Non-perturbative definition of the energy-momentum tensor on the lattice

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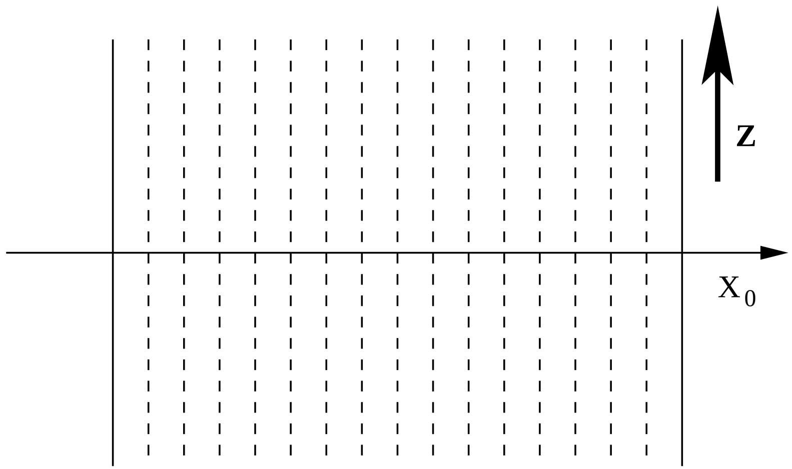
Based on: L. G. and H. B. Meyer 2011-2013; L. G. and M. Pepe 2014-2016

Outline

- Introduction
- Free-energy density with “shifted” boundary conditions

$$f\left(\sqrt{L_0^2+\boldsymbol{z}^2}\right) = -\lim_{V\to\infty}\frac{1}{L_0 V}\ln Z(L_0,\boldsymbol{z})$$

$$\phi(L_0,\boldsymbol{x}) = \phi(0,\mathbf{x}-\boldsymbol{z})$$



- Ward Identities in infinite and finite volume
- Definition of $T_{\mu\nu}$ on the lattice
- Non-perturbative measure of the entropy density
- Conclusions and outlook

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$$f\left(L_0\sqrt{1+\boldsymbol{\xi}^2}\right) = -\lim_{V\to\infty}\frac{1}{L_0 V}\ln Z(L_0,\boldsymbol{\xi})$$

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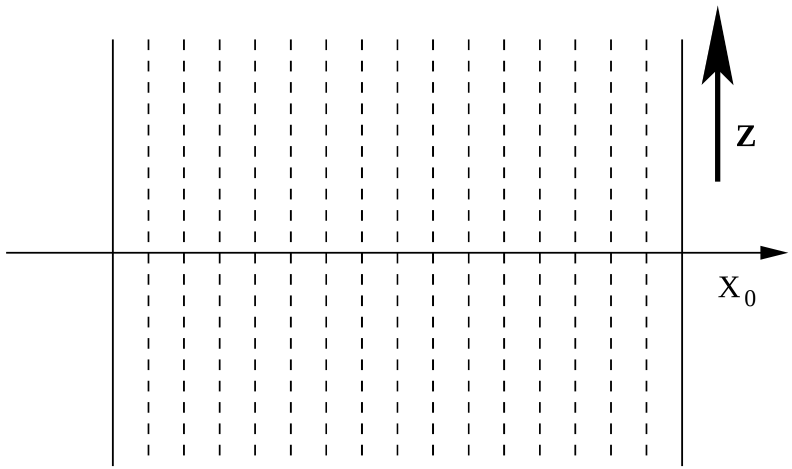

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Energy-momentum tensor on the lattice: the problem

- On the lattice the Poincaré group is broken down to a discrete group and standard discretizations of $T_{\mu\nu}$ acquire finite ultraviolet renormalizations

- We focus on the SU(3) Yang–Mills. The analysis applies to other theories as well

$$T^{\rm R}_{\mu\nu} = Z_T \Big\{ T^{[1]}_{\mu\nu} + z_T T^{[3]}_{\mu\nu} + z_S \big[T^{[2]}_{\mu\nu} - \langle T^{[2]}_{\mu\nu} \rangle_0 \big] \Big\} \, .$$

$$T^{[1]}_{\mu\nu} = (1-\delta_{\mu\nu}) \frac{1}{g_0^2} \Big\{ F^a_{\mu\alpha} F^a_{\nu\alpha} \Big\}$$

$$T^{[2]}_{\mu\nu} = \delta_{\mu\nu} \, \frac{1}{4 g_0^2} \, F^a_{\alpha\beta} F^a_{\alpha\beta}$$

$$T^{[3]}_{\mu\nu} = \delta_{\mu\nu} \frac{1}{g_0^2} \Big\{ F^a_{\mu\alpha} F^a_{\mu\alpha} - \frac{1}{4} F^a_{\alpha\beta} F^a_{\alpha\beta} \Big\}$$

- Non-perturbative definition of $T^{\rm R}_{\mu\nu}$ means knowing Z_T, z_T and z_S so that in the continuum limit

$$\partial_\mu \langle T^{\rm R}_{\mu\nu}(x)\, O(0) \rangle = 0 \, , \quad x \neq 0$$

Energy-momentum tensor on the lattice: recent progress

- Perturbative analysis and 1-loop computation
  [Caracciolo, Curci, Menotti, Pelissetto 88-92]

- Shifted boundary conditions in thermal theory and related WIs
  [LG, Meyer 11-13]

- Energy-momentum tensor from the Yang-Mills gradient flow
  [Makino, Suzuki 13-15; FlowQCD Coll. 14-16]

- Space-time symmetries and the Yang-Mills gradient flow
  [Del Debbio, Patella, Rago 13-15]

- Non-perturbative renormalization of $T_{\mu\nu}$
  [LG, Pepe 14-16]

Thermal field theories in the Euclidean path integral formalism

- From textbooks

$$\phi(x) = \phi(x + V_{\rm pbc} m) \quad m \in \mathbb{Z}^4$$

$$Z(L_0) = {\rm Tr}\left\{ e^{-L_0 \widehat{H}} \right\}$$

$$V_{\rm pbc} = \begin{pmatrix} L_0 & 0 & 0 & 0 \\ 0 & L_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{pmatrix}$$

where the temperature is $T = 1/L_0$

- The basic thermodynamic quantities are defined as

$$f = -\frac{1}{L_0 V} \ln Z(L_0) \qquad e = -\frac{1}{V} \frac{\partial}{\partial L_0} \ln Z(L_0) \qquad s = -\frac{L_0^2}{V} \frac{\partial}{\partial L_0} \left\{ \frac{1}{L_0} \ln Z(L_0) \right\}$$

which in the thermodynamic limit lead to

$$p = -f \qquad s = L_0(e + p) \qquad c_v = -L_0 \frac{\partial}{\partial L_0} s$$

- We are interested in the partition function

$$\phi(x) = \phi(x+V_{\rm sbc} m) \quad m \in \mathbb{Z}^4$$

$$Z(L_0, \boldsymbol{\xi}) = \mathrm{Tr}\left\{ e^{-L_0(\widehat{H} - i\boldsymbol{\xi}\cdot\widehat{\boldsymbol{P}})} \right\}$$

$$V_{\rm sbc} = \begin{pmatrix} L_0 & 0 & 0 & 0 \\ L_0\xi_1 & L_1 & 0 & 0 \\ L_0\xi_2 & 0 & L_2 & 0 \\ L_0\xi_3 & 0 & 0 & L_3 \end{pmatrix}$$

Path integrals with shifted boundary conditions: infinite-volume limit (I)

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where we have chosen $\boldsymbol{\xi} = \{\xi_1, 0, 0\}$

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- By making an Euclidean ”boost” rotation

$$\gamma_1 = \frac{1}{\sqrt{1+\xi_1^2}}$$

$$\Lambda = \begin{pmatrix} \gamma_1 & \gamma_1\xi_1 & 0 & 0 \\ -\gamma_1\xi_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Lorentz [SO(4)] invariance implies

$$Z(L_0,\boldsymbol{\xi}) = \mathrm{Tr}\left\{e^{-L_1\gamma_1(\widetilde{H}-i\xi_1\widetilde{P}_0)}\right\}$$

$$V'_{\rm sbc} = \Lambda\, V_{\rm sbc} = \begin{pmatrix} L_0/\gamma_1 & L_1\gamma_1\xi_1 & 0 & 0 \\ 0 & L_1\gamma_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{pmatrix}$$

Path integrals with shifted boundary conditions: infinite-volume limit (II)

- Assuming that $\widetilde{H}$ has a translationally-invariant vacuum and a mass gap [$\boldsymbol{\xi} = \{\xi_1, 0, 0\}$]

$$Z(L_0, \boldsymbol{\xi}) = \mathrm{Tr}\left\{e^{-L_1\gamma_1(\widetilde{H} - i\xi_1 \widetilde{P}_0)}\right\}$$

$$V'_{\mathrm{sbc}} = \Lambda V_{\mathrm{sbc}} = \begin{pmatrix} L_0/\gamma_1 & L_1\gamma_1\xi_1 & 0 & 0 \\ 0 & L_1\gamma_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{pmatrix}$$

the right hand side becomes insensitive to the phase in the limit $L_1 \to \infty$ at fixed ξ_1

$$f\left(L_0\sqrt{1+\xi_1^2}\right) = -\lim_{V\to\infty} \frac{1}{L_0 V} \ln Z(L_0, \boldsymbol{\xi})$$

$$V''_{\mathrm{sbc}} = \begin{pmatrix} L_0/\gamma_1 & 0 & 0 & 0 \\ 0 & L_1\gamma_1 & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_3 \end{pmatrix}$$

- Thanks to cubic symmetry (infinite volume)

$$f\left(L_0, \boldsymbol{\xi}\right) = f\left(L_0\sqrt{1+\boldsymbol{\xi}^2}, \mathbf{0}\right)$$

$$\phi(x_0, \boldsymbol{x}) = \phi(x_0 + L_0, \boldsymbol{x} + L_0\boldsymbol{\xi})$$

for a generic shift $\boldsymbol{\xi}$

Thermal field theory in a moving frame

- If $\widehat{H}$ and $\widehat{\boldsymbol{P}}$ are the Hamiltonian and the total momentum operator expressed in a moving frame, the standard partition function is

$$\mathcal{Z}(L_0, \boldsymbol{v}) \equiv \mathrm{Tr}\left\{e^{-L_0\,(\widehat{H} - \boldsymbol{v}\cdot\widehat{\boldsymbol{P}})}\right\}$$

- If we continue $\mathcal{Z}$ to imaginary velocities $\boldsymbol{v} = i\boldsymbol{\xi}$

$$Z(L_0, \boldsymbol{\xi}) = \mathrm{Tr}\,\{e^{-L_0(\widehat{H} - i\boldsymbol{\xi}\cdot\widehat{\boldsymbol{P}})}\}$$

- The functional dependence $f(L_0\sqrt{1+\boldsymbol{\xi}^2})$ is consistent with modern thermodynamic arguments on the Lorentz transformation of the temperature and the free-energy
[Ott 63; Arzelies 65; see Przanowski 11 for a recent discussion]

- In the zero-temperature limit the invariance of the theory (and of its vacuum) under the Poincaré group forces its free energy to be independent of the shift $\boldsymbol{\xi}$

- At non-zero temperature the finite length L_0 breaks SO(4) softly, and the free energy depends on the shift (velocity) but only through the combination $\beta = L_0\sqrt{1+\boldsymbol{\xi}^2}$

- The cumulants of the momentum distribution are

$$k_{\{2n,0,0\}} = \frac{1}{V}\langle \widehat{P}_1^{2n}\rangle_c = \frac{(-1)^{n+1}}{L_0^{2n-1}}\frac{\partial^{2n}}{\partial \xi_1^{2n}} f\Big(L_0\sqrt{1+\boldsymbol{\xi}^2}\Big)\Big|_{\boldsymbol{\xi}=0}$$

Cumulants of the energy and the momentum distributions

- The cumulants of the momentum distribution are

$$\frac{k_{\{2n,0,0\}}}{L_0} = (-1)^{n+1}\,(2n-1)!!\left\{\frac{1}{L_0}\frac{\partial}{\partial L_0}\right\}^n f\left(L_0\sqrt{1+\boldsymbol{\xi}^2}\right)\Big|_{\boldsymbol{\xi}=0}$$

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- The cumulants of the energy distribution are

$$c_n = \frac{1}{V}\,\langle\,\widehat{H}^n\,\rangle_c = (-1)^{n+1}\left[n\,\frac{\partial^{n-1}}{\partial L_0^{n-1}} + L_0\,\frac{\partial^n}{\partial L_0^n}\right] f\Big(L_0\sqrt{1+\boldsymbol{\xi}^2}\Big)\Big|_{\boldsymbol{\xi}=0} \qquad n = 2, 3\ldots$$

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which imply that

$$k_{\{2n,0,0\}} = \frac{(2n-1)!!}{(2L_0^2)^n}\sum_{\ell=1}^{n}\frac{(2n-\ell)!}{\ell!(n-\ell)!}\,(2L_0)^\ell\,c_\ell$$

the total energy and momentum distributions of a relativistic thermal theory are related

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Up to $n=4$ it reads

$$\begin{aligned}
L_0\,k_{\{2,0,0\}} &= c_1\\
L_0^3\,k_{\{4,0,0\}} &= 9\,c_1 + 3\,L_0\,c_2\ ,\\
L_0^5\,k_{\{6,0,0\}} &= 225\,c_1 + 90\,L_0\,c_2 + 15\,L_0^2\,c_3\ ,\\
L_0^7\,k_{\{8,0,0\}} &= 11025\,c_1 + 4725\,L_0\,c_2 + 1050\,L_0^2\,c_3 + 105\,L_0^3\,c_4
\end{aligned}$$

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- Thermodynamic potentials can be extracted from the momentum cumulants

$$k_{\{2,0,0\}} = T\,(e+p) = T^2\,s$$

$$k_{\{4,0,0\}} = 3T^4\,(c_v + 3s)$$

by remembering that $\frac{\partial p}{\partial T} = s$

Euclidean Ward identities for correlators of $\overline{T}_{\mu\nu}$

- In the path integral formalism

$$L_0 \, \langle \overline{T}_{01} \, T_{01} \rangle_c = \langle T_{00} \rangle - \langle T_{11} \rangle$$

$$L_0^3 \, \langle \overline{T}_{01} \, \overline{T}_{01} \, \overline{T}_{01} \, T_{01} \rangle_c = 9 \, \langle T_{11} \rangle - 9 \, \langle T_{00} \rangle + 3 \, L_0 \, \langle \overline{T}_{00} T_{00} \rangle_c$$

$$\dots$$

where $\langle T_{00} \rangle = -e$, $\langle T_{11} \rangle = p$, $\widehat{P}_1 \to -i\overline{T}_{01}$ and

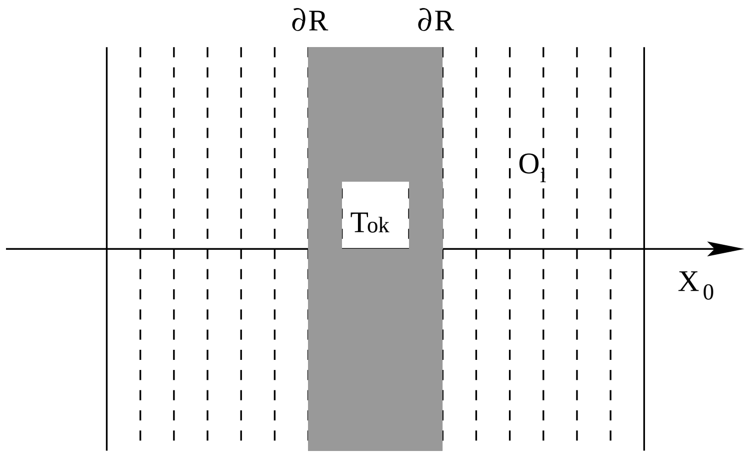
$$\overline{T}_{\mu\nu}(x_0) = \int d^3x \, T_{\mu\nu}(x)$$

- Note that:
  - ∗ All operators at non-zero distance
  - ∗ Number of EMT on the two sides different
  - ∗ On the lattice they can be imposed to fix the renormalization of $T_{\mu\nu}$

- The commutator of boost with momentum

$$[\hat{K}_k, \hat{P}_k] = i\hat{H}$$

is expressed in the Euclidean by the WIs



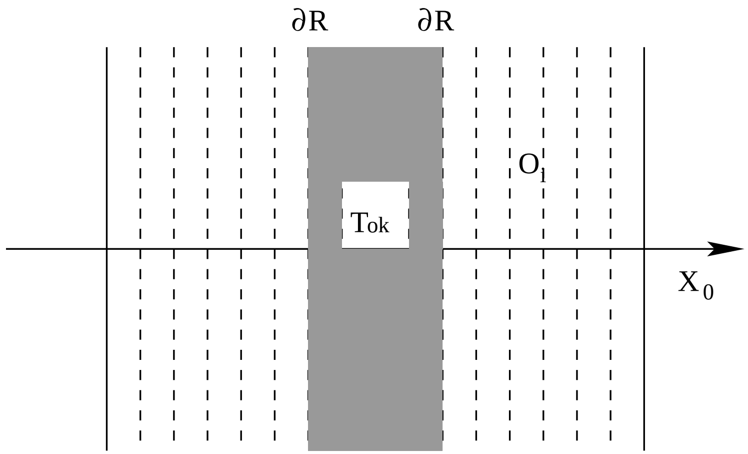
$$\int_{\partial R} d\sigma_\mu(x)\, \langle K_{\mu;0k}(x)\, \overline{T}_{0k}(y_0)\, O_1 \ldots O_n \rangle_c = \langle \overline{T}_{00}(y_0)\, O_1 \ldots O_n \rangle_c$$

when the O_i are localized external fields and, as usual,

$$K_{\mu;\alpha\beta} = x_\alpha T_{\mu\beta} - x_\beta T_{\mu\alpha}$$

- In a 4D box boost transformations are incompatible with (periodic) bc. WIs associated with $SO(4)$ rotations must be modified by finite-size contributions

Ward identities in thermal theory: where they come from ?



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- The finite-volume theory is translational invariant, and it has a conserved $T_{\mu\nu}$. Modified WIs associated to boosts constructed from those associated to translations. In infinite spatial volume

$$L_0\, \langle \overline{T}_{01}(x_0)\, T_{01}(y) \rangle_c = \langle T_{00} \rangle - \langle T_{kk} \rangle$$

Ward identities at non-zero shift

- When $\boldsymbol{\xi} \neq 0$ odd derivatives in the ξ_k do not vanish anymore, and new interesting WIs hold. The first non-trivial one is

$$\langle T_{0k} \rangle_{\boldsymbol{\xi}} = \frac{\xi_k}{1-\xi_k^2} \left\{ \langle T_{00} \rangle_{\boldsymbol{\xi}} - \langle T_{kk} \rangle_{\boldsymbol{\xi}} \right\}$$

which implies

$$s = - \frac{L_0 \, (1+\boldsymbol{\xi}^2)^{3/2}}{\xi_k} \, \langle T_{0k} \rangle_{\boldsymbol{\xi}}$$

- By deriving twice with respect to the ξ_k

$$\langle T_{0k} \rangle_{\boldsymbol{\xi}} = \frac{L_0 \xi_k}{2} \sum_{ij} \left\langle \overline{T}_{0i} \, T_{0j} \right\rangle_{\boldsymbol{\xi},\, c} \left[\delta_{ij} - \frac{\xi_i \, \xi_j}{\boldsymbol{\xi}^2} \right]$$

which implies for instance

$$s^{-1} = - \frac{1}{2(1+\boldsymbol{\xi}^2)^{3/2}} \sum_{ij} \frac{\left\langle \overline{T}_{0i} \, T_{0j} \right\rangle_{\boldsymbol{\xi},\, c}}{\langle T_{0i} \rangle_{\boldsymbol{\xi}} \langle T_{0j} \rangle_{\boldsymbol{\xi}}} \, \xi_i \xi_j \left[\delta_{ij} - \frac{\xi_i \xi_j}{\boldsymbol{\xi}^2} \right]$$

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$$\frac{c_v}{s^2} = -\frac{1}{2(1+\boldsymbol{\xi}^2)^{3/2}} \sum_{ij} \frac{\left\langle \overline{T}_{0i}\, T_{0j} \right\rangle_{\boldsymbol{\xi},\, c}}{\langle T_{0i} \rangle_{\boldsymbol{\xi}} \langle T_{0j} \rangle_{\boldsymbol{\xi}}} \frac{\xi_i \xi_j}{\boldsymbol{\xi}^2} \left[(1-2\boldsymbol{\xi}^2)\delta_{ij} - 3\frac{\xi_i \xi_j}{\boldsymbol{\xi}^2} \right]$$

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- Note that also in this case:

  ∗ All operators at non-zero distance

  ∗ Number or components of EMT on the two sides different

  ∗ On the lattice they can be imposed to fix the renormalization of $T_{\mu\nu}$

Entropy density from the response to the shift

- The Entropy density can be computed as

$$s = -\frac{L_0\,(1+\boldsymbol{\xi}^2)^{3/2}}{\xi_k}\,\langle T_{0k}\rangle_{\boldsymbol{\xi}}$$

or as

$$s = -\frac{(1+\boldsymbol{\xi}^2)^{3/2}}{\xi_k}\,\lim_{V\to\infty}\frac{1}{V}\frac{\partial}{\partial\xi_k}\ln Z(L_0,\boldsymbol{\xi})$$

- With respect to the standard technique:

  ∗ No ultraviolet power divergent subtraction (zero temperature subtraction)

  ∗ On the lattice finite multiplicative renormalization constant fixed non-perturbatively by WIs

Path integrals with shifted boundary conditions: finite-size effects

- The leading finite-size contributions to the free energy are

$$f(V_{\rm sbc}) - f\Big(L_0\sqrt{1+\boldsymbol{\xi}^2}\Big) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \cdots$$

where for $L_k = L$

$$\mathcal{I}_i = \frac{\gamma\nu}{2\pi L_0 L^3}\frac{1}{r}\frac{d}{dr}\Big[\frac{e^{-MLr}}{r}\Big]\Big|_{r=r_i}\,, \quad r_i = \frac{\gamma}{\bar\gamma_i}\,, \quad \bar\gamma_i = 1\Big/\sqrt{1+\sum_{k\neq i}\xi_k^2}$$

with M and ν being the mass and the multiplicity of the lightest screening state

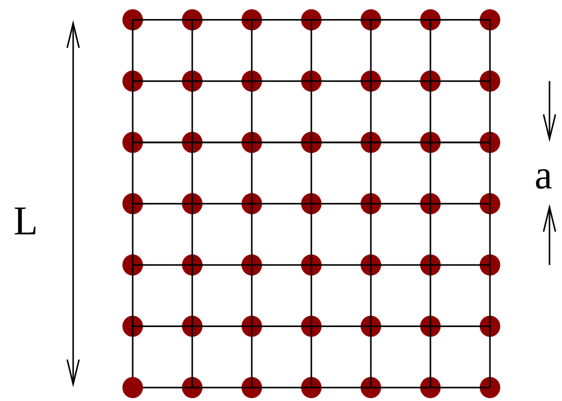
- Analogous formula for the entropy by noticing that

$$\langle T_{0k}\rangle_{V_{\rm sbc}} - \langle T_{0k}\rangle_{\boldsymbol{\xi}} = -\frac{\partial}{\partial\xi_k}\sum_{i=1}^{3}\mathcal{I}_i + \ldots$$

- WIs can be derived analogously in finite volume. They are modified by terms which vanish exponentially in the thermodynamic limit

Lattice gauge theory [Wilson 74]

- A Yang-Mills theory can be defined on a discretized space-time so that gauge invariance is preserved



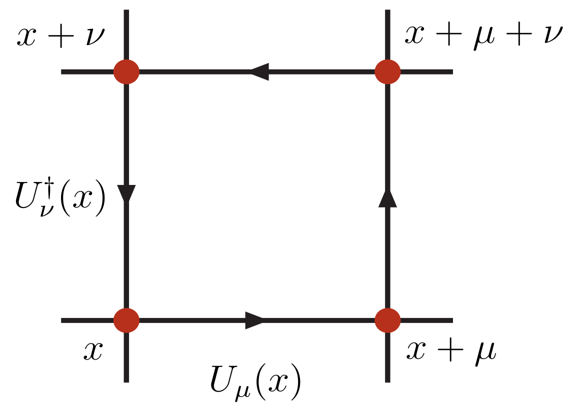
- The the gauge field $U_\mu \in \mathrm{SU}(3)$ resides on links

- The Wilson action is

$$S_G[U] = \frac{\beta}{2} \sum_x \sum_{\mu,\nu} \left[1 - \frac{1}{3}\mathrm{ReTr}\left\{U_{\mu\nu}(x)\right\}\right]$$

where $\beta = 6/g_0^2$ and the plaquette is

$$U_{\mu\nu}(x) = U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x)$$



- Discrete shifts in the boundary conditions can be implemented straightforwardly

Non-perturbative renormalization of $T_{\mu\nu}$

- On the lattice the Poincaré group is broken down to a discrete group and standard discretizations of $T_{\mu\nu}$ acquire finite ultraviolet renormalizations

- We focus on the SU(3) Yang–Mills. The analysis applies to other theories as well

$$T^{\rm R}_{\mu\nu} = Z_T \Big\{ T^{[1]}_{\mu\nu} + z_T\, T^{[3]}_{\mu\nu} + z_S \big[T^{[2]}_{\mu\nu} - \langle T^{[2]}_{\mu\nu} \rangle_0 \big] \Big\} \, .$$

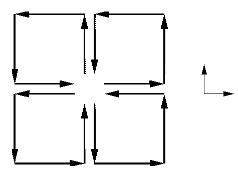
$$T^{[1]}_{\mu\nu} = (1-\delta_{\mu\nu}) \frac{1}{g_0^2} \Big\{ F^a_{\mu\alpha} F^a_{\nu\alpha} \Big\}$$

$$T^{[2]}_{\mu\nu} = \delta_{\mu\nu} \frac{1}{4 g_0^2} F^a_{\alpha\beta} F^a_{\alpha\beta}$$

$$T^{[3]}_{\mu\nu} = \delta_{\mu\nu} \frac{1}{g_0^2} \Big\{ F^a_{\mu\alpha} F^a_{\mu\alpha} - \frac{1}{4} F^a_{\alpha\beta} F^a_{\alpha\beta} \Big\}$$

where

$$F^a_{\mu\nu}(x) = -\frac{i}{4a^2} \mathrm{Tr}\Big\{ \Big[Q_{\mu\nu}(x) - Q_{\nu\mu}(x) \Big] T^a \Big\} \, , \qquad Q_{\mu\nu}(x) = \sum$$



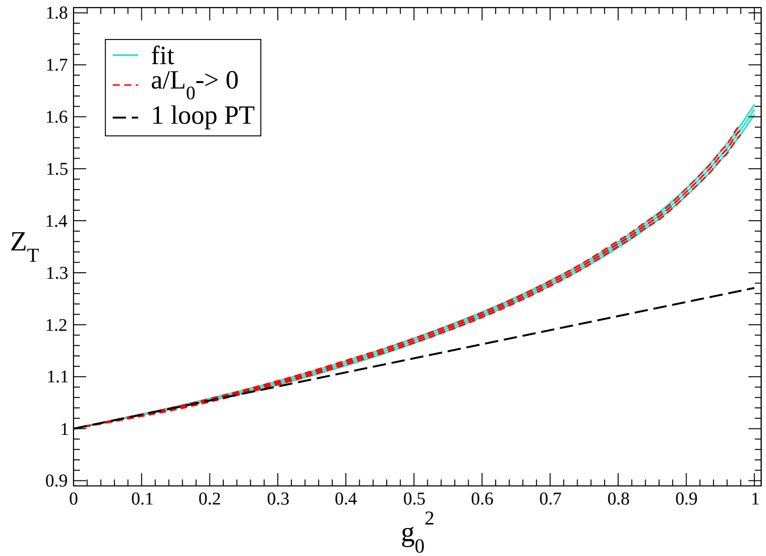
The sextet renormalization constant Z_T

- The continuum relation

$$\langle T_{0k}\rangle_{\boldsymbol{\xi}} = \frac{1}{L_0}\lim_{V\to\infty}\frac{1}{V}\frac{\partial}{\partial\xi_k}\ln Z(L_0,\boldsymbol{\xi})$$

can be imposed on the lattice to fix Z_T

$$Z_T(g_0^2) = -\frac{\Delta f}{\Delta\xi_k}\,\frac{1}{\langle T^{[1]}_{0k}\rangle_{\boldsymbol{\xi}}}$$



where the derivative in the shift is discretized by the symmetric finite difference

$$\frac{\Delta f}{\Delta\xi_k} = \frac{1}{2aV}\ln\Big[\frac{Z(L_0,\boldsymbol{\xi}-a\hat{k}/L_0)}{Z(L_0,\boldsymbol{\xi}+a\hat{k}/L_0)}\Big]$$

- The final results for $Z_T(g_0^2)$ are well represented by

$$Z_T(g_0^2) = \frac{1-0.4457\,g_0^2}{1-0.7165\,g_0^2} - 0.2543\,g_0^4 + 0.4357\,g_0^6 - 0.5221\,g_0^8$$

with the error that varies from 0.4% up 0.7% in the range $0 \le g_0^2 \le 1$

The sextet renormalization constant Z_T

- The continuum relation

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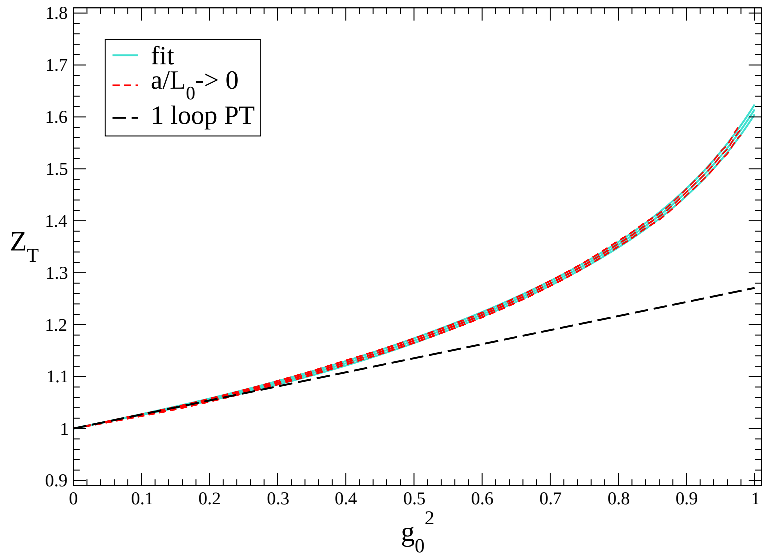

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- Within statistical errors, the non-perturbative determination starts to deviate significantly from the one-loop result [Caracciolo et al. 88, 90]

$$Z_T(g_0^2) = 1 + 0.27076\; g_0^2$$

already at $g_0^2 \sim 0.25$

The triplet renormalization constant z_T

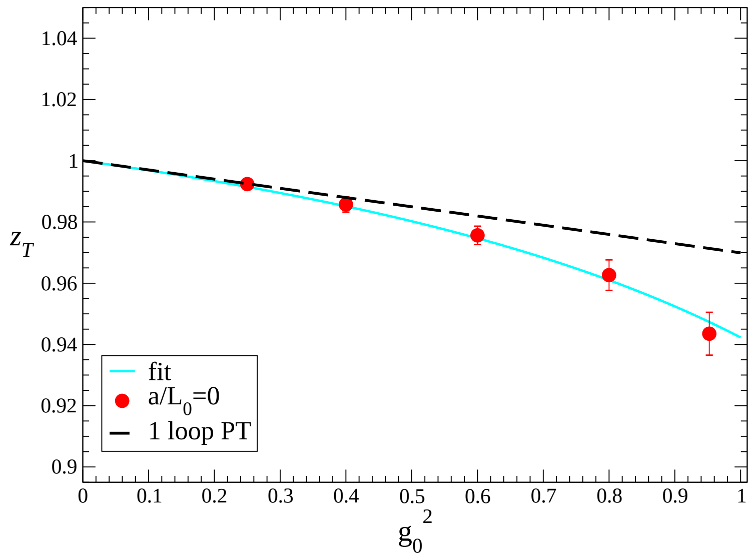
- The continuum relation

$$\langle T_{0k} \rangle_{\boldsymbol{\xi}} = \frac{\xi_k}{1-\xi_k^2} \left\{ \langle T_{00} \rangle_{\boldsymbol{\xi}} - \langle T_{kk} \rangle_{\boldsymbol{\xi}} \right\}$$

is enforced on the lattice to determine z_T

$$z_T(g_0^2) = \frac{1-\xi_k^2}{\xi_k} \frac{\langle T_{0k}^{[1]} \rangle_{\boldsymbol{\xi}}}{\langle T_{00}^{[3]} \rangle_{\boldsymbol{\xi}} - \langle T_{kk}^{[3]} \rangle_{\boldsymbol{\xi}}}$$

with the condition $\frac{L\,\xi_k}{L_0(1+\xi_k^2)} = q \in \mathbb{Z}$



- The results for $z_T(g_0^2)$ are well represented by

$$z_T(g_0^2) = \frac{1-0.5090\,g_0^2}{1-0.4789\,g_0^2}$$

where the error grows linearly from 0.15% to 0.75% in the interval $0 \le g_0^2 \le 1$

The triplet renormalization constant z_T

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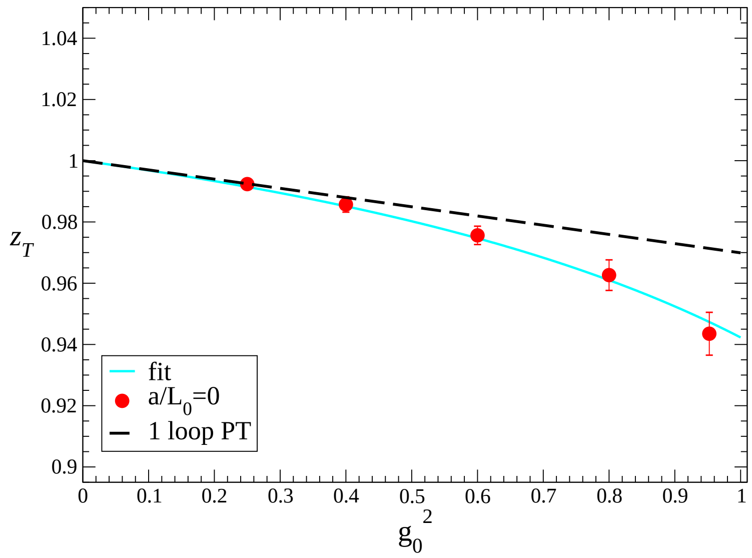


- Within statistical errors, the non-perturbative determination starts to deviate significantly from the one-loop result [Caracciolo et al. 88, 90]

$$z_T(g_0^2) = 1 - 0.03008\; g_0^2$$

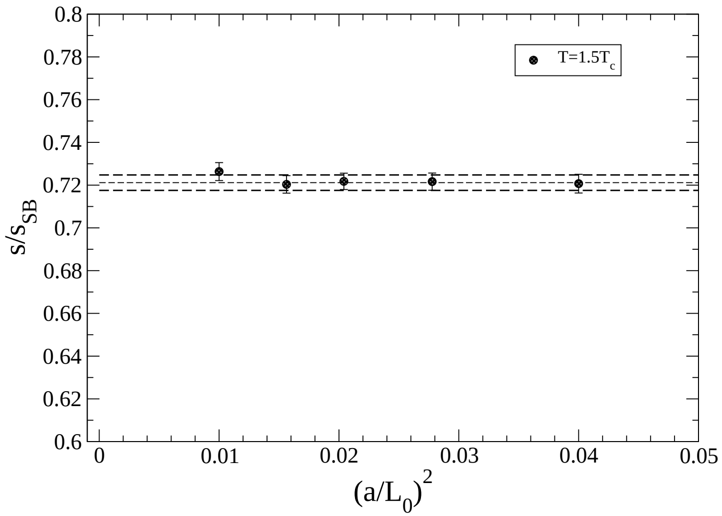
already at $g_0^2 \sim 0.4$

Entropy density in the continuum

- At all temperatures the entropy density is obtained by extrapolating

$$\frac{s}{s_{\rm SB}} = -\frac{45}{32\pi^2}\,\frac{(1+\xi^2)}{\xi_k}\,\frac{Z_T\,\langle T^{[1]}_{0k}\rangle_{\boldsymbol{\xi}}}{T^4}$$

to the continuum limit



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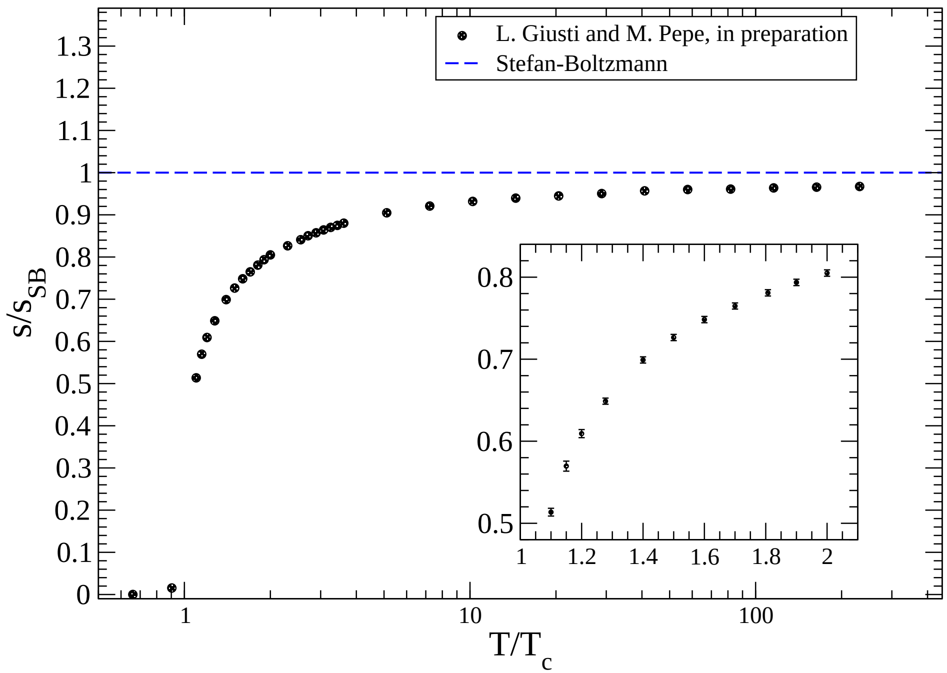

- Precision of $\sim 0.5\%$ for all points
- At $T \sim 230\,T_c$ the entropy still differs from the Stefan-Boltzmann value by $\sim 3\%$

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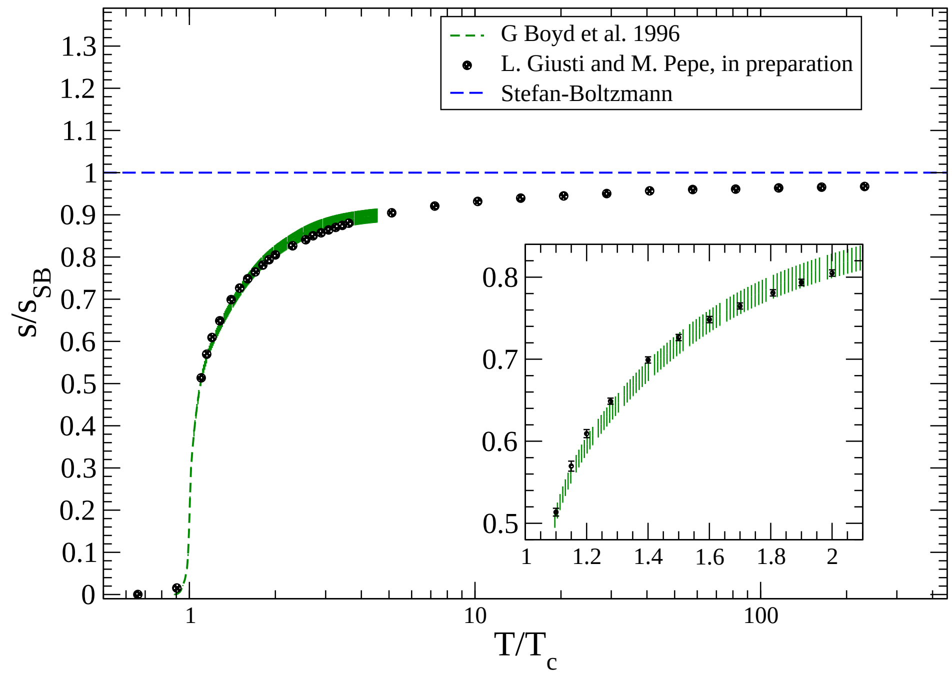

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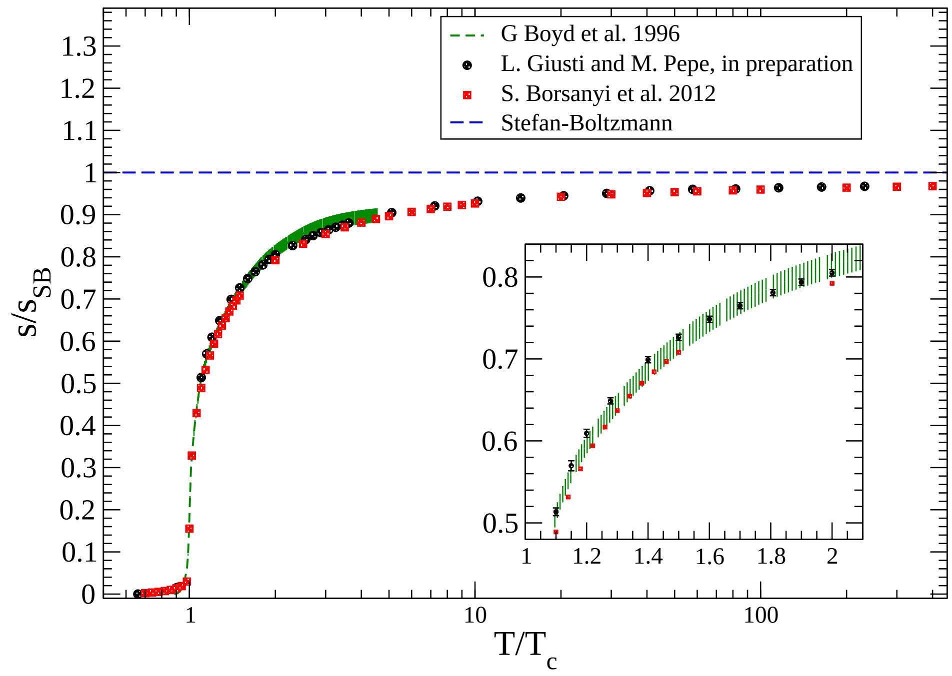

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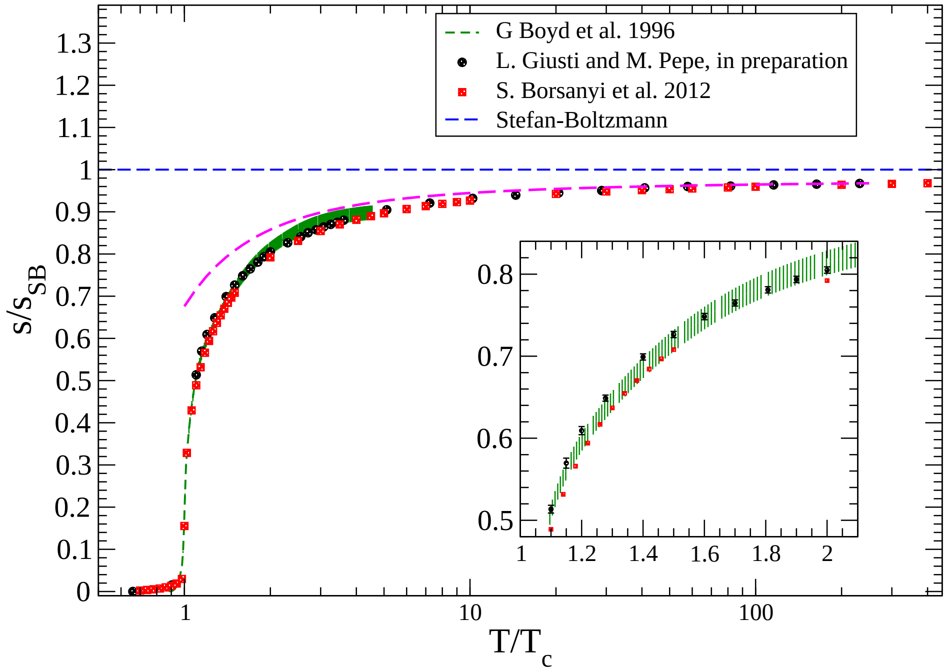

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- When matching with perturbation theory (blue line), the series has oscillating coeffs. At $T \sim 230\,T_c$, the $O(g^6)$ is roughly 50% of total correction with respect to SB

Conclusions and outlook (I)

- Lorentz invariance implies a great degree of redundancy in defining a relativistic thermal theory in the Euclidean path-integral formalism

- In the thermodynamic limit, the orientation of the compact periodic direction with respect to the coordinate axes can be chosen at will and only its length is relevant

$$f\left(L_0\sqrt{1+\boldsymbol{\xi}^2}\right) = -\lim_{V\to\infty} \frac{1}{L_0 V} \ln Z(L_0, \boldsymbol{\xi})$$

- The redundancy in the description implies that the total energy and momentum distributions in the canonical ensemble are related

- For a finite-size system, the lengths of the box dimensions break this invariance. Being a soft breaking, however, interesting exact Ward Identities survive

- As in the standard case, if the lightest screening mass $M \neq 0$, leading finite-size corrections are exponentially small in (ML)

Conclusions and outlook (II)

- When the theory is regularized on a lattice, the overall orientation of the periodic directions with respect to the lattice coordinate system affects renormalized observables at the level of lattice artifacts

- As the cutoff is removed, the artifacts are suppressed by a power of the spacing

- The flexibility in the lattice formulation added by the introduction of a triplet $\boldsymbol{\xi}$ of (renormalized) parameters has interesting consequences:

  $*$ WIs to renormalize non-perturbatively $T_{\mu\nu}$

  $*$ Simpler ways to compute thermodynamic potentials

$$s = -\frac{Z_T L_0 (1 + \boldsymbol{\xi}^2)^{3/2}}{\xi_k} \langle T_{0k} \rangle_{V_{\rm sbc}}$$

  $* \ldots$

- In the Yang–Mills theory we defined non-perturbatively $T_{\mu\nu}$, and we computed the entropy density over several orders of magnitude in T. Discretization and statistical errors are at the level of a few per mille in both cases

Singlet and trace anomaly

$$\frac{\partial}{\partial \xi_k} \langle T_{\mu\mu} \rangle_{\boldsymbol{\xi}} = \frac{1}{(1+\xi^2)^2} \frac{\partial}{\partial \xi_k} \left[\frac{(1+\xi^2)^3}{\xi_k} \langle T_{0k} \rangle_{\boldsymbol{\xi}} \right] \; .$$

where

$$T_{\mu\mu} = \frac{b_0}{2} \{F^a_{\alpha\beta} F^a_{\alpha\beta}\}^{\rm RGI} \; .$$

with

$$\{F^a_{\alpha\beta} F^a_{\alpha\beta}\}^{\rm RGI} = -\frac{\beta}{b_0 g^3} \{F^a_{\alpha\beta} F^a_{\alpha\beta}\}^R$$