The Effective Bootstrap

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Introduction and Motivation

Conformal Field Theories (CFTs) play a fundamental role in theoretical physics

They are the starting and ending points of renormalization group flows in QFTs

They describe second order phase transitions in critical phenomena

They are an essential part of string theory and quantum gravity by means of the AdS/CFT correspondence

In particular, using AdS/CFT, one might hope to shed light on deep and hard quantum gravity problems by rephrasing them in a CFT context

Several emergent phenomena arise in the IR end-point CFT

Typically such CFTs are strongly coupled and difficult to study

Expansions such as large N, ϵ- expansion, very useful but still approximations

It would be desirable to exploit the full power of conformal symmetry to get first principles results

Luckily enough, this is possible and goes under the name of the conformal bootstrap or simply bootstrap approach

Advocated in the 70's by [Ferrara, Grillo, Gatto '73; Polyakov, '74] it didn't receive too much attention until recently, where it has been shown to be of interest for CFTs with d>2 [Rattazzi et al, 0807.0004]

Great progress has been achieved recently using bootstrap techniques, using both analytical and numerical techniques (no time to review them here)

In this talk we will focus on aspects related to the numerical bootstrap

Basic idea in numerical analysis is to make some assumption about the structure of some CFT and check if it is consistent with fundamental principles such as unitarity and crossing symmetry. If it is not, that CFT is ruled out.

The CFT does **not** need to have a Lagrangian description. In fact, in bootstrap analysis a CFT is axiomatically defined by the so called CFT data.

Recall that the conformal group in d euclidean dimensions is SO(d+1,1)

In addition to the usual translation and rotation generators $P_M, \ J_{MN}$

we have now the dilatations and the special conformal generators $D, \ K_M$

The basic operators of a CFT are the so called primary operators

$$[K_M, \mathcal{O}(0)] = 0$$

They are characterized by their scaling dimension and SO(d) "spin" quantum numbers

$$[D, \mathcal{O}(0)] = i\Delta\mathcal{O}(0)$$

The spectrum of a CFT is given by the set of its primary operators.

The interactions of a CFT are given by the set of three-point functions among its primary operators. These are uniquely fixed by conformal symmetry, up to some constants, the three-point coefficients λ_{ijk}

The CFT data are defined as

- Spectrum of primary operators (scaling dimensions Δ_i and spins l_i)
- Their three point function coefficients λ_{ijk}

Conformal symmetry allows us to fix all the higher point functions of the theory in terms of the CFT data. If they are known, in principle the theory is solvable

Four-point correlation functions are not kinematically determined, because with four points we can construct two independent conformal invariant scalar quantities, called cross ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$x_{ij}^2 = (x_i - x_j)_M (x_i - x_j)^M$$

Simplest four-point function: identical scalars.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}}$$

The dynamical information of the CFT is encoded in the function $g(u,v)$

Using the Operator Product Expansion (OPE), we can relate g(u,v) to the CFT data

$$\phi(x)\phi(0) = x^{-2\Delta} + \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} \mathcal{T}^{M_1 \ldots M_l}(x) \mathcal{O}_{M_1 \ldots M_l}(0) + \ldots$$

Sum over all symmetric traceless primary operators in the OPE of the two scalars.

$\lambda_{\phi\phi\mathcal{O}}$ is the coefficient of the $\langle \phi\phi\mathcal{O} \rangle$ three-point function

$\mathcal{T}^{M_1 \ldots M_l}(x)$ are determined tensor structure coefficients

Taking the OPE between two pairs of operators gives

$$g(u,v) = 1 + \sum_{l,\Delta} \lambda^2_{\phi\phi\mathcal{O}_{\Delta,l}} g_{\Delta,l}(u,v)$$

The functions $g_{\Delta,l}(u,v)$, contrary to $g(u,v)$, are kinematically determined

For each primary of dimension Δ and spin l, $g_{\Delta,l}(u,v)$ encodes the contribution of all its descendants in the exchange.

These functions are called conformal blocks and are the key players in the conformal bootstrap

The conformal blocks for external scalar operators exchanging traceless symmetric operators of spin s is known for any s. For even space-time dimension d=2,4,6 they are known in a closed and compact form.

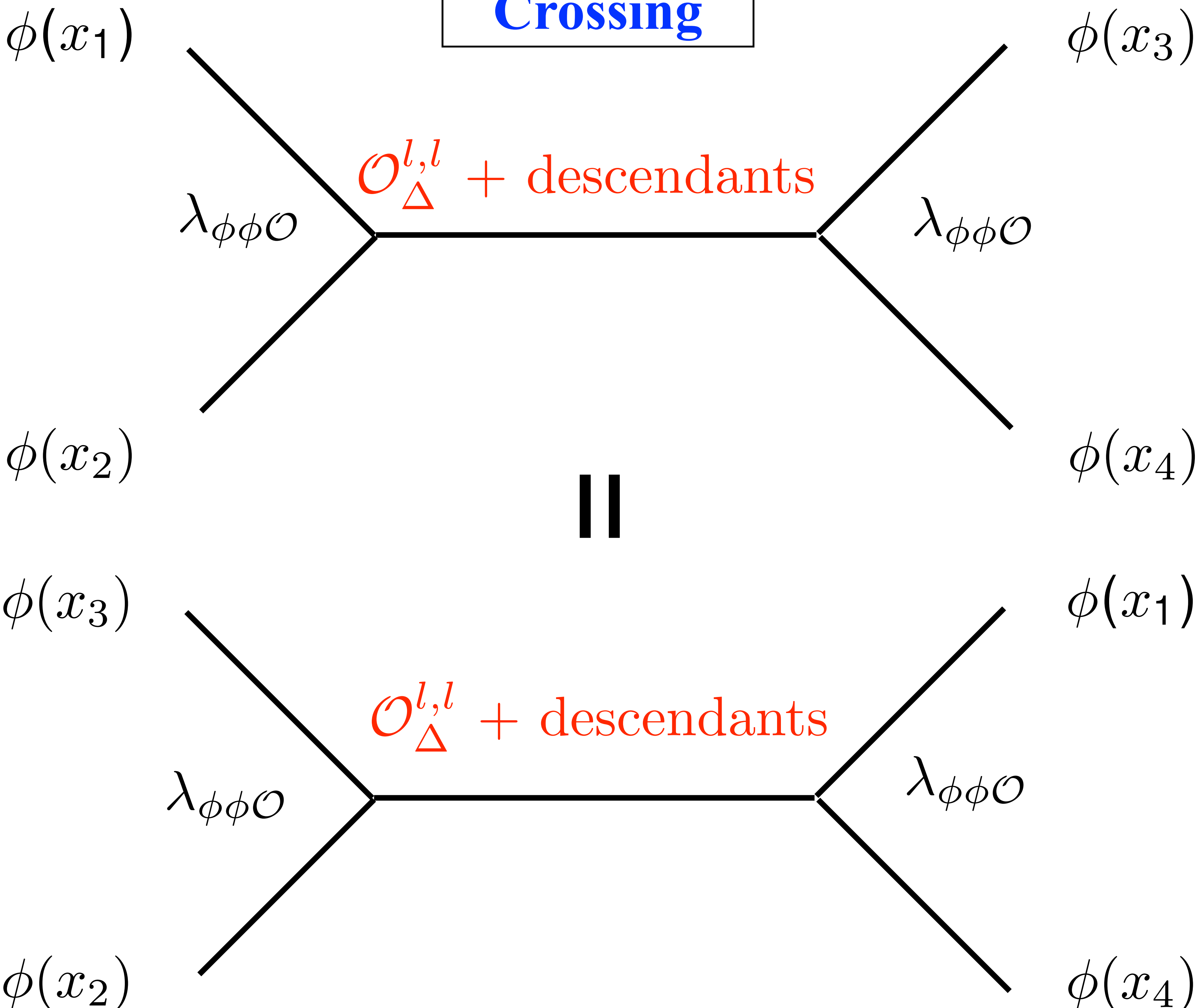
[Dolan,Osborn, hep-th/0011040, hep-th/0309180]

In odd dimensions they can be obtained recursively

The two key principles of the bootstrap are **crossing symmetry** and **unitarity** (the latter not always necessary)

Unitarity: $\lambda^2_{\phi\phi\mathcal{O}} > 0$

Crossing



$\phi(x_1)$
$\phi(x_3)$
$\mathcal{O}_{\Delta}^{l,l}$ + descendants
$\lambda_{\phi\phi\mathcal{O}}$
$\lambda_{\phi\phi\mathcal{O}}$
$\phi(x_2)$
$\phi(x_4)$
=
$\phi(x_3)$
$\phi(x_1)$
$\mathcal{O}_{\Delta}^{l,l}$ + descendants
$\lambda_{\phi\phi\mathcal{O}}$
$\lambda_{\phi\phi\mathcal{O}}$
$\phi(x_2)$
$\phi(x_4)$

Demanding that the OPE in two different pairings (s and t channels) gives the same result, we get a crossing symmetry constraint

$$\sum_{\Delta,l} \lambda^2_{\phi\phi\mathcal{O}} F_{d,\Delta,l}(u,v) = 1$$

$$F_{d,\Delta,l}(u,v) = \frac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}$$

Useful variables: $\quad u = z\bar{z} \qquad v = (1-z)(1-\bar{z})$

In euclidean space, $\bar{z} = z^*$.

In Minkowski space, z and $\bar{z}$ independent real variables.

In terms of $\lambda^2_{\phi\phi\mathcal{O}}$ crossing equation is a linear equation

Crossing (bootstrap) equation typically studied at a single point:

$$u = v = \tfrac{1}{4}, \text{ i.e. } z = \bar{z} = \tfrac{1}{2}$$

Different numerical approaches, based on well developed linear programming techniques

A possible way to get bounds is by applying a functional to the bootstrap equation

$$\alpha(f(z,\bar{z})) = \sum_{m+n\leq N_D} a_{mn}\partial_z^m \partial_{\bar{z}}^n f(z,\bar{z})|_{z=\bar{z}=1/2}$$

Assume a specific operator $\mathcal{O}_0$ is in the theory.

Normalize α: $\alpha(F_{\Delta_0}(z,\bar{z})) = 1$

Look for α such that $\alpha(F_\Delta(z,\bar{z})) > 0, \qquad \Delta \neq \Delta_0$

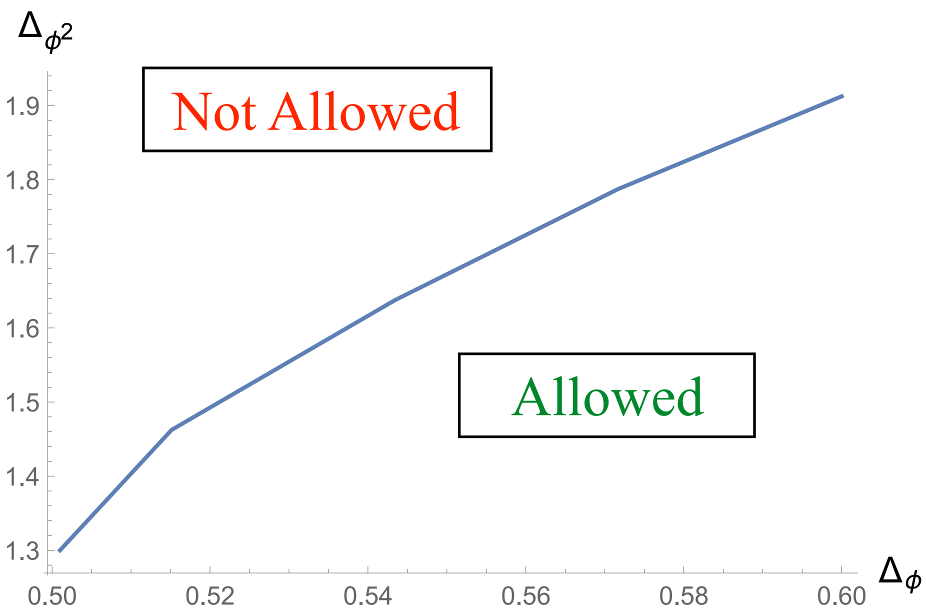
$$\lambda^2_{\phi\phi\mathcal{O}_0} = \alpha(1) - \sum_{\Delta\neq\Delta_0} \lambda^2_{\phi\phi\mathcal{O}}\alpha(F_\Delta(z,\bar{z})) \leq \alpha(1)$$

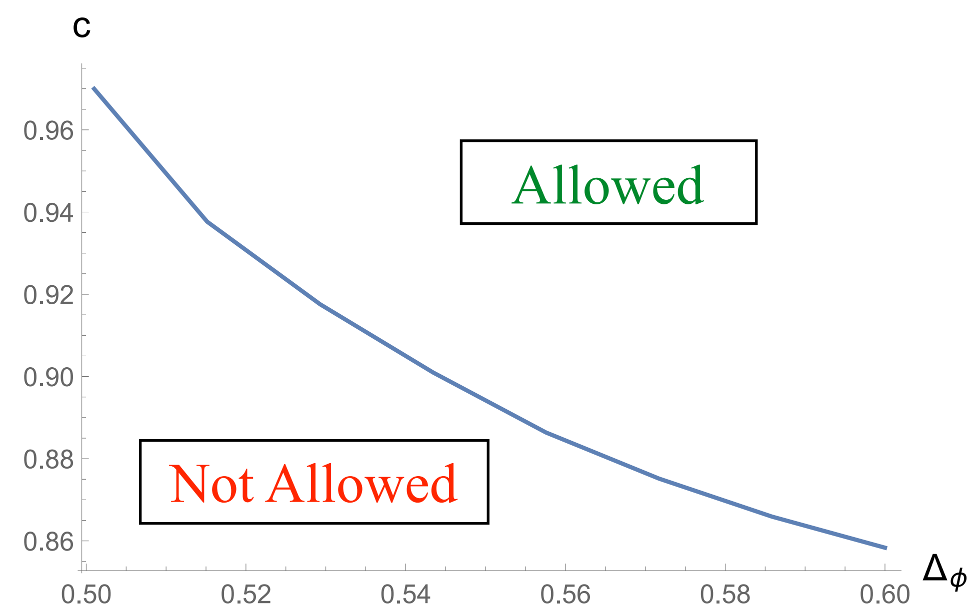
If $\alpha(1) < 0$ - assumption wrong. The operator $\mathcal{O}_0$ cannot be in the theory

If $\alpha(1) > 0$ - The operator $\mathcal{O}_0$ can be in the theory.

Upper bound on OPE coefficient.

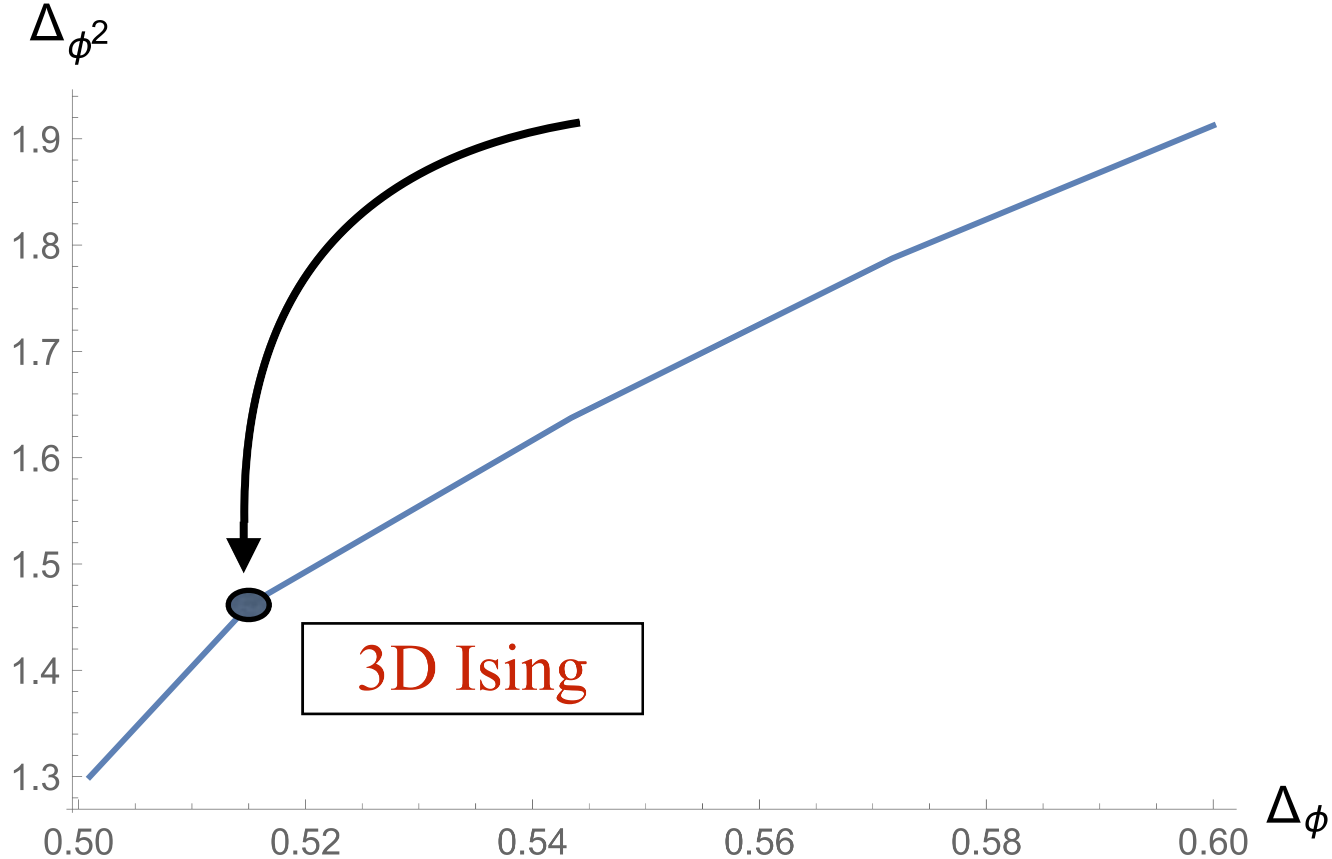
Typical generic bootstrap results





$c \propto \dfrac{1}{\lambda^2_{\phi\phi T}}$ Upper bound on OPE coefficient turns into a lower bound for central charge

One can also study a specific CFT if one knows where to look at



Drawback

Large number of operators should be included in the bootstrap equation, numerics quite demanding (but way less than Monte Carlo)

Any, even limited, analytical understanding difficult

It would be desirable to have a more "effective" approach where less operators are included

An approach of this kind where a few operators are considered has been proposed by Gliozzi, but it is not very rigorous

Another approach, with more operators than Gliozzi's idea, was advocated by Hogervorst and Rychkov, 1303.1111, but never implemented so far

Aim of this talk is to implement Hogervorst and Rychkov's proposal and show that it works!

Convergence of the OPE

Recall $$g(u,v) = 1 + \sum_{l,\Delta} \lambda^2_{\phi\phi\mathcal{O}_{\Delta,l}} g_{\Delta,l}(z,\bar{z})$$

Sum converges for any value of z and $\bar{z}$, but the real line $z = \bar{z} = x \in [1,\infty)$.

The convergence of the above sum is exponential.

The estimate of its remainder when truncated is known for any d>2 for $\Delta_* \gg 1$

$$\Big| \sum_{\Delta \geq \Delta_*} \lambda^2_{\mathcal{O}}\, g_{\Delta,l}(z,\bar{z}) \Big| \leq \mathcal{R}(z,\bar{z})$$

$$\mathcal{R}(z,\bar{z}) \equiv \frac{(-\log|\rho(z)|)^{-4\Delta_\phi} 2^{4\Delta_\phi}}{\Gamma(4\Delta_\phi+1)} \Gamma(4\Delta_\phi+1, -\Delta_* \log|\rho(z)|)$$

$$\rho(z) = \frac{z}{(1+\sqrt{1-z})^2}$$

[Pappadopulo et al, 1208.6449]

Estimate of the remainder recently improved for d=3 and d=4 CFTs

[Rychkov, Yvernay 1510.08486]

When $z = \bar{z} \to 1$

$$\frac{(-\log|\rho(z)|)^{-4\Delta_\phi} 2^{4\Delta_\phi}}{\Gamma(4\Delta_\phi + 1)} \Gamma(4\Delta_\phi + 1, -\Delta_* \log|\rho(z)|) \to |1-z|^{-2\Delta_\phi}$$

The limit is saturated by generalized free fields (i.e. free fields in AdS)

$$g(u,v)_{GFT} = 1 + |z|^{2\Delta_\phi} + \left(\frac{|z|}{|1-z|}\right)^{2\Delta_\phi} \to |1-z|^{-2\Delta_\phi}$$

$\boxed{\text{For large scaling dimensions any CFT resembles a generalized free theory}}$

Euclidean version of a known result in Minkowski signature

[Fitzpatrick et al, 1212.3616; Komargodski, Zhiboedov, 1212.4103]

Bootstrapping with Multiple Points

Recall bootstrap equation:

$$\sum_{\Delta,l} \lambda^2_{\phi\phi\mathcal{O}} F_{d,\Delta,l}(u,v) = 1$$

$$F_{d,\Delta,l}(u,v) = \frac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}$$

Rewrite as

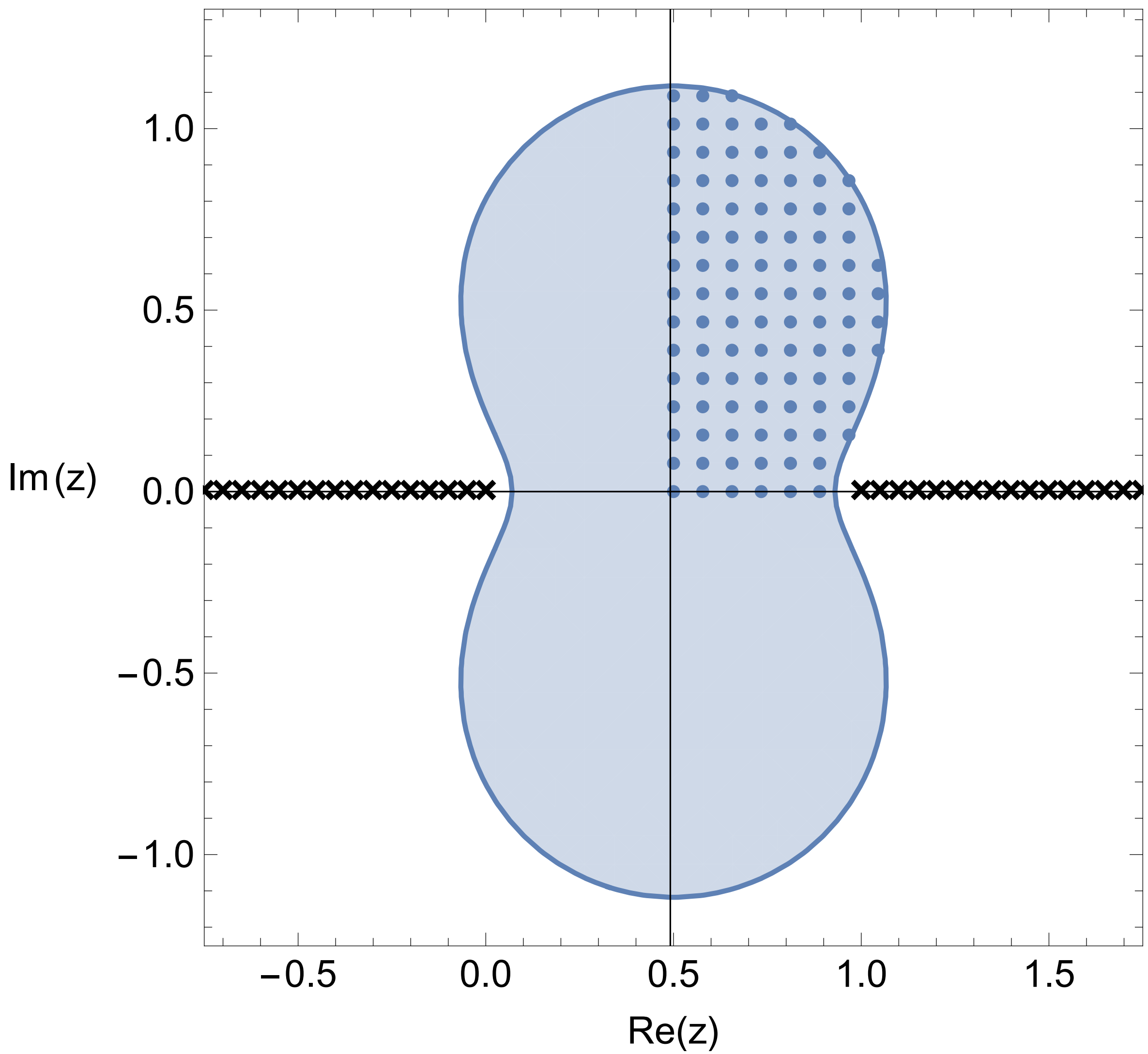
$$\sum_{\Delta<\Delta_*} \lambda^2_{\mathcal{O}} \Big(v^{\Delta_\phi} g_{\Delta,l}(u,v) - u^{\Delta_\phi} g_{\Delta,l}(v,u) \Big) = u^{\Delta_\phi} - v^{\Delta_\phi} + \mathcal{E}(z,\bar{z})$$

and use estimate of the remainder

$$|\mathcal{E}(z,\bar{z})| \leq \mathcal{E}_{\max}(z,\bar{z}) \equiv v^{\Delta_\phi} \mathcal{R}(z,\bar{z}) + u^{\Delta_\phi} \mathcal{R}(1-z,1-\bar{z})$$

We have now an effective set of bootstrap inequalities involving operators up to some given dimension Δ_*

Evaluate the equation for different values of z and numerically solve for the set of inequalities (no need of functional)



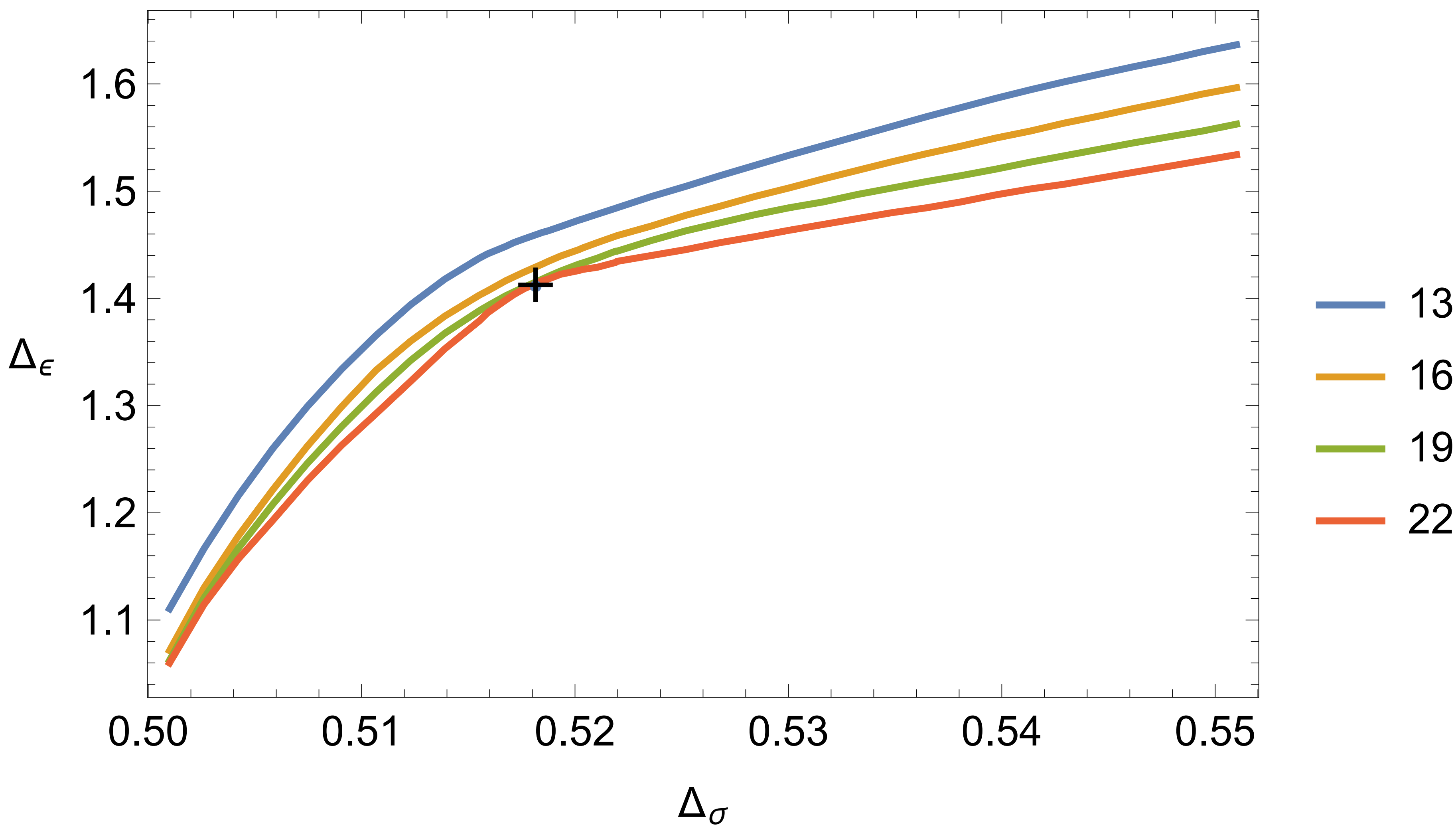
Results

There are two key parameters in the effective bootstrap:

- N the number of points sampled in the z plane
- Δ_* the cut-off in the operator dimension

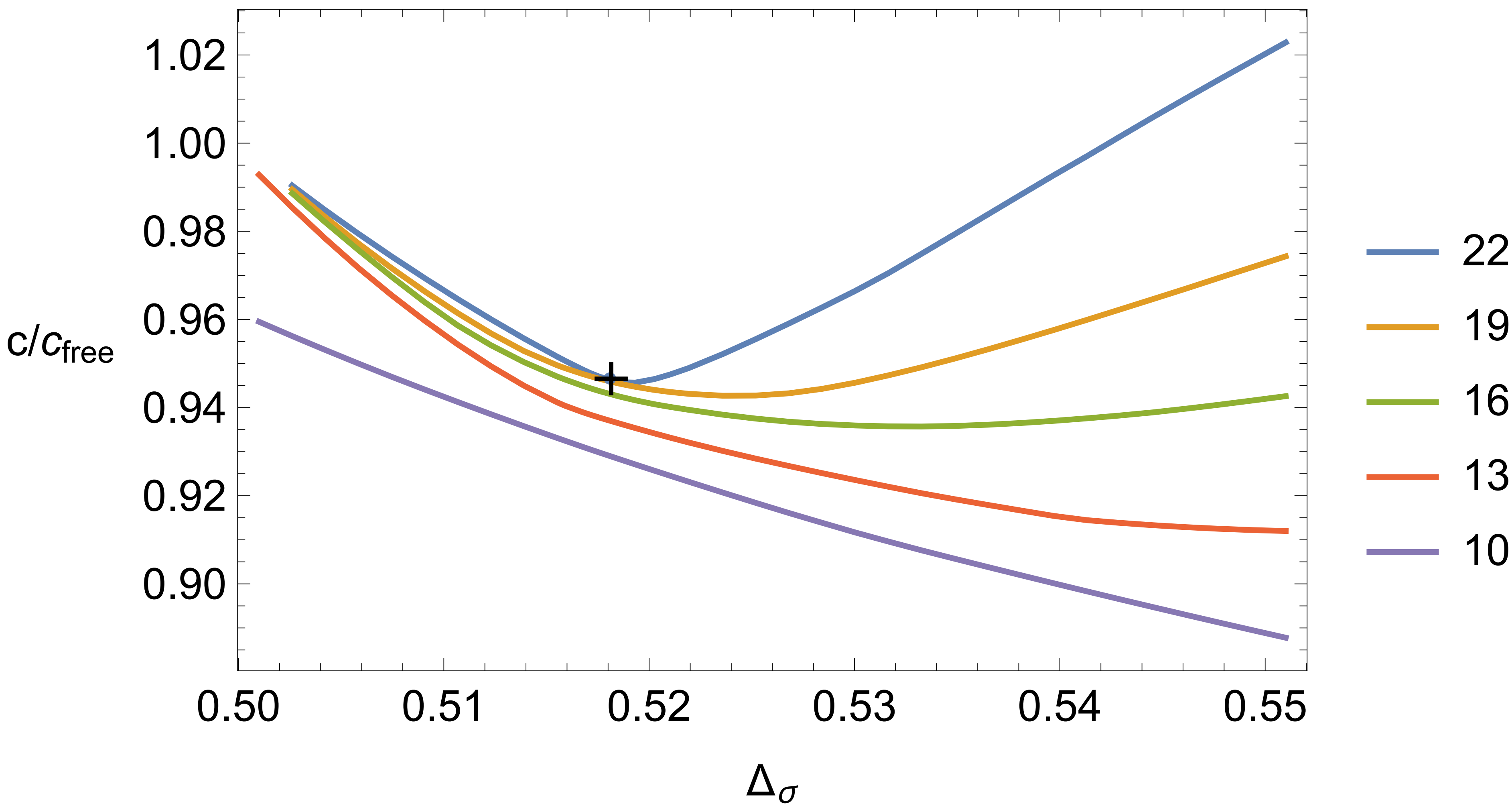
Below some results for d=3 CFTs with or without a global O(n) global symmetry

No global symmetry (Ising)



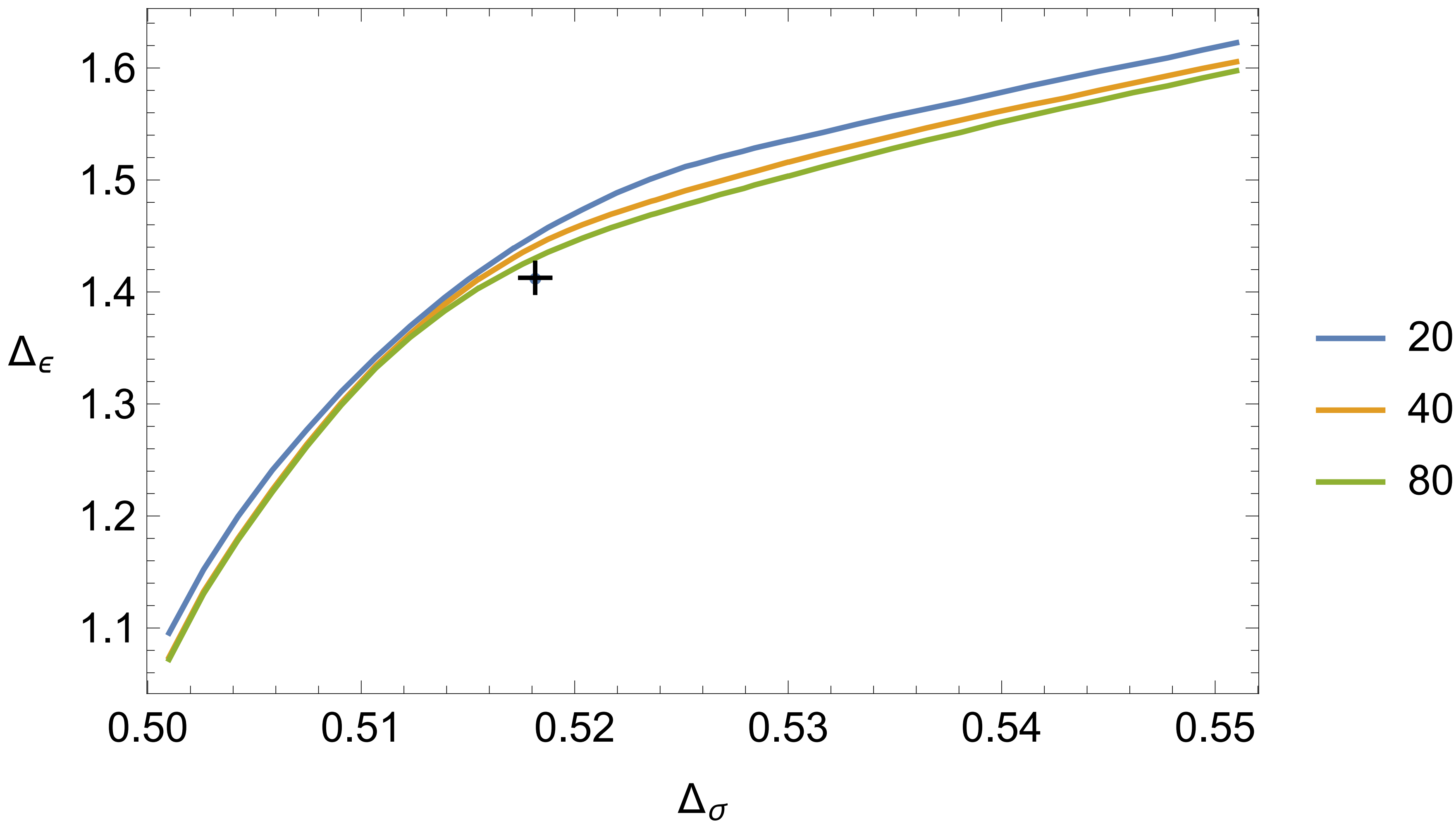
N=100 points

No global symmetry (Ising)



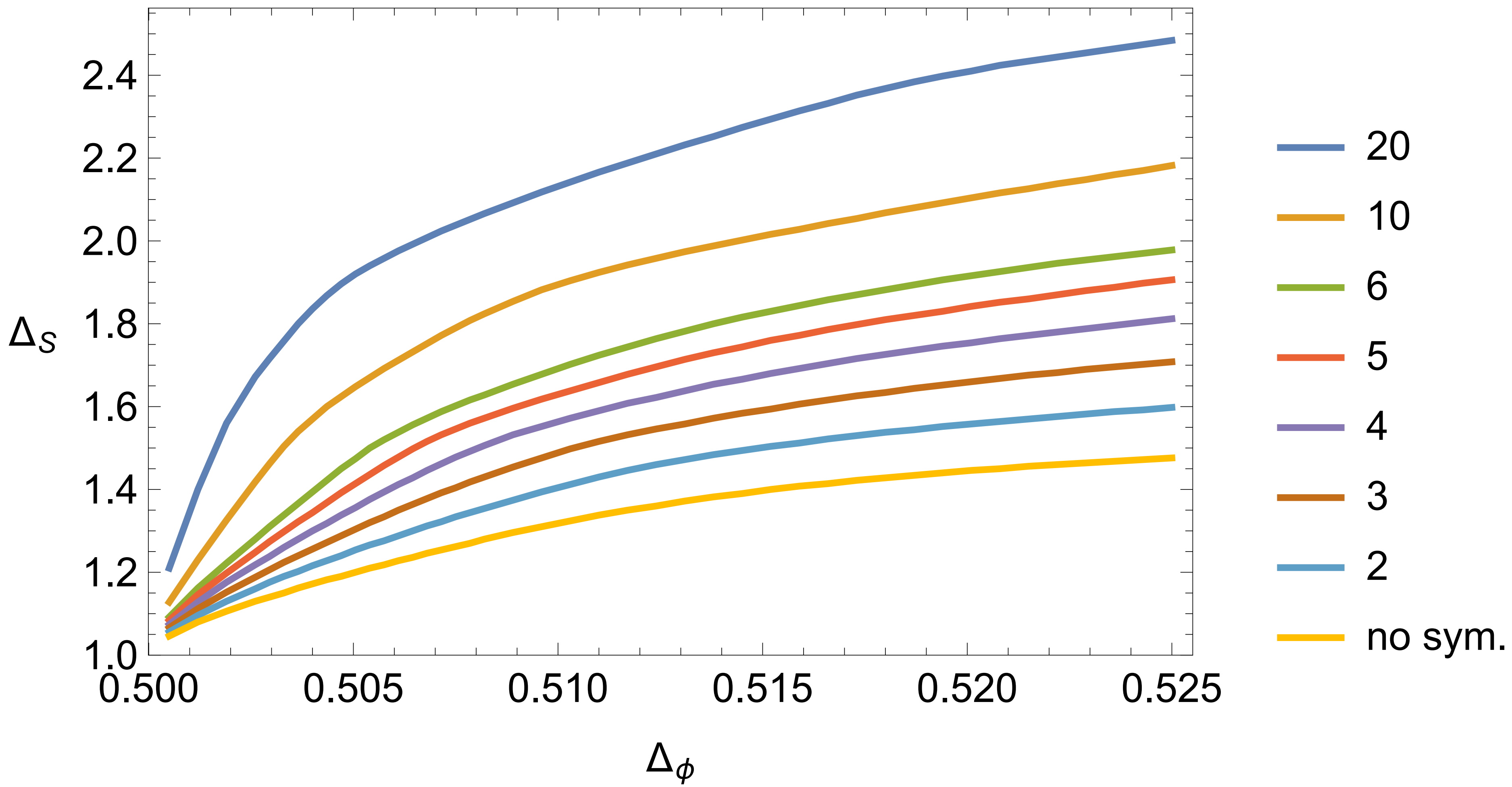
N=100 points

No global symmetry (Ising)



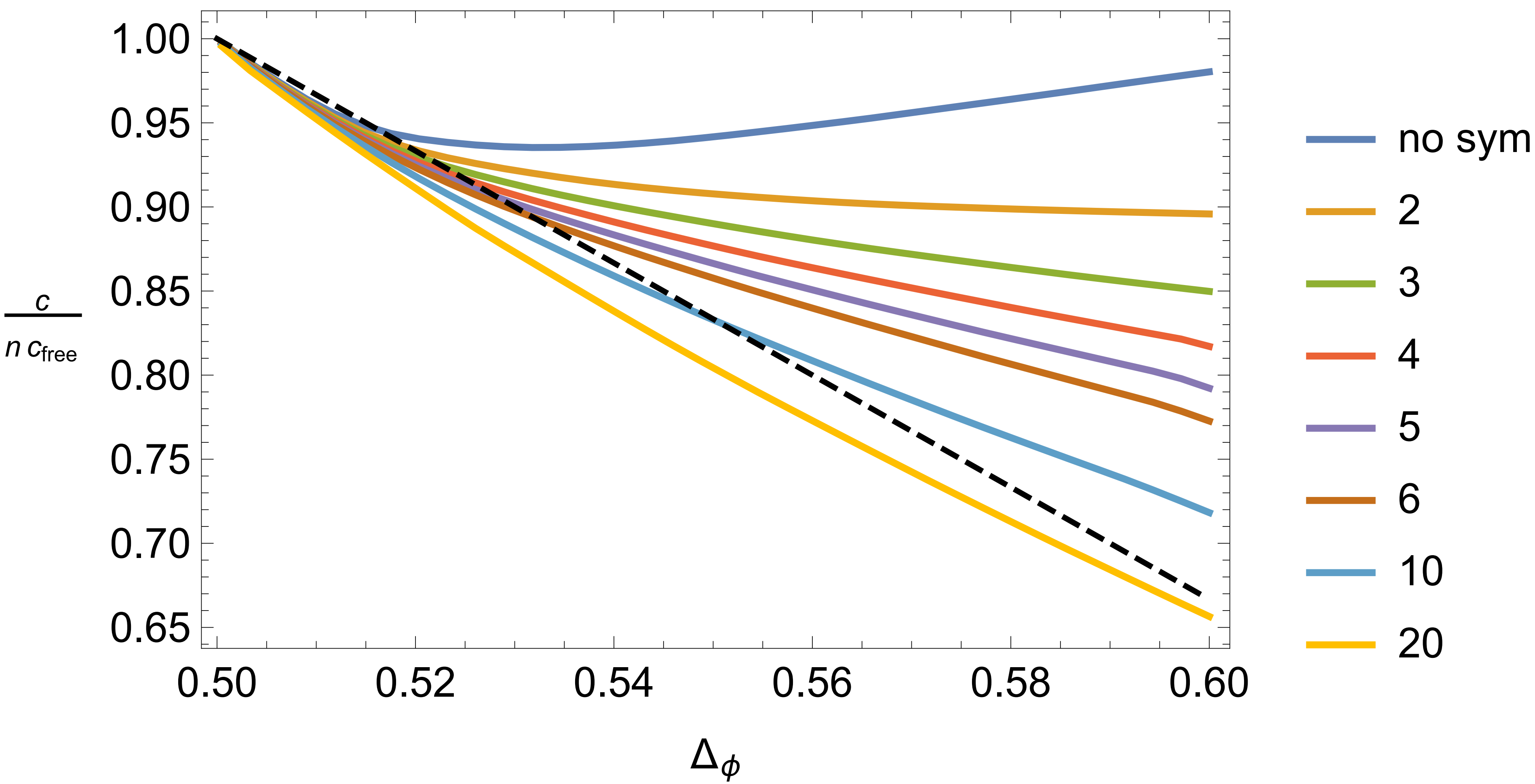
$$\Delta_* = 16$$

O(n) global symmetry

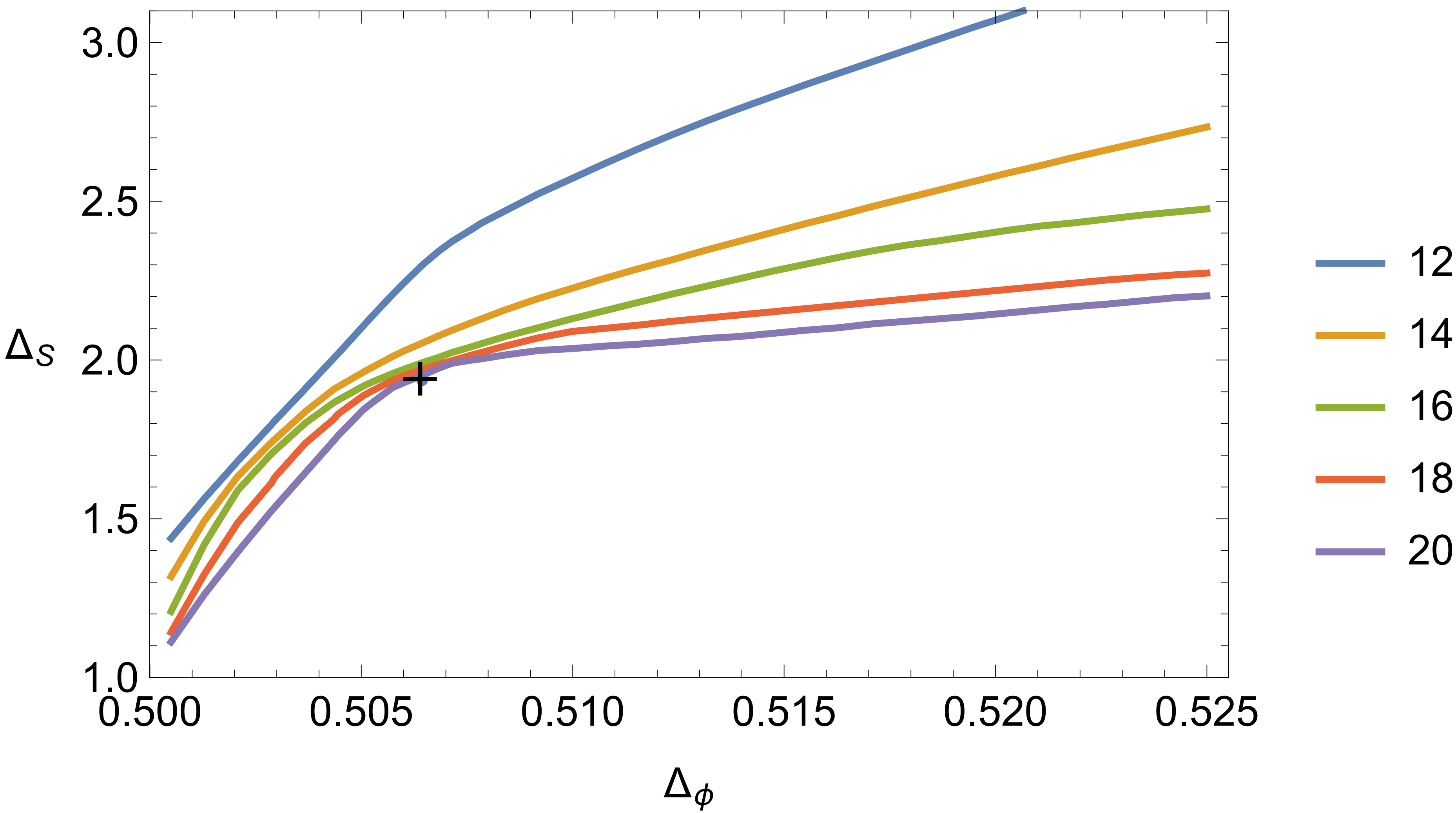


$\Delta_* = 16$ N=80 points

O(n) global symmetry



$\Delta_* = 16$ N=80 points



O(20), N=100 points

n	Δ_ϕ	Δ_S	$\Delta_{S'}$	Δ_T	$\Delta_{T'}$
2	0.51905(10) [1]	$1.5118^{+0.0012}_{-0.0022}$ [2]	3.802(18) [3]	$1.23613^{+0.00058}_{-0.00158}$ [2]	3.624(10) [4]
3	0.51875(25) [5]	$1.5942^{+0.0037}_{-0.0047}$ [2]	3.794(18) [3]	$1.2089^{+0.0013}_{-0.0023}$ [2]	3.550(14) [4]
4	0.51825(40) [6]	$1.6674^{+0.0077}_{-0.0087}$ [2]	3.795(30) [3]	$1.1864^{+0.0024}_{-0.0034}$ [2]	3.493(14) [4]

n	Δ_ϕ	Δ_S	$\Delta_{S'}$	Δ_T	$\Delta_{T'}$
2	0.51905(10) [1]	1.5124(10)	3.811(10)	1.2365(16)	3.659(7)
3	0.51875(25) [5]	1.5947(35)	3.791(22)	1.2092(22)	3.571(12)
4	0.51825(40)[6]	1.668(6)	3.817(30)	1.1868(24)	3.502(16)

[1] Campostrini, Hasenbusch, Pelissetto and Vicari, cond-mat/0605083.

[2] Kos, Poland and Simmons-Duffin, 1406.4858.

[3] Guida and Zinn-Justin, cond-mat/9803240.

[4] Calabrese, Pelissetto and Vicari, cond-mat/0209580.

[5] Campostrini, Hasenbusch, Pelissetto, Rossi and Vicari, cond-mat/0110336.

[6] M. Hasenbusch, cond-mat/0010463

Conclusions

Understanding CFTs is essential to the study of emergent phenomena, being end-points of RG flows where the emergent symmetry arises

Great progress has been obtained lately using the conformal bootstrap

It is hard to get some understanding of the results coming out from the numerical bootstrap, since too many operators are involved and the system is too complicated

In analogy to effective field theories in QFT, we have shown that one can “integrate out” higher dimensional operators and get an effective set of bootstrap equations

Three main reasons to do that:

1) Numerics much faster

2) Better understanding of the sensitivities of various quantities to the higher dimensional operators

3) First step towards an analytic understanding

Point 3) is perhaps the most interesting but the most complicated. Hopefully there is room for progress here

Thank You