Quantum error correction and the information structure of holography

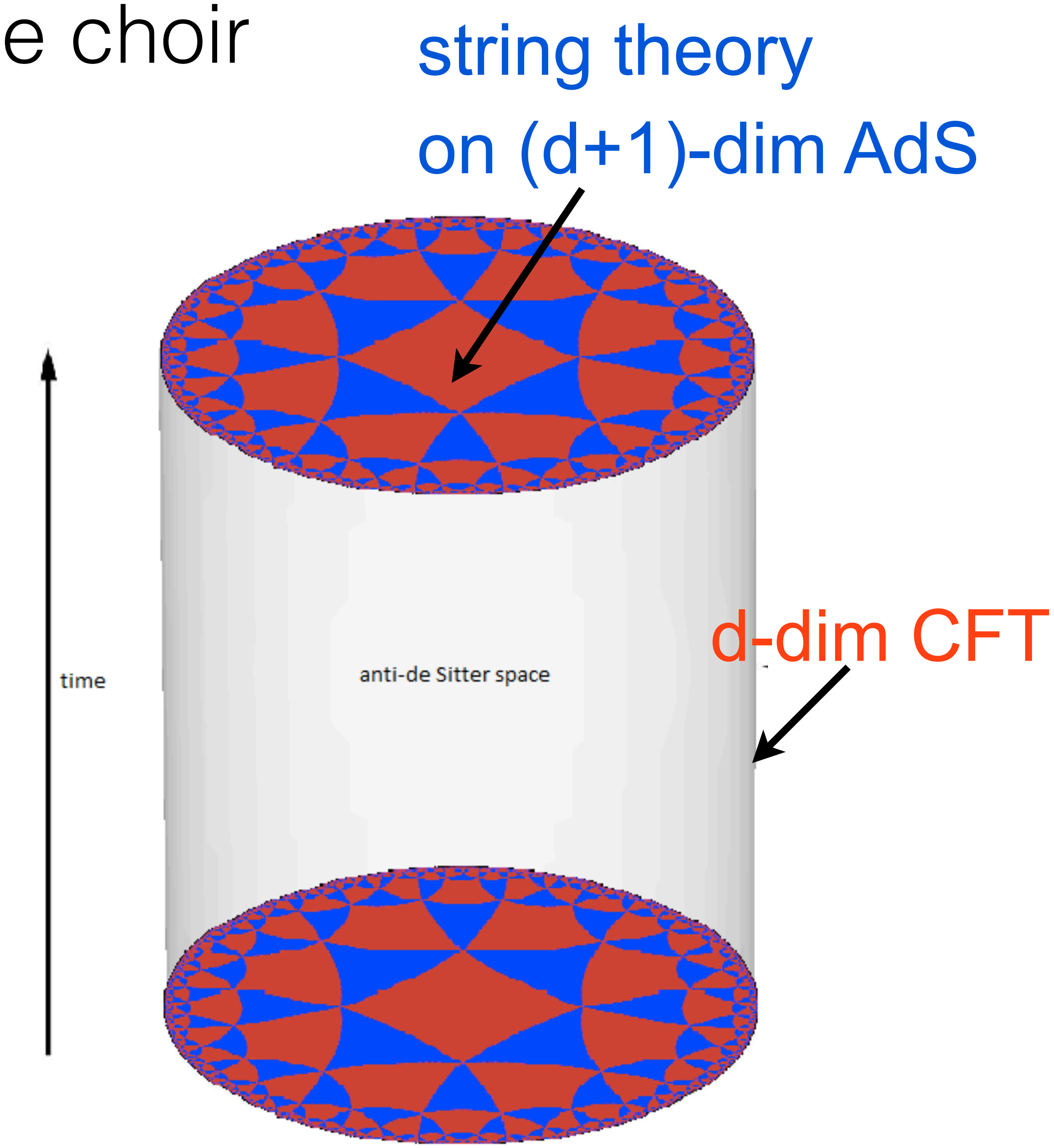
Fernando Pastawski
based on joint work with Beni Yoshida, Daniel Harlow and John Preskill

What does QEC have to do with holography?

AdS/CFT

preaching to the choir

AdS	CFT
Weakly coupled gravity	Strongly coupled
Geometric minimal surface	Entanglement entropy
Bulk operators	Boundary operators
Gravitational dynamics	Entanglement thermodynamics

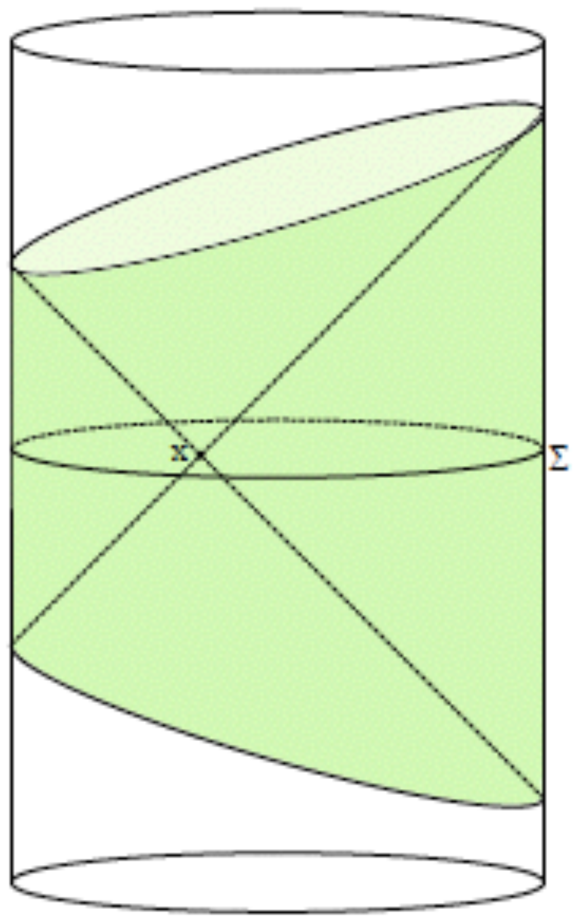


Powerful framework to study strongly-interacting systems

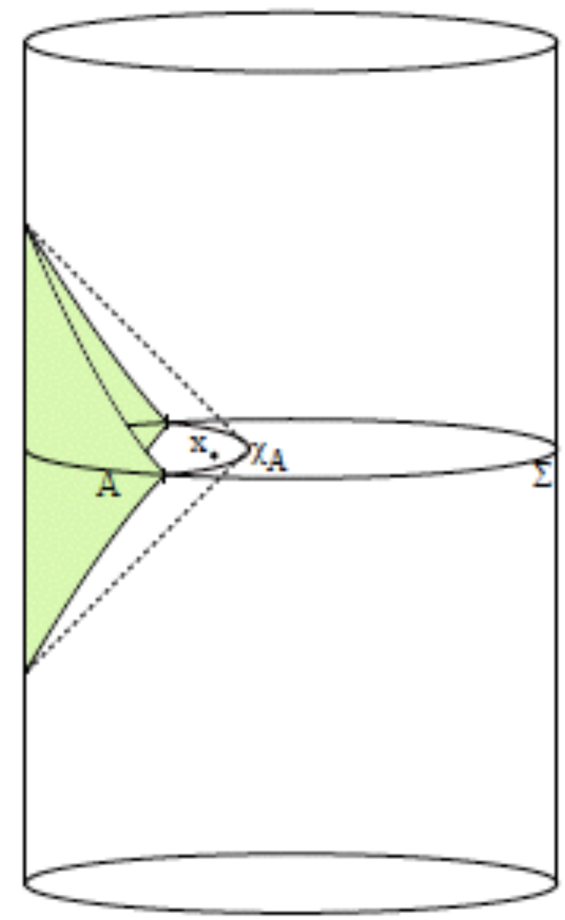
Advanced our understanding of quantum gravity

Maldacena, J. The Large-N Limit of Superconformal Field Theories and Supergravity. IJTP, 38(4), 1113–1133.

Boundary reconstruction of bulk operators



Global reconstruction

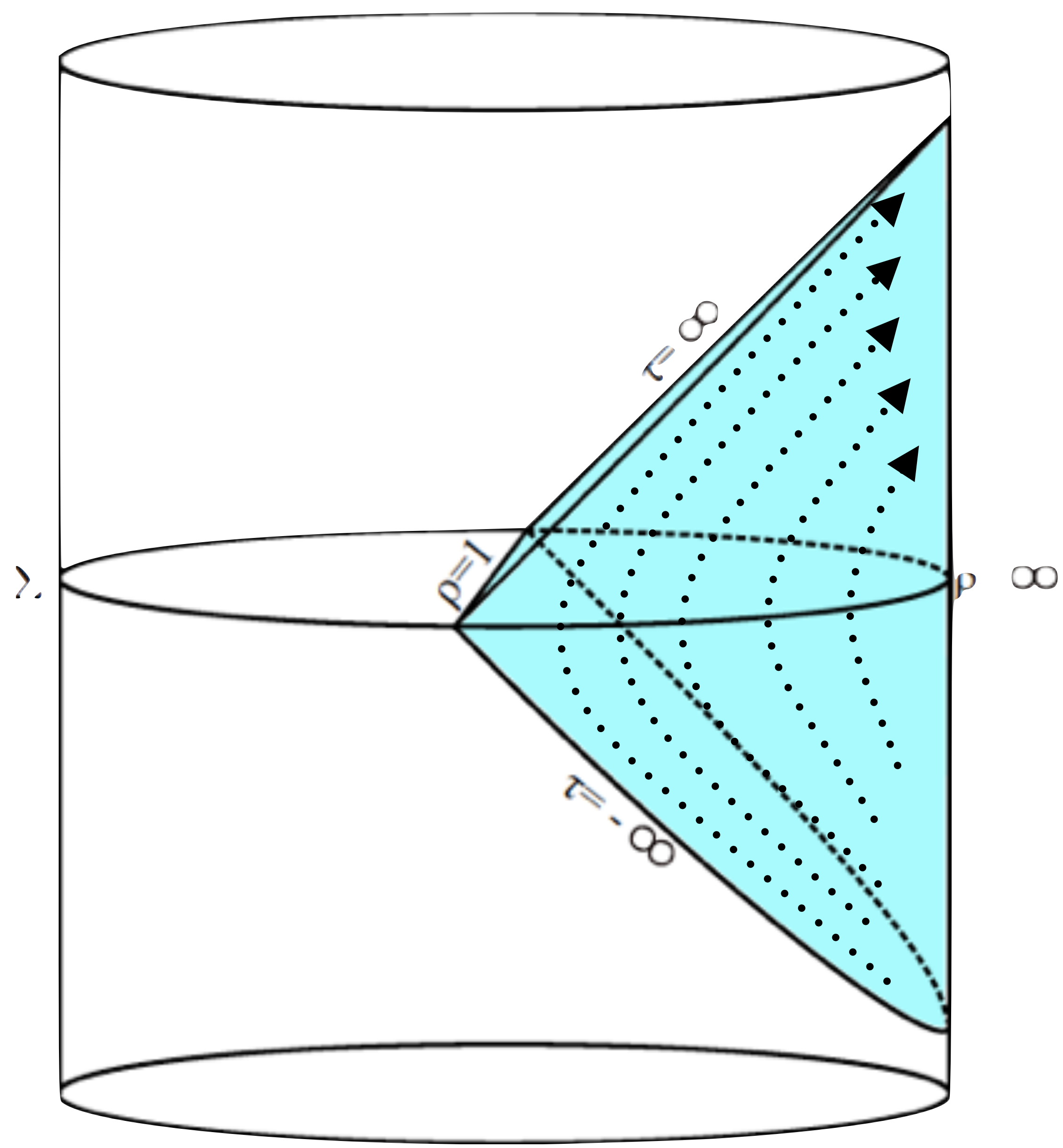


on AdS-Ridler wedge

Hamilton, A., Kabat, D., Lifschytz, G., & Lowe, D. (2006).
Holographic representation of local bulk operators. PRD, 74(6), 066009.

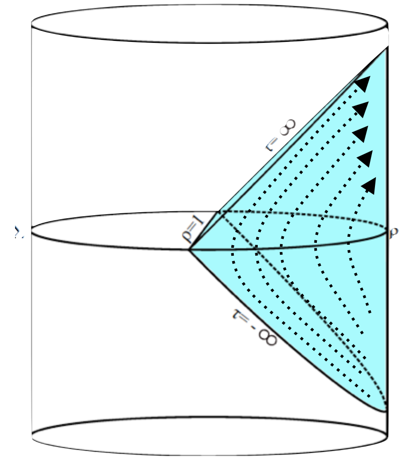
Explicit solution in BG metric

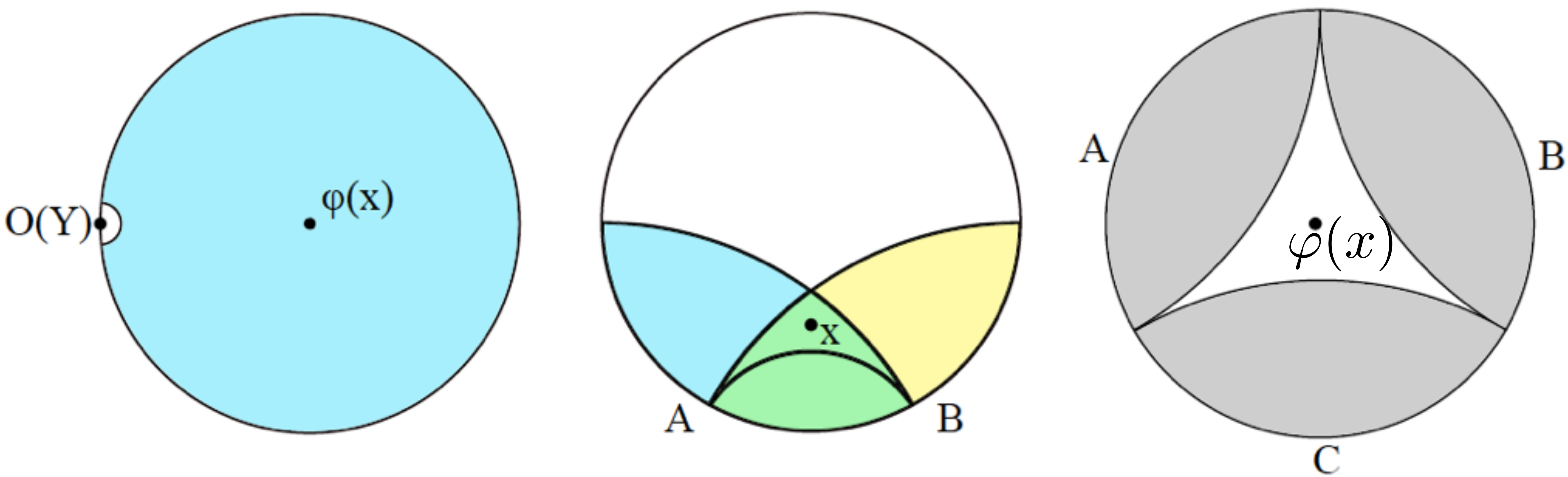
$$\mathcal{W}_C[A] \equiv \mathcal{J}^+[D[A]] \cap \mathcal{J}^-[D[A]].$$



$$\phi(x) = \int_{\mathbb{S}^{d-1}\times\mathbb{R}} dY\, K(x;Y)\mathcal{O}(Y)$$

Reduction to spacelike slice





Solve boundary EOMs

$\varphi(x) \to \Phi_{AB}(x)$ $\varphi(x) \to \Phi_{BC}(x)$ $\varphi(x) \to \Phi_{CA}(x)$

Bulk locality and quantum error correction in AdS/CFT

JHEP04(2015)163

Ahmed Almheiri,[a] Xi Dong[a] and Daniel Harlow[b]

[a] *Stanford Institute for Theoretical Physics, Department of Physics, Stanford University, Stanford, CA 94305, U.S.A.*

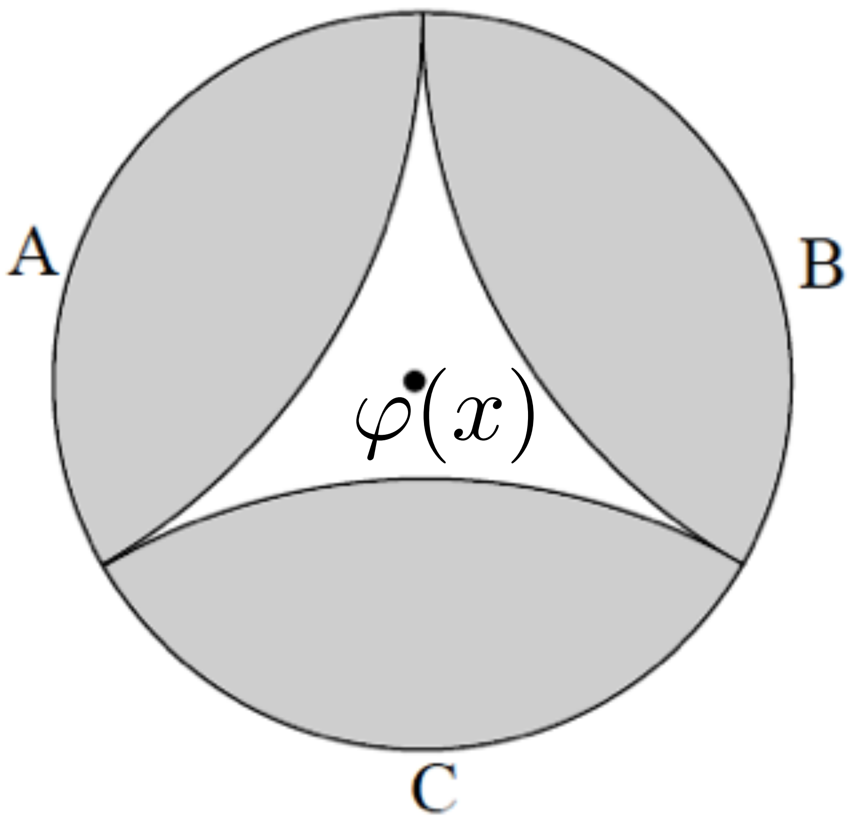
[b] *Princeton Center for Theoretical Science, Princeton University, Princeton NJ 08540, U.S.A.*

E-mail: almheiri@stanford.edu, xidong@stanford.edu, dharlow@princeton.edu

ABSTRACT: We point out a connection between the emergence of bulk locality in AdS/CFT and the theory of quantum error correction. Bulk notions such as Bogoliubov transformations, location in the radial direction, and the holographic entropy bound all have natural CFT interpretations in the language of quantum error correction. We also show that the question of whether bulk operator reconstruction works only in the causal wedge or all the way to the extremal surface is related to the question of whether or not the quantum error correcting code realized by AdS/CFT is also a "quantum secret sharing scheme", and suggest a tensor network calculation that may settle the issue. Interestingly, the version of quantum error correction which is best suited to our analysis is the somewhat nonstandard "operator algebra quantum error correction" of Beny, Kempf, and Kribs. Our proposal gives a precise formulation of the idea of "subregion-subregion" duality in AdS/CFT, and clarifies the limits of its validity.

Almheiri, A., Dong, X., & Harlow, D. (2015).
Bulk locality and quantum error correction in AdS/CFT. JHEP, 2015(4), 163.

Sharpening the paradox



$$|\Omega'\rangle = \varphi(x)|\Omega\rangle$$

$$A \cup B \cup C = Boundary$$

$$\rho'_A = \rho_A = \mathrm{tr}_{BC}[|\Omega\rangle\langle\Omega|]$$
$$\rho'_B = \rho_B = \mathrm{tr}_{CA}[|\Omega\rangle\langle\Omega|]$$
$$\rho'_C = \rho_C = \mathrm{tr}_{AB}[|\Omega\rangle\langle\Omega|]$$

$$\rho'_{AB} \neq \rho_{AB} \quad \rho'_{BC} \neq \rho_{BC} \quad \rho'_{AC} \neq \rho_{AC}$$

The effect of $\varphi(x)$ is encoded in non-local correlations.

Entanglement and Operator "teleportation"

Singlet $$|\Psi^-\rangle := \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}$$

Stabilizer equations

$$X \otimes X|\Psi^-\rangle = Y \otimes Y|\Psi^-\rangle = Z \otimes Z|\Psi^-\rangle = -|\Psi^-\rangle$$

Operator "teleportation"

$$O_A|\Psi^-\rangle = O_B|\Psi^-\rangle$$

Resolution: Entangled ground state and low energy sector.

Operator Algebra Quantum Error Correction (OAQEC)

Bény, C., Kempf, A., & Kribs, D. (2007).
Generalization of Quantum Error Correction via the Heisenberg Picture.
PRL, 98(10), 100502.

Definition: OAQEC

Code space: $\mathcal{H}_C = P\mathcal{H}$ $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$

Noise map:

$$\mathcal{N}(\rho) = \sum_j N_j \rho N_j^\dagger$$

Noise span:

$$\mathcal{N} = \mathsf{span}\{N_a^\dagger N_b\}_{a,b}$$

$\mathcal{N}$ is correctable with respect to $\mathcal{A}$ in the code subspace $\mathcal{H}_C$ iff

i) ∃ a recovery map $\mathsf{tr}[X\mathcal{R}\circ\mathcal{N}(\rho)] = \mathsf{tr}[X\rho]$ $\rho = P\rho P$

ii) Algebraic condition $[PN_a^\dagger N_b P, X] = 0$ $\forall X \in \mathcal{A}$

Region R is correctable iff the span of supported operators is.

Distance d : size of the smallest non-correctable region.

Example: Repetition code

(Ferromagnetic Ising)

$$\mathcal{H}_C = \text{span}\{|0\rangle^{\otimes N}, |1\rangle^{\otimes N}\}$$

$$\bar{X} = \bigotimes_j X_j$$

$$d(\bar{X}) = 1$$

N-fold decoherence increase!

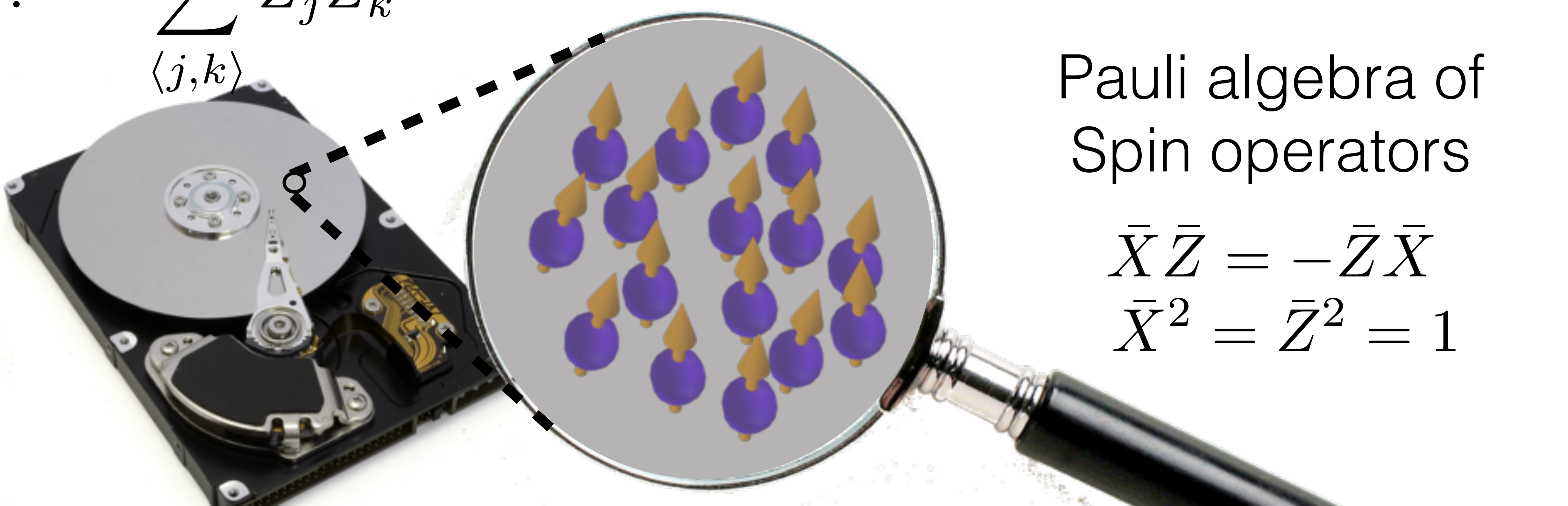
$$H := -\sum_{\langle j,k\rangle} Z_j Z_k$$

Conserved Quantities

$$\bar{Z} = \Theta\left(\sum_j Z_j\right)$$

$$d(\bar{Z}) = N$$

$$Z_j \sim_C \bar{Z}$$



Pauli algebra of Spin operators

$$\bar{X}\bar{Z} = -\bar{Z}\bar{X}$$

$$\bar{X}^2 = \bar{Z}^2 = 1$$

Example:[[3,1,2]]$_3$ quantum code

[[n,k,d]]   Protect non-commuting observables

$$\mathcal{H}_C = \mathsf{span}\{|\tilde{0}\rangle, |\tilde{1}\rangle, |\tilde{2}\rangle\}$$

$$|0\rangle \to |\tilde{0}\rangle = |000\rangle + |111\rangle + |222\rangle$$
$$|1\rangle \to |\tilde{1}\rangle = |012\rangle + |120\rangle + |201\rangle$$
$$|2\rangle \to |\tilde{2}\rangle = |021\rangle + |102\rangle + |210\rangle$$

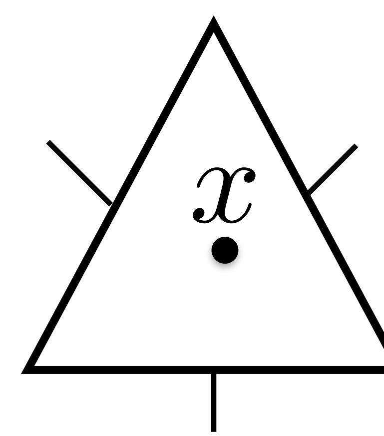
$$X|j\rangle = |j+1\rangle$$
$$Z|j\rangle = \omega|j\rangle \qquad \omega = e^{\frac{2i\pi}{3}}$$

$$E = \sum_j |\tilde{j}\rangle\langle j| \qquad EE^\dagger = P_C \qquad \mathcal{E}nc(\rho) = E\rho E^\dagger$$

$$\bar{Z} \sim_C Z \otimes Z^\dagger \otimes 1 \sim_C 1 \otimes Z \otimes Z^\dagger \sim_C Z^\dagger \otimes 1 \otimes Z$$

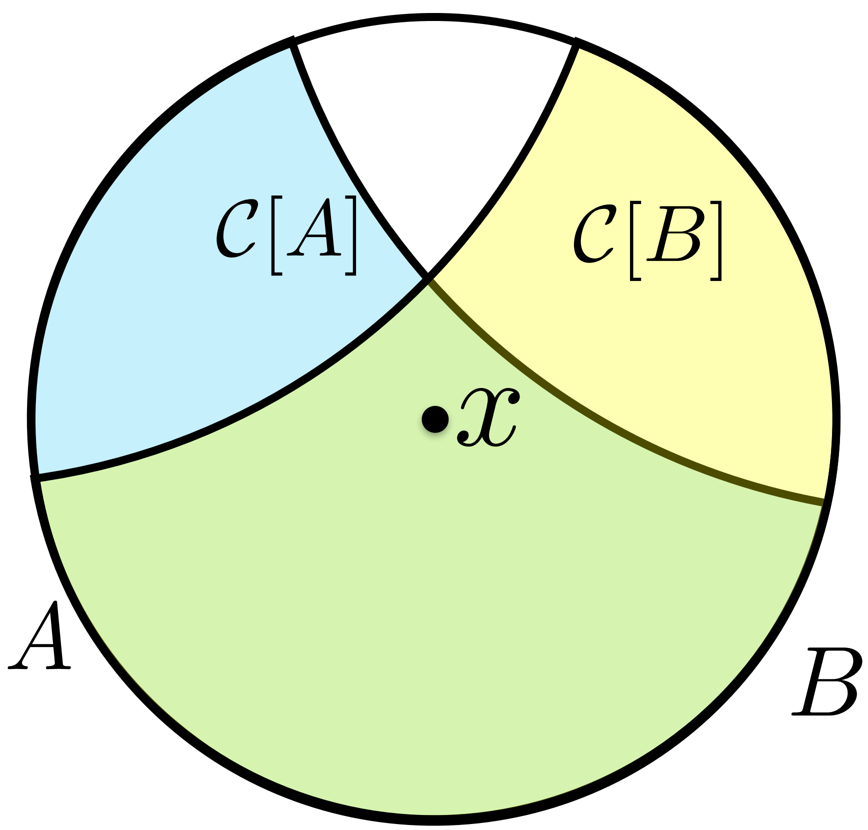
$$\bar{X} \sim_C X \otimes X^\dagger \otimes 1 \sim_C 1 \otimes X \otimes X^\dagger \sim_C X^\dagger \otimes 1 \otimes X$$

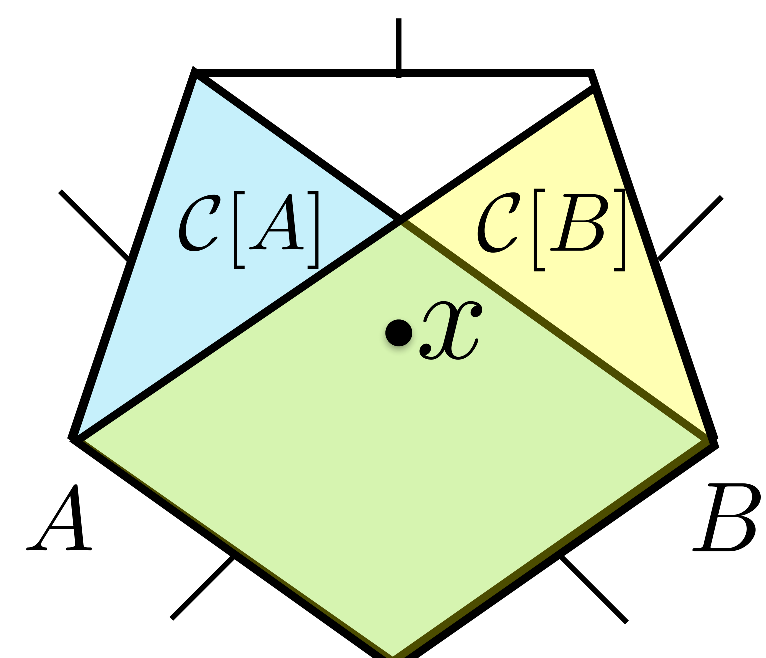
$$d(\bar{X}) = d(\bar{Z}) = d = 2$$



Dictionary

Holography	QECC
Bulk operators	Logical operators
Boundary operators	Physical operators
AdS-Rindler reconstruction	Systematic correctable regions
Vacuum geometry assumption	Code subspace definition
x in the causal wedge of $\mathcal{C}[R]$	R^c correctable with respect to $\mathcal{A}_x$
Bulk locality	Factorable logical algebra



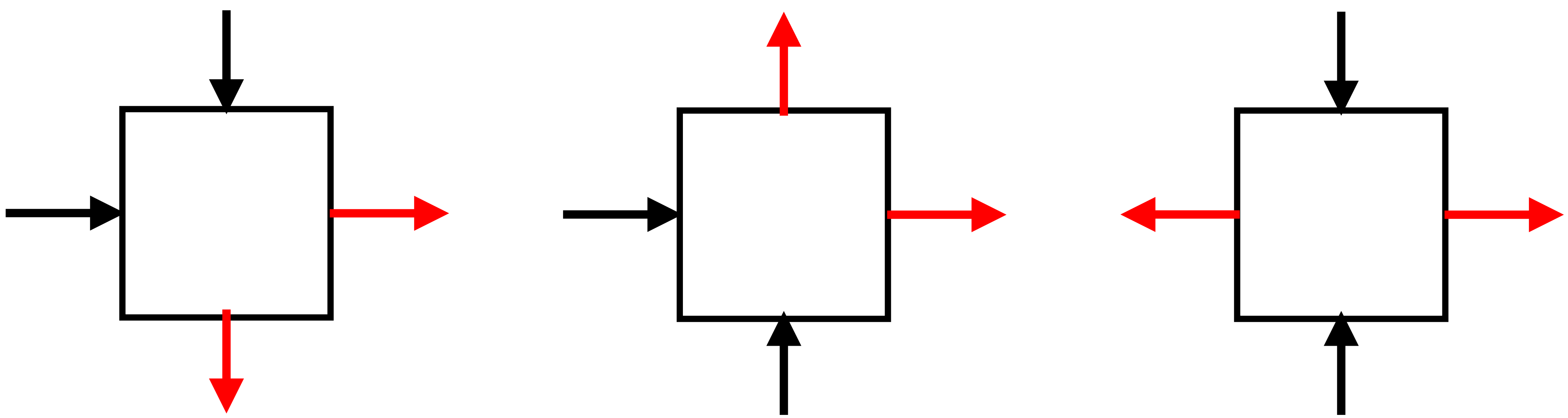


$[[5,1,3]]_2$ code
Single bulk location

Perfect tensors & codes

[[n,k,d]] codes

- Perfect tensors (states) [[2n,0,n+1]] are maximally entangled along any bipartition



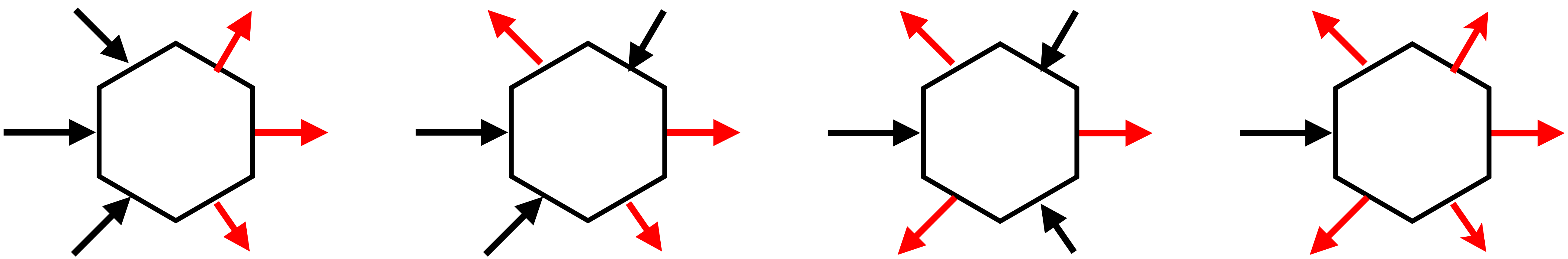
$$|\psi\rangle = \sum_{i_1,i_2,\ldots,i_{2n}} T_{i_1,i_2,\ldots,i_{2n}} |i_1, i_2, \ldots, i_{2n}\rangle$$

Helwig, W. (2013). Absolutely Maximally Entangled Qudit Graph States.

Goyeneche, D., Alsina, D., Latorre, J. I., Riera, A., & Życzkowski, K. (2015). Absolutely maximally entangled states, combinatorial designs, and multiunitary matrices. PRA, 92(3), 032316.

Perfect tensors & codes

- Proportional to unitary on balanced bipartition
- Proportional to [[2n-k,k,n+1-k]] isometry (encoder)



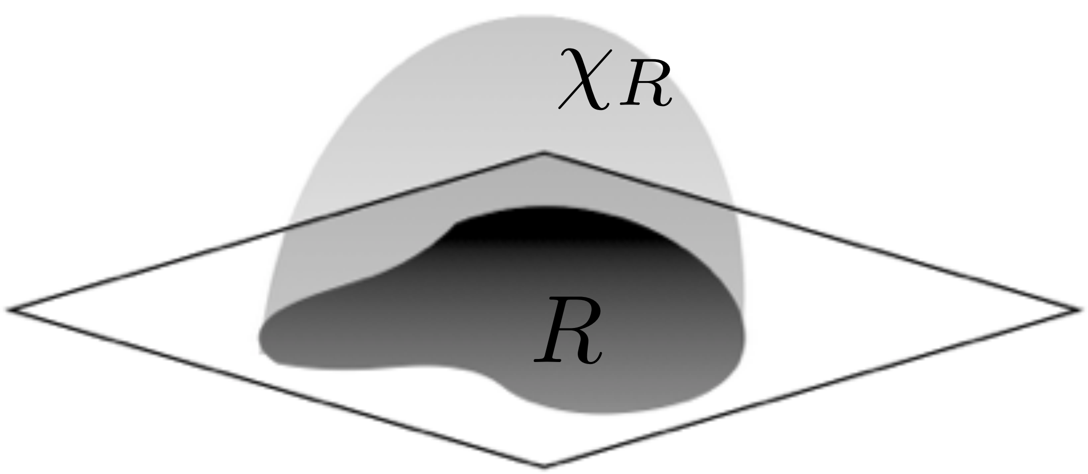
$$|\psi\rangle = \sum_{i_1,i_2,\ldots,i_{2n}} T_{i_1,i_2,\ldots,i_{2n}} |i_1, i_2, \ldots, i_{2n}\rangle$$

PF, Yoshida, B., Harlow, D., & Preskill, J. (2015). Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence. JHEP, 2015(6), 149.

Ryu-Takayanagi formula

Bulk/Boundary duality to Geometry/Entanglement duality

Entanglement ⟷ Geometry (Space-time)



$$S(R) = \frac{1}{4G_N} \min_{\partial\chi_R = \partial R} \mathrm{area}(\chi_R)$$

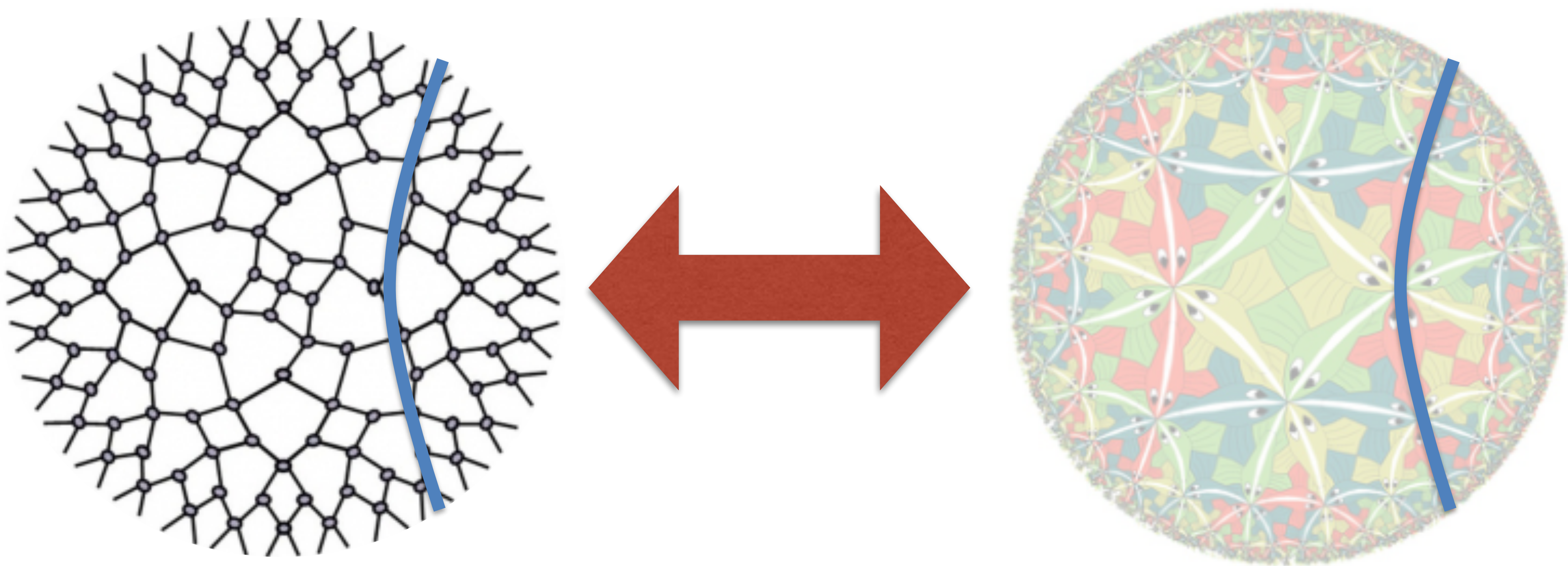
Generalization of Bekenstein-Hawking black hole entropy

Ryu, S., & Takayanagi, T. (2006). Holographic Derivation of Entanglement Entropy from the anti–de Sitter Space/Conformal Field Theory Correspondence. PRL, 96(18), 181602.

MERA

Multiscale entanglement renormalization ansatz
tensor network representations for critical states

$\text{minimal cut length} \geq S_R$ $\text{area}(\chi_R)$

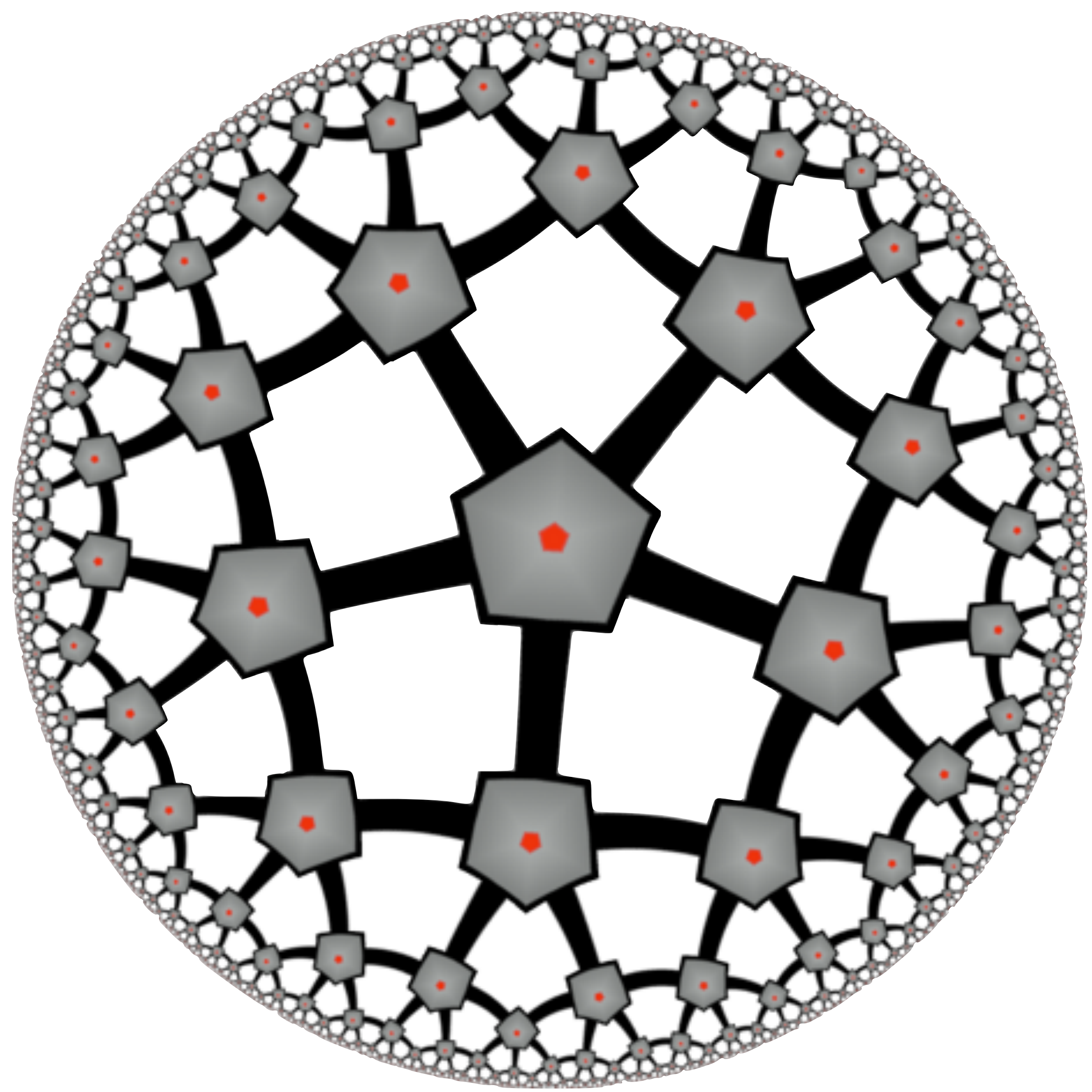


R

Vidal, G. (2008). Class of Quantum Many-Body States That Can Be Efficiently Simulated. PRL, 101(11), 110501.

Swingle, B. (2012). Entanglement renormalization and holography. PR D, 86(6), 065007.

Regular hyperbolic tilings



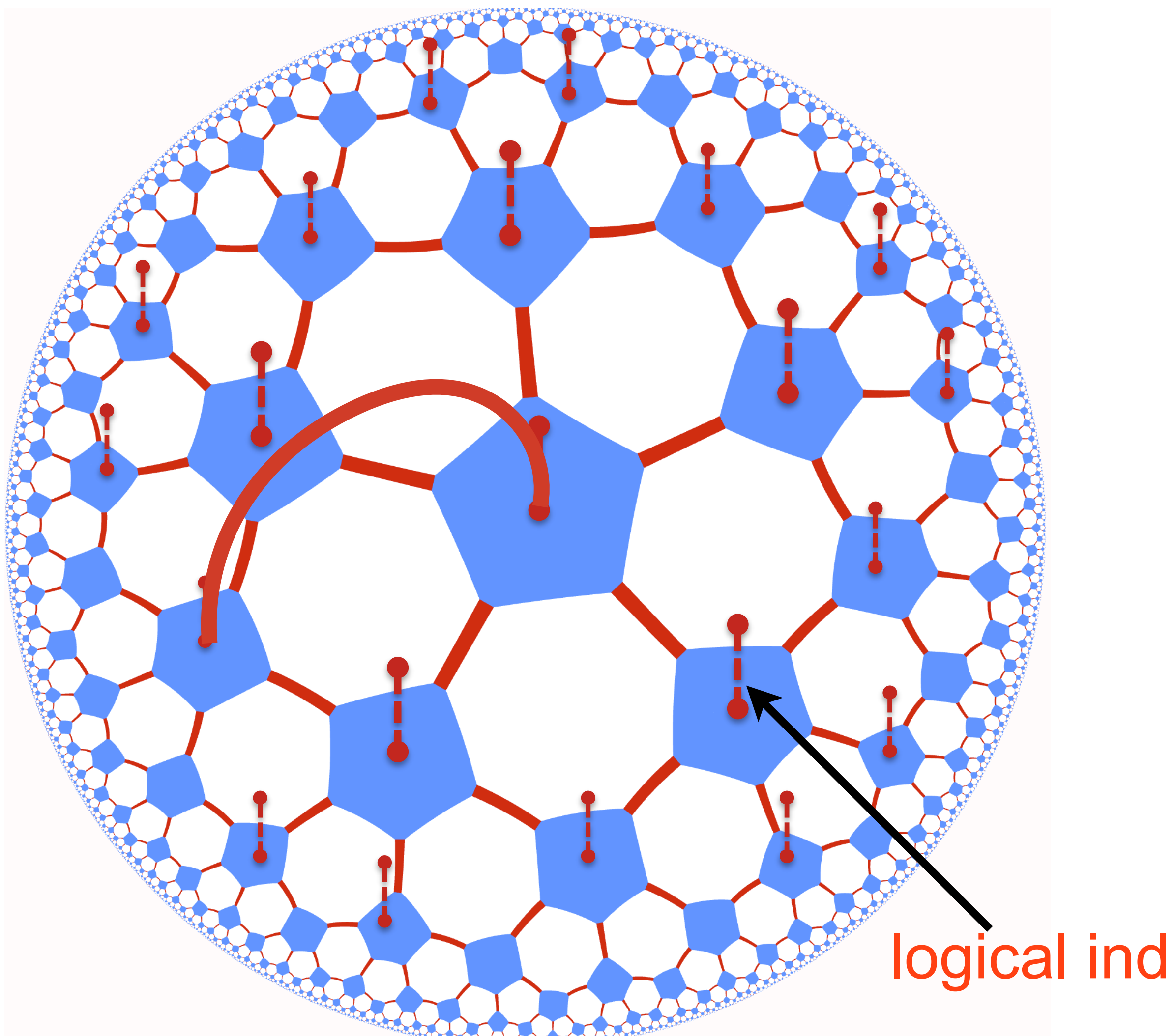
Minimal breaking of bulk symmetries
Almost isotropic bulk -> (no preferred direction)

PF, Yoshida, B., Harlow, D., & Preskill, J. (2015). Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence. JHEP, 2015(6), 149.

Holographic

codes

Has bulk input indices



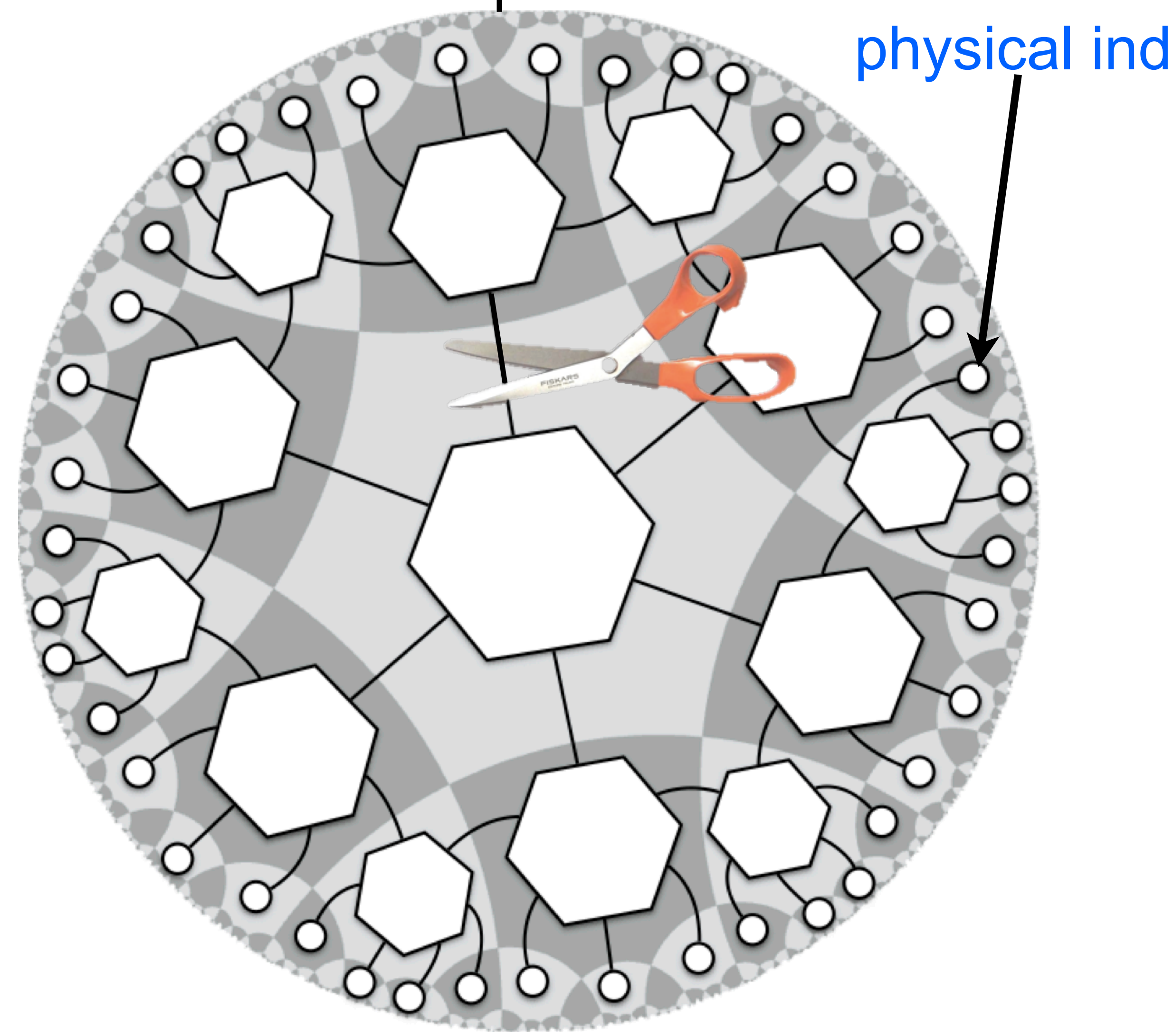
Bulk to boundary isometry

Code -> State

Input local EPR pairs

states

No bulk input indices



Entropy =* Minimal cut

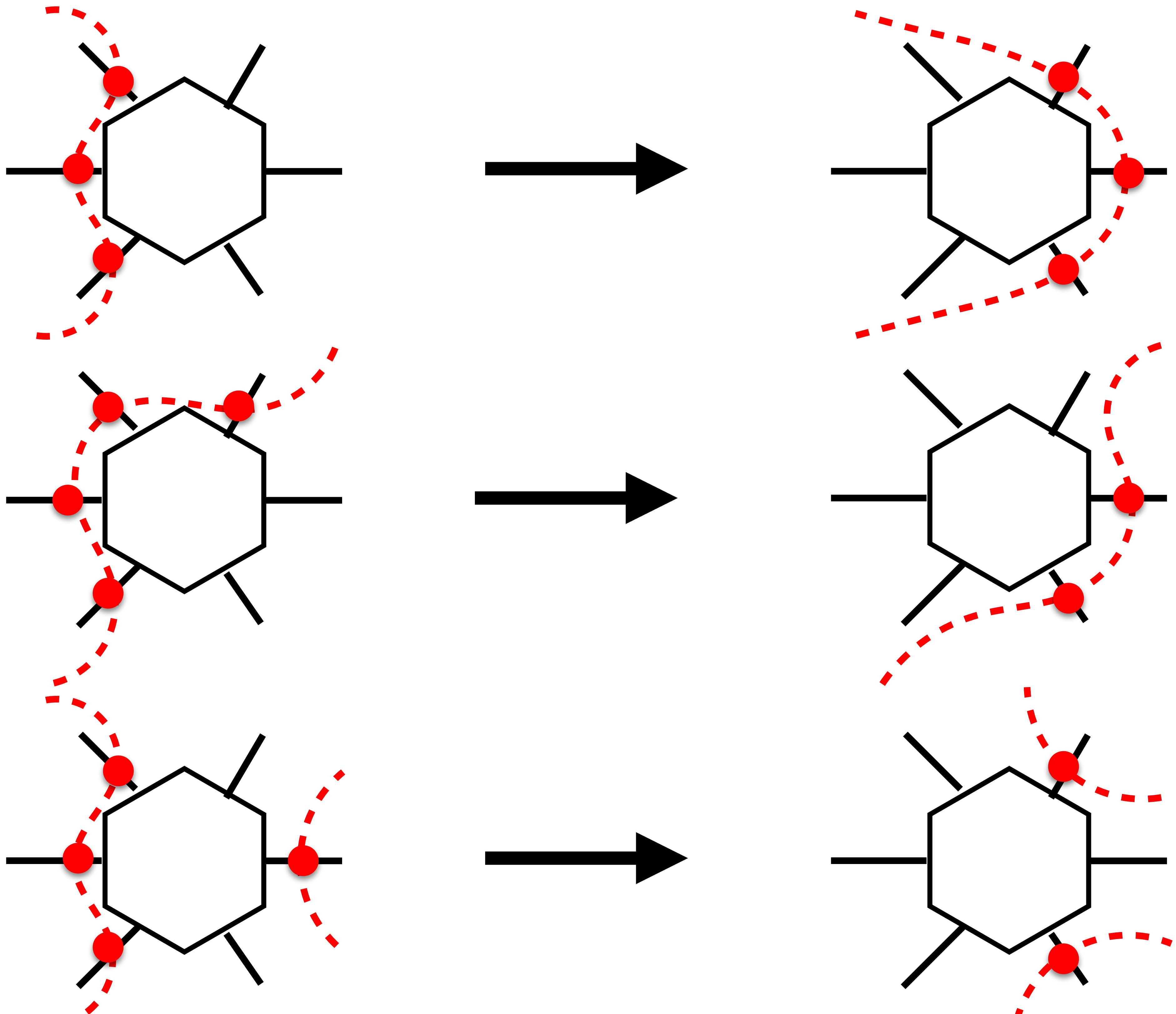
*Guaranteed for contiguous boundary regions

State -> Code

Cut/open virtual bonds

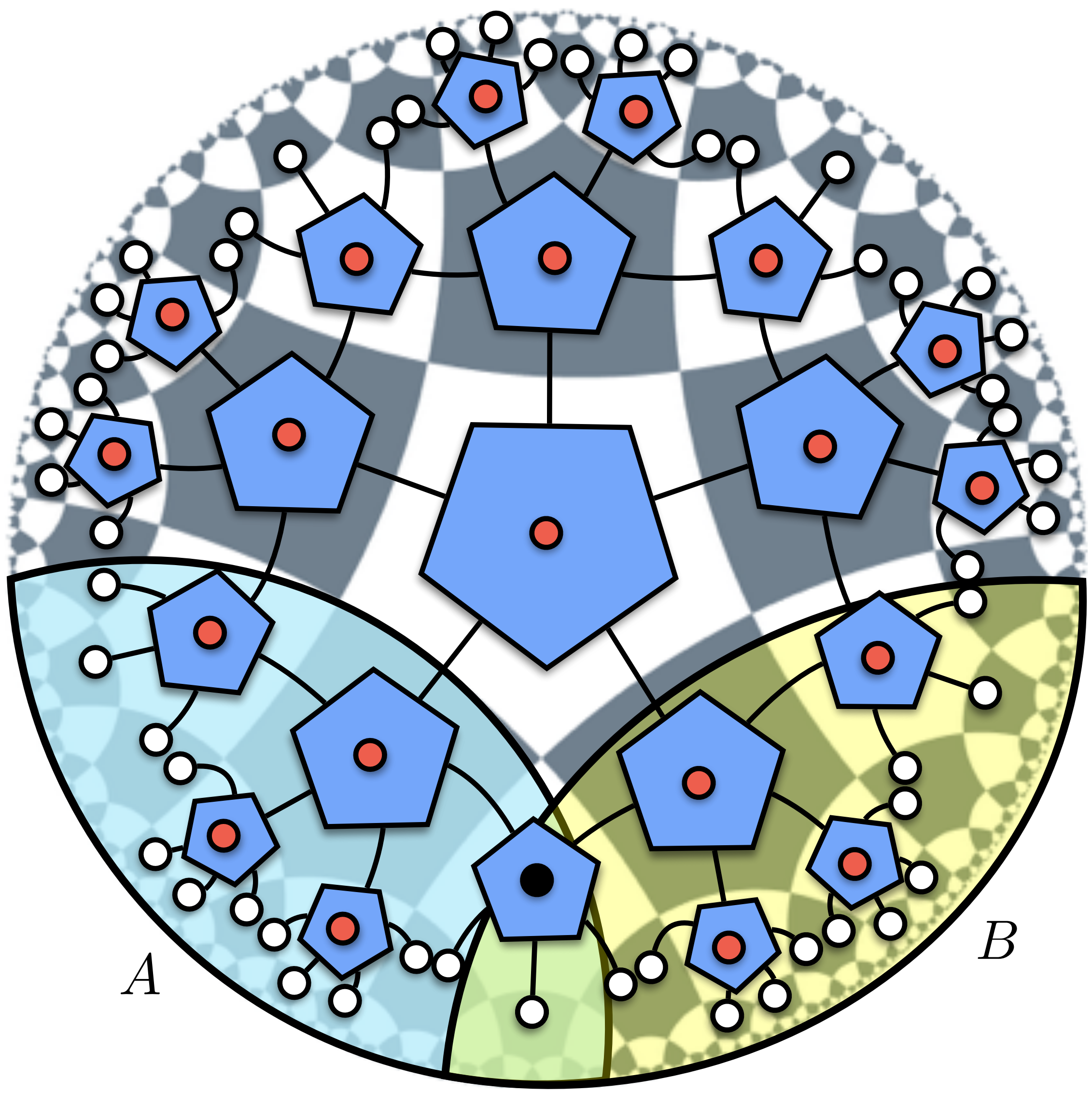
Greedy algorithm

Search minimal surface through local moves



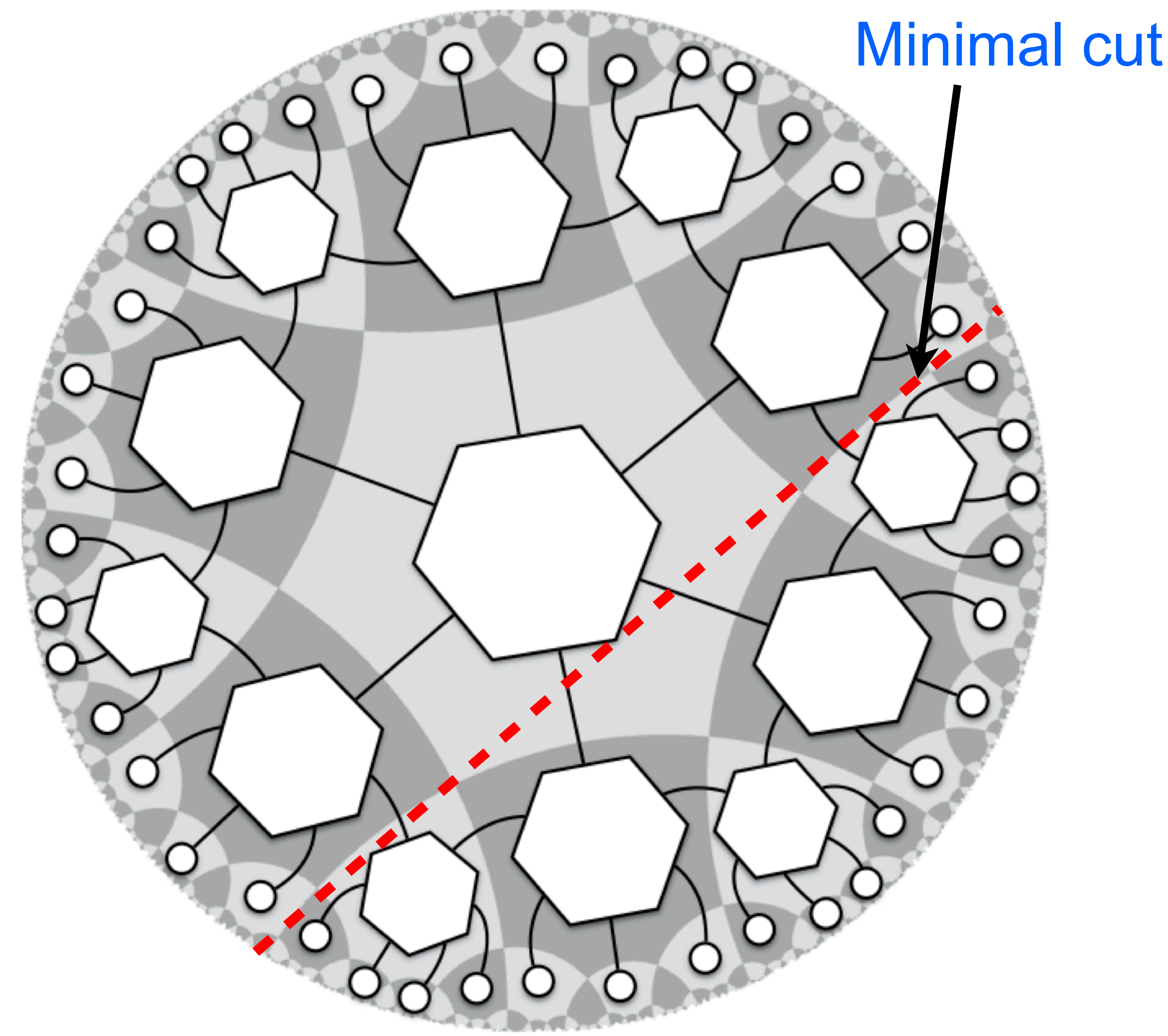
Greedy algorithm

codes



Qualitative analog to AdS-Rindler reconstructions

states

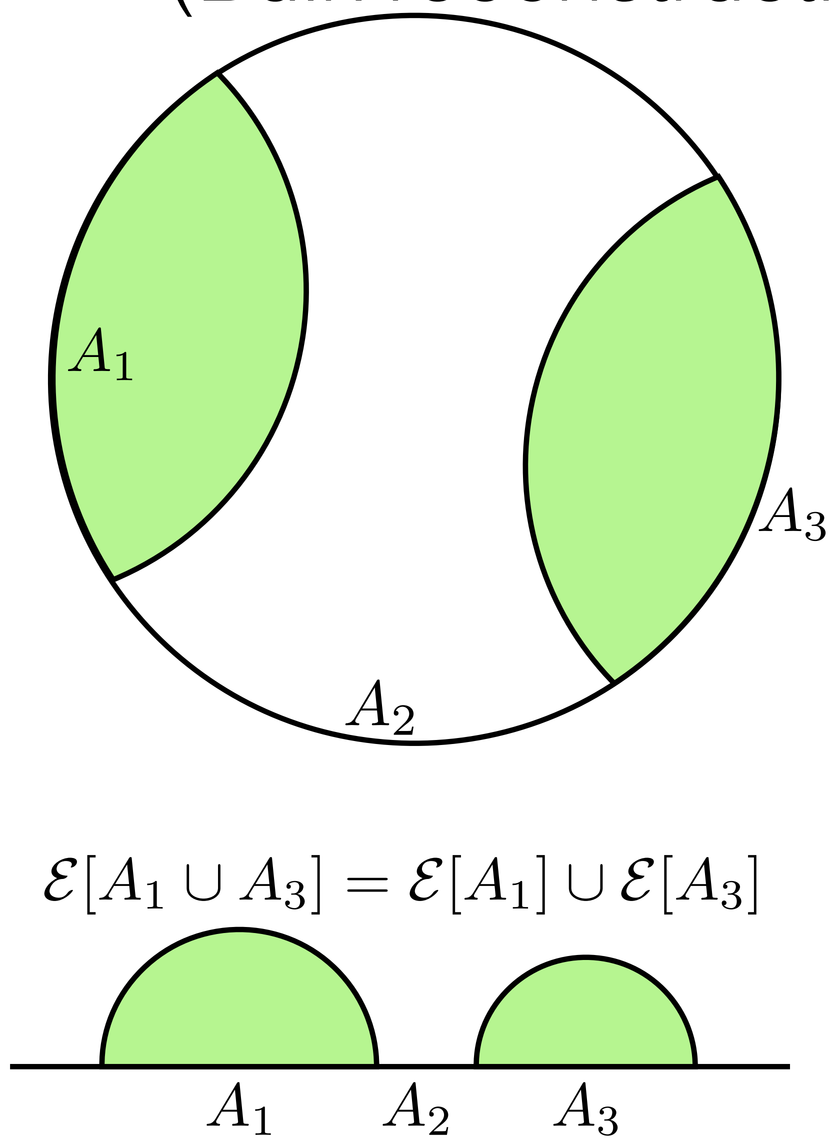


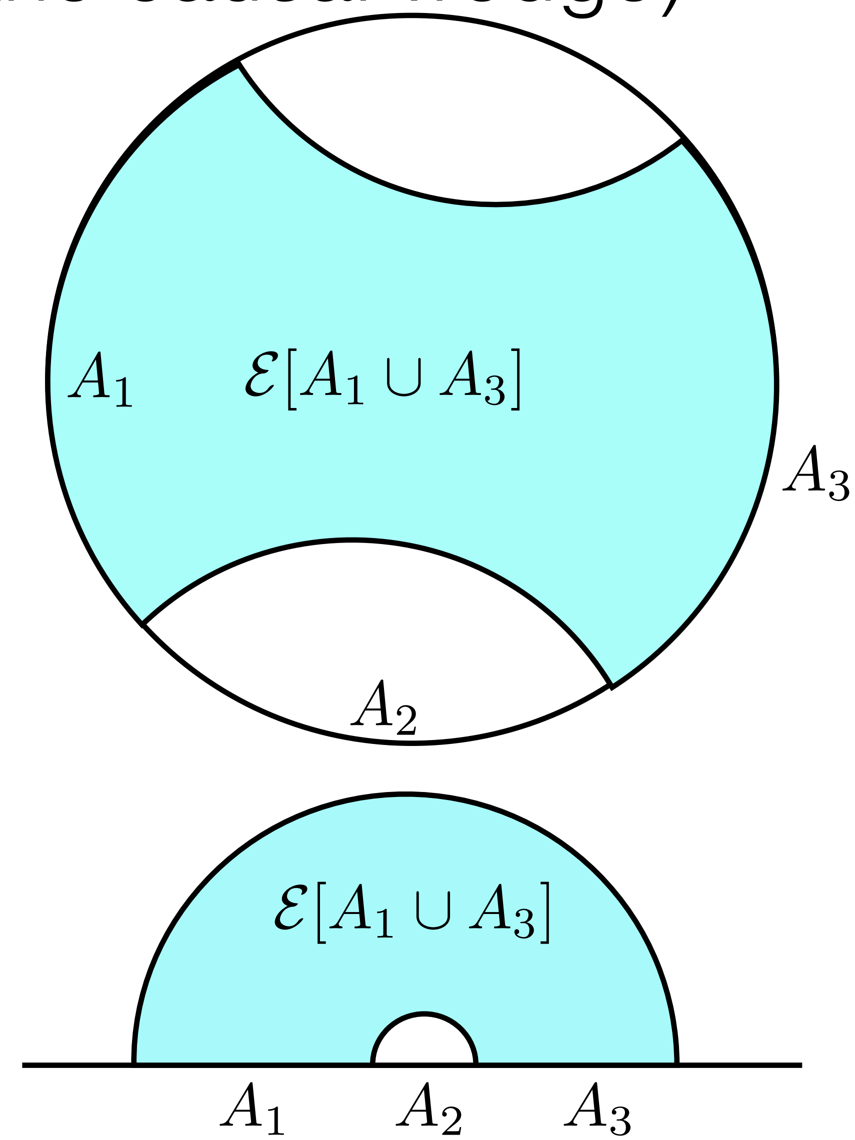
Entanglement distillation*

*Guaranteed for contiguous boundary regions in non-positive curvature planar graph

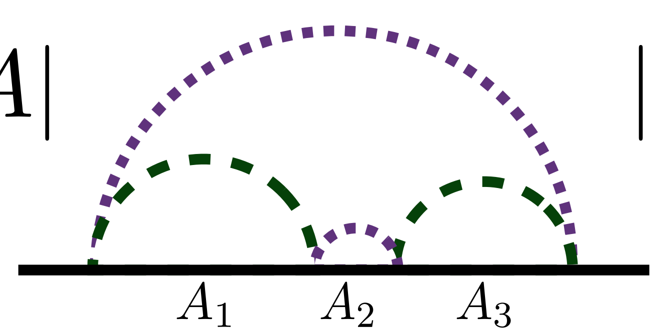
Entanglement wedge

(Bulk reconstruction beyond the causal wedge)



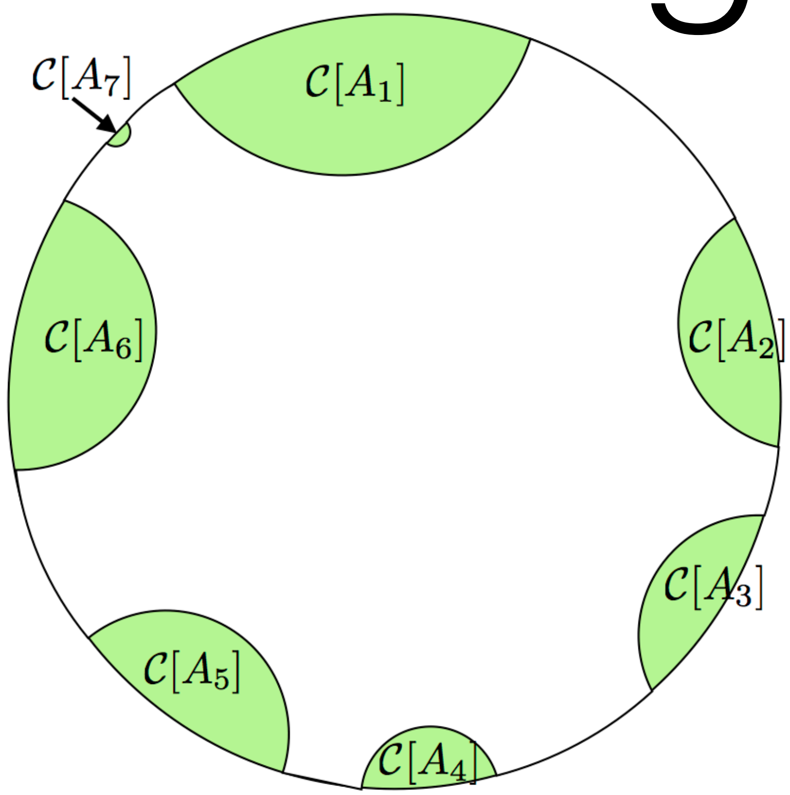


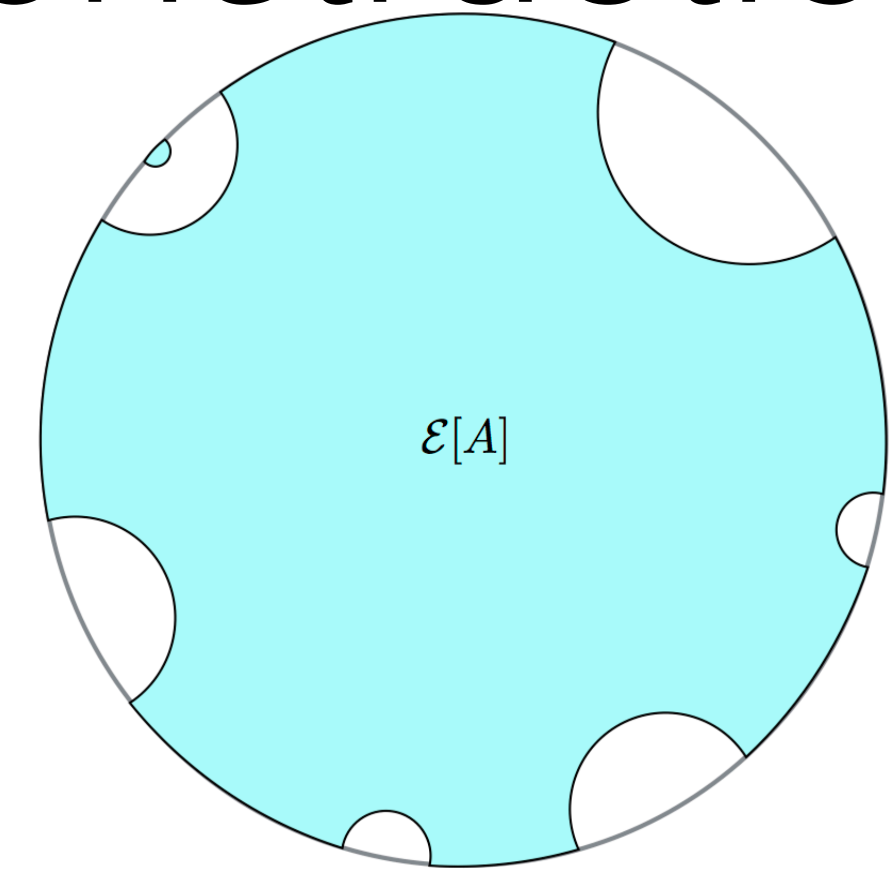
$$|A_1||A_3| \leq |A_2||A|$$



$$|A_1||A_3| \geq |A_2||A|$$

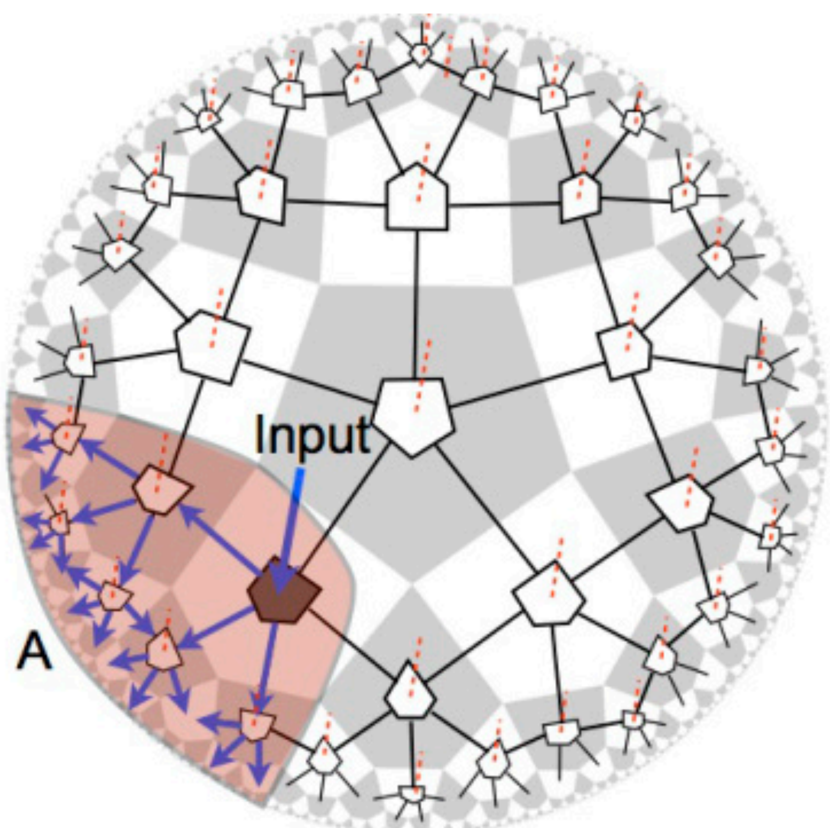
Ent. wedge reconstruction

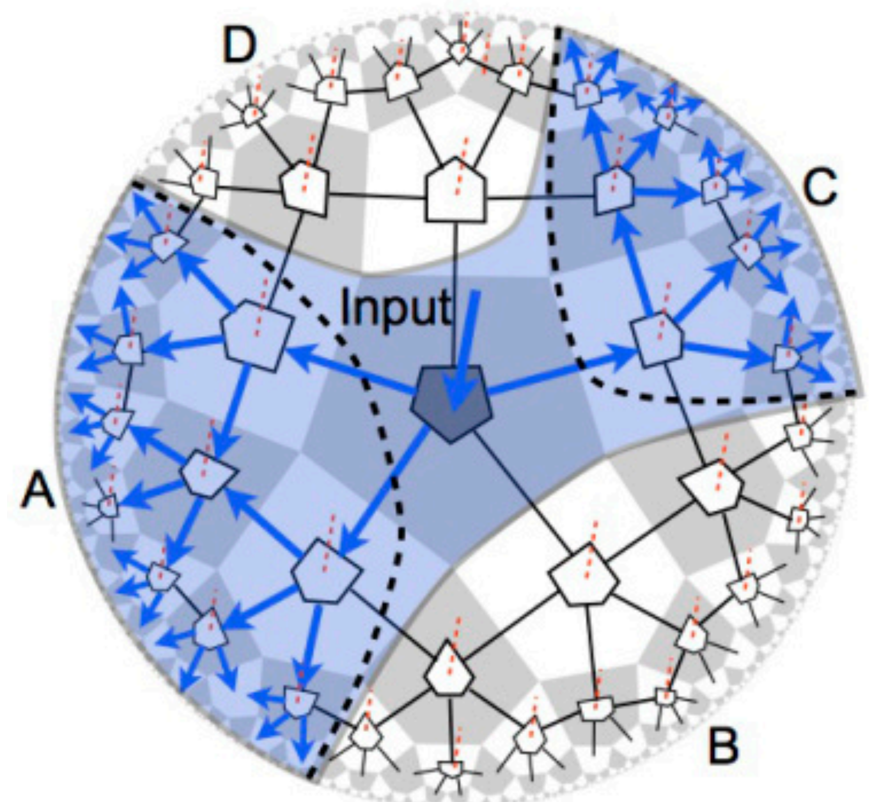




Causal wedge

Entanglement wedge





Why perfect tensors?

(Emergence of unitarity)

Perfect tensor = Maximal entanglement

⇕ Canonical typicality

Typical random tensor ≈ Maximal entanglement

⇕ Quantum chaos

Typical random tensor ≈ Scrambling at AdS scale

Hayden, P., Nezami, S., Qi, X.-L., Thomas, N., Walter, M., & Yang, Z. (2016). Holographic duality from random tensor networks.

Conclusions

- AdS/CFT can be interpreted as a QECC
- AdS scale TN models can reproduce exact properties
- Exact bulk locality
- Bulk-boundary locality
- Geometrization of codes is a fruitful avenue for QECCs