\nabla_{\nu}\Phi = 0$$

$\nabla_{\mu}$ and $\mathcal{R}$ are the target spacetime covariant derivative and the scalar curvature respectively.

The vanishing of $\beta^{G}_{\mu\nu}$ and $\beta^{\Phi}$ leads to effective equations of motion for the background fields $G_{\mu\nu}$ and $\Phi$.

# Non Trivial Solutions. Linear Dilaton Background

A consistent background solution to the equations of motion for arbitrary $D$ consists in a flat target spacetime and a linear dilaton,

$$G_{\mu\nu}(X) = \eta_{\mu\nu}$$

$$\Phi(X) = V_\mu X^\mu$$

$$V_\mu V^\mu = \frac{26 - D}{12\alpha'} = Q^2$$

For $D < D_{crit} = 26$, the dilaton gradient is spacelike.

# 2D String Theory. Linear Dilaton Background

2D String Theory: the worldsheet theory is Liouville theory + $c = 1$ conformal matter. It is considered *the baby cousin* of AdS/CFT (Klebanov, Ginsparg-Moore, Polchinski)

We consider the case for $D = 2$ and $\Phi(X)$ lying along $X^1$, i.e.,

$$\Phi(X) = Q\,X^1; \quad Q^2 = 2/\alpha'.$$

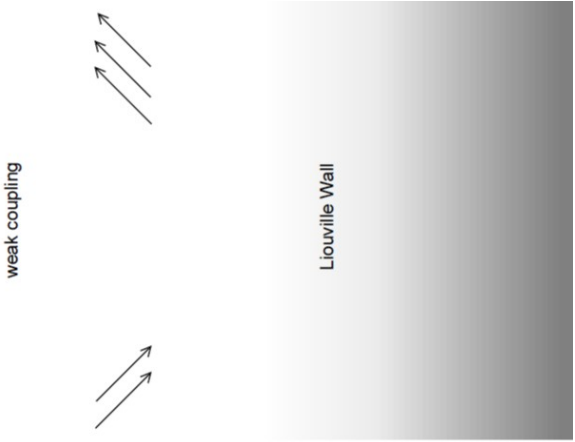
- The strength of the string interactions varies as a function of the $X^1$ coordinate as

$$g_{eff} = e^{\Phi(X)} = e^{QX^1}$$

- in the $X^1 \to \infty$ region of the target spacetime $g_{eff}$ diverges and string perturbation theory fails.

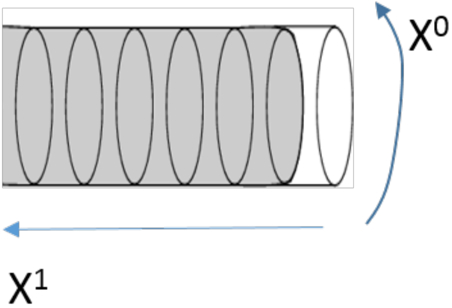
# 2D String Theory. Linear Dilaton Background

Two dimensional spacetime with dilaton and tachyon backgrounds



# 2D String Theory Linear Dilaton Background

- The geometry seen by the propagating string is a two dimensional flat spacetime with a dilaton linearly varying along its $X^1$ direction.



# cMERA Linear Dilaton Background

May the cMERA action $S_{eff}$ be interpreted as the dilaton term of the worldsheet action? Let us recall the variational solution

$$\Phi(u) = -\frac{1}{4} \log \frac{e^{-2u} + \bar{m}^2}{1 + \bar{m}^2}$$

When $\bar{m} = 0$,

$$\Phi(u) = Q\, u$$

with $Q = 1/2$ and $g_{uu} = 1/4$.

- Choosing the target spacetime coordinates as $X^\mu = (X^0, X^1) = (x, u)$ and $G_{\mu\nu}(X) = diag(1, g_{uu})$, $S_{eff}$ reads as,

$$S_{eff} = \frac{1}{4} \int d^2\sigma \sqrt{g}\, \mathcal{R}^{(2)} Q\, X^1$$

# cMERA Linear Dilaton Background. Comments

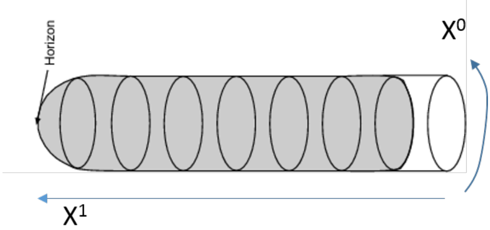
- When $\bar{m} = 0$, it is suggested that the cMERA effective action describes a linear dilaton background with $Q = 1/2$ .
- The consistency condition $Q^2 = 2/\alpha'$ imposes that one has to work in units where $\alpha' \equiv 8$.
- This amounts to define a fiducial string interaction strength $g_0^2 \sim \alpha'$ which landmarks the regime $g_{eff}^2 = e^{X^1} \ll g_0$ where perturbation theory is valid.

# cMERA Linear Dilaton Background. Comments

- In CMERA parlance, states $|\Phi(X^1)\rangle$ close to the UV point, (entanglement at all length scales) $\Longrightarrow$ regions where perturbation theory is valid.
- Meanwhile, states located at the IR region (those that have been devoid of their entanglement at small length scales) $\Longrightarrow$ strong coupling region.
- The inverse string coupling limits the number of the left-right moving entangled modes at the scale $u$ to those with momentum $k \leq \Lambda g_{eff}^{-2}$ .

# Two Dimensional Black Hole.

- String theory also describes strong gravitational fields like black holes. A non trivial solution is the 2D black hole (Witten 91).
- Here, the spacetime manifold parametrizes the coset $SL(2,R)/U(1)$.
- The spacetime geometry seen by the string looks like a cigar.



# Two Dimensional Black Hole.

The non-trivial fields in spacetime are the metric and the dilaton given by,

$$G_{11}(X) = \frac{1}{4}\tanh^2\left(2QX^1 + \log \; M\right)$$

$$\Phi(X) = -\frac{1}{2}\log 2M - \frac{1}{2}\log(2QX^1 + log\, M)$$

with $G_{00}(X) = 1$ and $M$ being a mass constant. As $M \to 0$, the background is the linear dilaton which is also recovered when $X^1 \to 0$ .

# cMERA Two Dimensional Black Hole.

Now we look at the CMERA variational solution

$$\Phi(u) = -\frac{1}{4} \log \frac{e^{-2u} + \bar{m}^2}{1 + \bar{m}^2}$$

when $\bar{m} \neq 0$. If we choose $M = \bar{m}$ and the target spacetime metric and coordinates as before, as long as $X^1 < -\log \bar{m}$, one may write

$$G_{11}(X) = \frac{1}{4} \tanh^2 \left(2QX^1 + \log \ \bar{m}\right) \left(= g_{uu}(u)\right)$$

and

$$\Phi(X^1) = -\frac{1}{2} \log 2\bar{m} - \frac{1}{2} \log(2QX^1 + log \ \bar{m})$$

# cMERA Two Dimensional Black Hole.

- The scalar curvature of $g_{ab}$,

$$\mathcal{R}^{(2)} = -8 + 8\bar{m}^4 e^{4X^1}$$

  remains constant along the $X^1$ coordinate before it exponentially vanishes when reaching $X^1_\bullet \sim -\log \bar{m}$.

- Breakdown of the linear dilaton behaviour at $X^1_\bullet$ where,

$$\Phi(X^1_\bullet) \approx -\frac{1}{2} \log \bar{m}$$

  and $G_{11}$ changes its sign, which might be interpreted by the presence of an horizon.

- Here, the effective string coupling $g^2_{eff} = e^{2QX^1_\bullet} \sim 1/\bar{m} \gg g^2_0$.

# Conclusions.

- The cMERA of different ground states correspond to non-trivial backgrounds of 2D string theory.
- Insights on how the background fields $G_{\mu\nu}$ and $\Phi$ arise from the structure of the LREE.
- How general is the idea that gravitational systems, such as black holes or other cosmological spaces are represented by composite entities of microscopic entangled quantum constituents?
- Excited states, dynamical processes (covariant formulation of cMERA), interacting and/or chiral theories, cMERA in higher dimensions?