Entanglement Entropy in CGC

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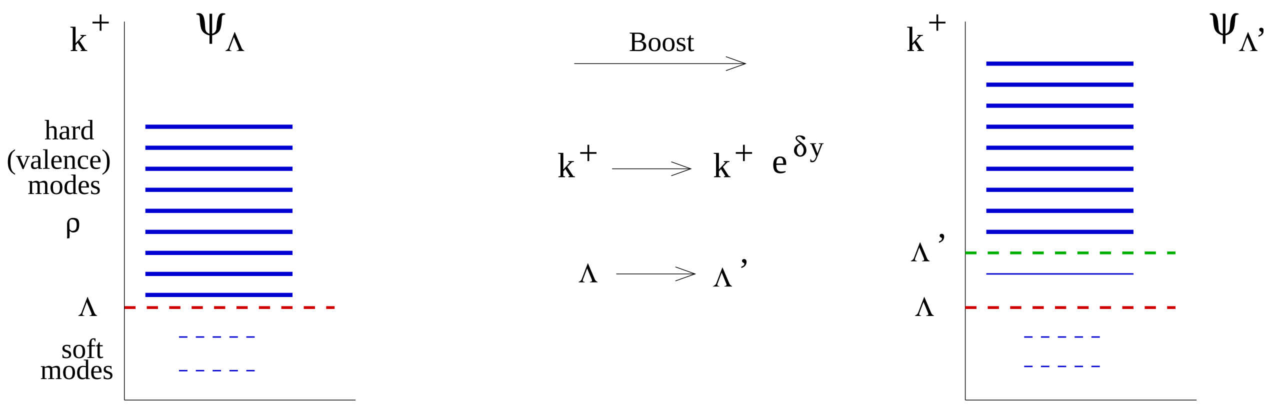
Alex Kovner and ML; Phys.Rev. D92 (2015) 3, 034016

T. Altinoluk, N. Armesto, G. Beuf, A. Kovner and ML arXiv:1503.07126 (PLB)
T. Altinoluk, N. Armesto, G. Beuf, A. Kovner and ML arXiv:1509.03223 (PLB)

Light Cone Wave Function



Hard particles with $k^+ > \Lambda$ scatter of the target. Hard (valence) modes are described by the valence density $\rho(x_\perp)$.

The boost opens a window above Λ with the width $\sim \delta y$. The window is populated by soft modes, which became hard after the boost. These newly created hard modes do scatter off the target.

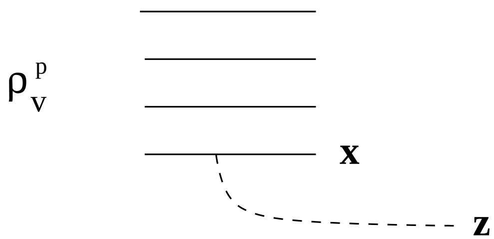
In the dilute limit $\rho \sim 1$; gluon emission $\sim \alpha_s \, \rho$, LO = one gluon, NLO = 2 gluons

In the dense limit $\rho \sim 1/\alpha_s$, we have $\alpha_s \, \rho \sim 1$, and the number of gluons in the window can be very large.

The wave function coming into the collision region at time $t = 0$

$$|\mathbf{\Psi_{in}}\rangle \; = \; \mathbf{\Omega_Y}\, |\rho, \mathbf{0_a}\rangle \; .$$

$$\mathbf{\Omega_Y}(\rho \;\to\; \mathbf{0}) \;\equiv\; \mathbf{C_Y} \;=\; \mathrm{Exp}\left\{\mathbf{i} \int \mathbf{d^2z\, b^a_i(z)} \int_{\mathbf{e^{Y_0}\Lambda}}^{\mathbf{e^{Y}\Lambda}} \frac{\mathbf{dk^+}}{\pi^{1/2}|\mathbf{k^+}|^{1/2}} \left[\mathbf{a^a_i(k^+, z)} \;+\; \mathbf{a^{\dagger a}_i(k^+, z))}\right]\right\}$$



Linear evolution means $\delta\rho \;\propto\; \rho^p$

Emission amplitude is given by the Weizsaker-Williams field

$$\mathbf{b^a_i(z)} \;=\; \frac{\mathbf{g}}{\mathbf{2}\,\pi} \int \mathbf{d^2x}\, \frac{\mathbf{(x \;-\; z)_i}}{\mathbf{(x \;-\; z)^2}}\; \rho^{\mathbf{a}}\mathbf{(x)}$$

The operator $\mathbf{C}$ dresses the valence charges by a cloud of the WW gluons

In the dense regime: $\quad \mathbf{\Omega}(\rho \;\sim\; \mathbf{1}/\alpha_s) \;=\; \mathbf{C\;B}$ $\qquad$ $\mathbf{B}$ is a Bogolyubov operator

$$\mathbf{B} \;=\; \exp[\,\mathbf{\Lambda}(\rho)\,(\mathbf{a^2} \;+\; \mathbf{a^{\dagger\,2}}) \;+\cdots]$$

Altinoluk, Kovner, ML, Peressutti, Wiedemann (2007-2009)

From LCWF to observables: CGC-type approach

For any observable $\mathcal{O}$

$$\langle \mathcal{O} \rangle \;=\; \langle\langle\, \mathbf{\Psi_{in}} |\; \mathcal{O}(\rho,\, \mathbf{a})\; | \mathbf{\Psi_{in}} \rangle\rangle_\rho \;=\; \int \mathbf{D}\rho\;\; \tilde{\mathcal{O}}[\rho]\;\; \mathbf{W_Y}[\rho]$$

$\mathbf{W}$ is a probability distribution, subject to high energy evolution equations

McLerran-Venugopalan model for dense systems:

$$\mathbf{W^{MV}}[\rho] \;=\; \mathcal{N}\;\exp\left[-\int_{\mathbf{k}} \frac{\mathbf{1}}{\mathbf{2}\mu^{\mathbf{2}}(\mathbf{k})} \rho(\mathbf{k})\rho(-\mathbf{k})\right]$$

where $\mathbf{Q_s^2} = \frac{\mathbf{g}^4}{\pi}\mu^2$

Density Matrix of soft modes

Standard CGC formalism: first integrate out the soft modes and then average over ρ

Q: Can we learn something if we do in the opposite order? A: probably Yes.

Define the reduced density matrix of soft modes

$$\hat{\rho} = \int \mathbf{D}\rho\, \mathbf{W}[\rho]\, |\mathbf{\Psi}_{\mathbf{in}}\rangle \langle \mathbf{\Psi}_{\mathbf{in}}|$$

"Dilute/Dense mix approximation": $\Omega = C$ and $W = W^{MV}$ (Gaussian), $\hat{\rho}$ is computable analytically

T. Altinoluk, N. Armesto, G. Beuf, A. Kovner and ML, arXiv:1503.07126

$$\hat{\rho} = \sum_n \frac{1}{n!} e^{-\frac{1}{2}\phi_i M_{ij}\phi_j} \left[\prod_{m=1}^{n} M_{i_m j_m} \phi_{i_m} |0\rangle \langle 0| \phi_{j_m} \right] e^{-\frac{1}{2}\phi_i M_{ij}\phi_j}$$

Here we have introduced compact notations:

$$\phi_{\mathbf{i}} \equiv \left[\mathbf{a}_{\mathbf{i}}^{\dagger \mathbf{a}}(\mathbf{x}) + \mathbf{a}_{\mathbf{i}}^{\mathbf{a}}(\mathbf{x}) \right] \; ; \qquad \mathbf{M_{ij}} \equiv \frac{\mathbf{g^2}}{\mathbf{4\pi^2}} \int_{\mathbf{u},\mathbf{v}} \mu^{\mathbf{2}}(\mathbf{u},\mathbf{v}) \frac{(\mathbf{x}-\mathbf{u})_{\mathbf{i}}}{(\mathbf{x}-\mathbf{u})^{\mathbf{2}}} \frac{(\mathbf{y}-\mathbf{v})_{\mathbf{j}}}{(\mathbf{y}-\mathbf{v})^{\mathbf{2}}} \delta^{\mathbf{ab}}$$

M **bears two polarisation, colour, and coordinate indices, collectively denoted as $\{ij\}$.**

Entanglement Entropy

Alex Kovner and ML, arXiv:1506.05394

Entanglement Entropy of soft modes

$$\sigma^{\mathbf{E}} \;=\; -\,\mathbf{tr}[\hat{\rho}\ln\hat{\rho}]$$

How to calculate $\ln$? The "replica trick":

$$\ln\hat{\rho} = \lim_{\epsilon\to\mathbf{0}}\frac{\mathbf{1}}{\epsilon}\left(\hat{\rho}^{\epsilon}-\mathbf{1}\right)$$

Calculate ρ^N and take $N\to 0$. N copies of the field - replicas.

The result

$$\sigma^{\mathbf{E}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{tr}\left\{\ln\frac{\mathbf{M}}{\pi}+\sqrt{\mathbf{1}+\frac{\mathbf{4M}}{\pi}}\ln\left[\mathbf{1}+\frac{\pi}{\mathbf{2M}}\left(\mathbf{1}+\sqrt{\mathbf{1}+\frac{\mathbf{4M}}{\pi}}\right)\right]\right\}$$

Translationally invariant limit ($\mu = const$):

$$\mathbf{M^{ab}_{ij}(p) = g^2 \mu^2 \frac{p_i p_j}{p^4} \delta^{ab}}$$

For small M, or the UV contribution (formally UV divergent)

$$\sigma^{\mathbf{E}}_{\mathbf{UV}} = \mathbf{tr} \left[\frac{\mathbf{M}}{\pi} \ln \frac{\pi \mathbf{e}}{\mathbf{M}} \right] = \frac{\mathbf{Q_s^2}}{\mathbf{4}\pi \mathbf{g^2}} (\mathbf{N_c^2} - \mathbf{1}) \mathbf{S} \left[\ln^2 \frac{\mathbf{g^2 \Lambda^2}}{\mathbf{Q_s^2}} + \ln \frac{\mathbf{g^2 \Lambda^2}}{\mathbf{Q_s^2}} \right]$$

The large M, IR contribution is

$$\sigma^{\mathbf{E}}_{\mathbf{IR}} \simeq \frac{\mathbf{1}}{\mathbf{2}} \mathbf{tr} [\ln \frac{\mathbf{e^2 M}}{\pi}] == \frac{\mathbf{3(N_c^2 - 1)}}{\mathbf{8}\pi \mathbf{g^2}} \mathbf{S Q_s^2}$$

Properties of σ^E

UV divergent: the divergence is cutoff physically at $\mathbf{\Lambda \sim Me^{Y_0} \gg M}$, where eikonal approximation breaks down.

σ^E is not extensive in rapidity: only one longitudinal mode (rapidity independent) is entangled with valence degrees of freedom.

Similar to "topological entropy": insensitive to boundary region between the modes.

But not quite what we would like to know.

We need to address scattering process

Semi-inclusive reactions

The wave function coming into the collision region at time $t = 0$

$$|\mathbf{\Psi}_{\mathrm{in}}\rangle \;=\; \mathbf{\Omega}_{\mathbf{Y}}\,|\rho, \mathbf{0}_{\mathrm{a}}\rangle \; .$$

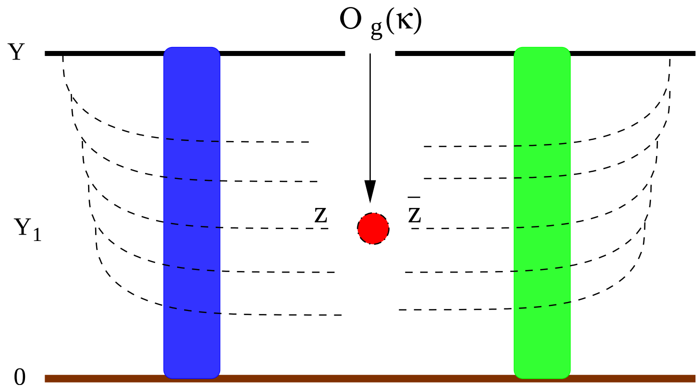
The system emerges from the collision region with the wave function

$$|\mathbf{\Psi}_{\mathrm{out}}\rangle \;=\; \hat{\mathbf{S}}\,\mathbf{\Omega}_{\mathbf{Y}}\;|\rho, \mathbf{0}_{\mathrm{a}}\rangle \; .$$

The system keeps evolving after the collision to the asymptotic time $t \to +\infty$, at which point the measurement of an observable $\hat{\mathcal{O}}$ is made

$$\langle\hat{\mathcal{O}}\rangle_{\mathbf{P},\mathbf{T}} \;=\; \langle\;\; \langle\mathbf{0}_{\mathrm{a}}|\mathbf{\Omega}^{\dagger}_{\mathbf{Y}}\;(\mathbf{1}\;-\;\hat{\mathbf{S}}^{\dagger})\,\mathbf{\Omega}_{\mathbf{Y}}\;\;\hat{\mathcal{O}}\;\;\mathbf{\Omega}^{\dagger}_{\mathbf{Y}}\;(\mathbf{1}\;-\;\hat{\mathbf{S}})\,\mathbf{\Omega}_{\mathbf{Y}}\,|\mathbf{0}_{\mathrm{a}}\rangle\;\rangle_{\mathbf{P},\mathbf{T}}$$

Single inclusive gluon production



The observable

$$\hat{\mathcal{O}}_{\mathbf{g}} \;\sim\; \mathbf{a}_{\mathbf{i}}^{\dagger\,\mathbf{a}}(\mathbf{k})\,\mathbf{a}_{\mathbf{i}}^{\mathbf{a}}(\mathbf{k})$$

$$\frac{\mathbf{dN}}{\mathbf{d^2kd}\eta} \;=\; \langle\, \sigma(\mathbf{k})\,\rangle_{\mathbf{P,T}}$$

After soft gluon averaged using dilute projectile approximation ($\boldsymbol{\Omega} \rightarrow \mathbf{C}$)

$$\sigma(\mathbf{k}) = \int_{\mathbf{z},\bar{\mathbf{z}},\mathbf{x_1},\bar{\mathbf{x}}_1} \mathbf{e}^{\mathbf{ik(z-\bar{z})}} \frac{(\bar{\mathbf{z}}-\bar{\mathbf{x}}_1)_{\mathbf{i}}}{(\bar{\mathbf{z}}-\bar{\mathbf{x}}_1)^2} \frac{(\mathbf{x}_1-\mathbf{z})_{\mathbf{i}}}{(\mathbf{x}_1-\mathbf{z})^2} \left\{\rho(\mathbf{x}_1)[\mathbf{S}^\dagger(\mathbf{x}_1)-\mathbf{S}^\dagger(\mathbf{z})][\mathbf{S}(\bar{\mathbf{x}}_1)-\mathbf{S}(\mathbf{z})]\rho(\bar{\mathbf{x}}_1)\right\}$$

Entropy production

$$\sigma^{\mathbf{P}} = \frac{\mathbf{1}}{\mathbf{2}}\langle \mathbf{tr} \left\{ \ln\frac{\mathbf{M^P}}{\pi} + \sqrt{\mathbf{1} + \frac{\mathbf{4M^P}}{\pi}} \ln\left[\mathbf{1} + \frac{\pi}{\mathbf{2M^P}}\left(\mathbf{1} + \sqrt{\mathbf{1} + \frac{\mathbf{4M^P}}{\pi}}\right)\right]\right\}\rangle_{\mathbf{T}}$$

with

$$\mathbf{M^P} \equiv \mathbf{g}^2 \int_{\mathbf{u},\mathbf{v}} \mu^2(\mathbf{u},\mathbf{v}) \frac{(\mathbf{x}-\mathbf{u})_\mathbf{i}}{(\mathbf{x}-\mathbf{u})^2} \frac{(\mathbf{y}-\mathbf{v})_\mathbf{j}}{(\mathbf{y}-\mathbf{v})^2} \left[(\mathbf{S}(\mathbf{u}) - \mathbf{S}(\mathbf{x}))(\mathbf{S}^\dagger(\mathbf{v}) - \mathbf{S}^\dagger(\mathbf{y}))\right]^{\mathbf{ab}}$$

T-averaging is complicated Let expand σ^P around $\bar{\mathrm{M}} \equiv \langle \mathrm{M^P}\rangle_{\mathrm{T}}$ (dilute projectile limit)

$$\bar{\mathbf{M}} = \delta^{\mathbf{ab}} \frac{\mathbf{Q_P^2}\pi}{\mathbf{g}^2} \int_{\mathbf{z}} \frac{(\mathbf{x}-\mathbf{z})_\mathbf{i}}{(\mathbf{x}-\mathbf{z})^2} \frac{(\mathbf{y}-\mathbf{z})_\mathbf{j}}{(\mathbf{y}-\mathbf{z})^2} [\mathbf{P_A}(\mathbf{x},\mathbf{y}) + \mathbf{1} - \mathbf{P_A}(\mathbf{x},\mathbf{z}) - \mathbf{P_A}(\mathbf{z},\mathbf{y})]$$

P_A - S-matrix of an adjoint dipole Q_p - saturation momentum of the projectile.

$\bar{M}$ is *almost* single inclusive gluon, but it is not summed over ij

$$\sigma^{\mathbf{P}} = \mathbf{tr}\left[\frac{\bar{\mathbf{M}}}{\pi}\ln\frac{\pi\mathbf{e}}{\bar{\mathbf{M}}}\right] - \frac{\mathbf{1}}{\mathbf{2}\pi}\mathbf{tr}\left[\left\{\langle(\mathbf{M}^{\mathbf{P}} - \bar{\mathbf{M}})\ (\mathbf{M}^{\mathbf{P}} - \bar{\mathbf{M}})\rangle_{\mathbf{T}}\right\}\bar{\mathbf{M}}^{-1}\right]\ldots.$$

First term is $almost\ -n\ln n$**, where** n **is a multiplicity per unit rapidity (**$dN/d\eta$**)**

it depends on the production probabilities of longitudinally and transversely (with respect to the direction of their transverse momentum) polarized gluons separately

Second term - $almost$ **correlated part of double inclusive gluon production.**

Correlations between gluons decrease entropy of the produced state. consistent with the view of entropy as measuring disorder in the final state.

For a parametrically large number of produced particles ($\alpha_s dN/d\eta \sim 1$**), the entropy is parametrically of order** $1/\alpha_s$

"Temperature" of produced system

We can naturally define temperature through:

$$\mathbf{T^{-1} \;=\; \frac{d\sigma}{dE_T}}$$

$$\mathbf{E_\perp \;\propto\; \int d^2k\,|k|\,M^P(k) \;\propto\; (N_c^2 - 1)\,S\,\frac{Q_P^2}{g^2}Q_T}$$

Keeping only mean field term in the entropy:

$$\mathbf{T \;=\; \frac{\pi}{2}\,\langle k_T\rangle}$$

.

$$\mathbf{\langle k_T\rangle \;=\; E_\perp / N_{total}} \qquad\qquad \mathbf{N_{total} \;=\; \int d^2k\, M^P(k)}$$

Outlook

- **What we did was just a pilot project on "Quasi-Thermodynamics"**

- **Improve approximations:** $\Omega \to CB$**,** $\quad W^{MV} \to$ **BK/JIMWLK induced** W

- **Time dependence:** $\quad \hat{\rho}(t) \quad \to \quad \sigma^p(t), \; T_{\mu\nu}(t) = tr[\hat{\rho}^p(t) T_{\mu\nu}]$