A Consistent Calculation of Sommerfeld Effect

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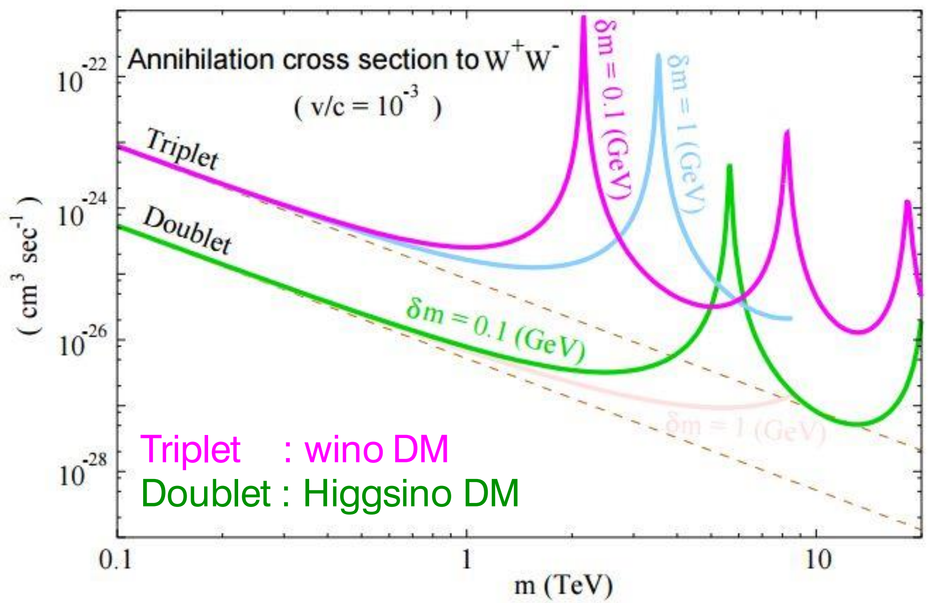
Sommerfeld enhancement

[Sommerfeld (1931)]
[Hisano, Matsumoto, Nojiri (2003)]

If dark matter couples with a light force carrier ($m_{force} \ll m_{DM}$),
There is large enhancement of the cross section in non-relativistic regime.



Dotted lines : LO results
Solid lines : Resummed results

[Hisano, Matsumoto, Nojiri (2003)]

Non-perturbative resummation is required to calculate this effect.

"Usual" way to calculate and its problem

See, e.g., [Hisano, Matsumoto, Nojiri, Saito (2004)], [Cirelli, Strumia, Tamburini (2007)]

Long range force distorts wave function from plane-wave.

1. Solve Schrodinger Eq. with long-range force ($V(r) \sim \exp(-m_{force} r)/r$)

$$\left(-\frac{1}{2\mu}\nabla^2 + V(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0 \quad \text{with} \quad \psi \to e^{ipz} + f(\theta)\frac{e^{ipr}}{r}$$

2. Compare the result with $V(r) = 0$ ⟶ $\boldsymbol{\sigma} = \boldsymbol{\sigma_0} \times \boldsymbol{S(v)}$

LO cross section : σ_0

Enhancement factor : $S(v) = |\psi(\mathbf{0})|^2 / |\psi_0(\mathbf{0})|^2$ with $\psi_0(\boldsymbol{x}) = e^{ipz}$

$\boldsymbol{\sigma_0}$ and $\boldsymbol{S(v)}$ are irrelevant each other.

⇕ Could be inconsistent in some cases

Unitarity bound on s-wave : $\sigma \leq \dfrac{\pi}{p^2}$ [Griest, Kamionkowski (1992)] [Landau-Lifshits's textbook]

What is a formula which is consistent with unitarity bound?

Our calculation

Correct calculation in QM gives consistent answer.

1. Annihilation effect is taken as delta function potential with complex coefficient u.

$$V_{eff}(\boldsymbol{x}) = V(\boldsymbol{x}) + u\delta^3(\boldsymbol{x})$$

2. Solve the Schrodinger equation:

$$\left(-\frac{1}{2\mu}\nabla^2 + V(\boldsymbol{x}) + u\delta^3(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

3. Determine asymptotic form ($r \to \infty$) of s-wave solution :

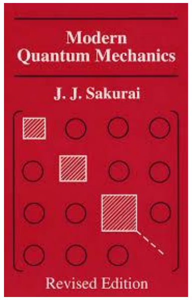
$$\psi(r) \;\to\; S_0\frac{e^{ipr}}{r} - \frac{e^{-ipr}}{r}$$

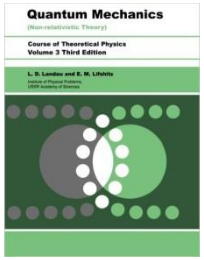
4. Calculate the cross section from the above S_0.

$$\sigma_{sc} = \frac{\pi}{p^2}|S_0 - 1\,|^2, \quad \sigma_{ann} = \frac{\pi}{p^2}(1 - |S_0|^2)$$

DM-DM elastic scattering — DM-DM annihilation

What we need is s-wave solution. See also





etc.

Solving the Schrodinger equation

c.f. [Jackiw (1992)]

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) + u\delta^3(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

Write ψ by ψ_0 and Green's function $G(x,0)$

Wave function w/o short range effect :

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) - \frac{p^2}{2\mu}\right)\psi_0(\boldsymbol{x}) = 0$$

Green's function :

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) - \frac{p^2}{2\mu}\right)G_p(\boldsymbol{x},0) = \frac{1}{2\mu}\delta^3(\boldsymbol{x}), \qquad G_p(\infty,0) \propto \frac{e^{ipr}}{4\pi r}$$

$$\psi(x) = \psi_0(x) - 2\mu u G_p(x,0)\psi(0)$$

solve a consistency condition at $x = 0$.

$$\psi(x) = \psi_0(x) - G_p(x,0)\psi_0(0)\left(\frac{1}{2\mu u} + G_p(0,0)\right)^{-1}$$

Renormalize coupling u ($G_p(0,0)$ is divergent)

$$\psi(x) = \psi_0(x) - G_p(x,0)\psi_0(0)\left(\frac{k_{p_0}}{4\pi} - \mathrm{Re}G_{p_0}(0,0) + G_p(0,0)\right)^{-1}$$

Asymptotic behavior and cross sections

$$\psi(\infty) \propto \frac{1}{r}\left(\frac{d_p}{d_p^*} \frac{k_{p_0} - \mathrm{Re}\, g'_{p_0}(0) + {g'_p}^*(0)}{k_{p_0} - \mathrm{Re}\, g'_{p_0}(0) + g'_p(0)} e^{ipr} - e^{-ipr} \right)$$

amplitude of DM-DM s-wave scattering S_0

$G_p(x,0) = \frac{g_p(x)}{4\pi x}$ with $g_p(0) = 1,\ g_p(x) \to d_p e^{ipx}$

We can take $\psi_0(x) = c\,\mathrm{Im}G_p(x,0)$

$$\sigma_{sc} = \frac{4\pi}{p^2}\left|\frac{1 - S_0}{2i}\right|^2, \qquad \sigma_{ann} = \frac{4\pi}{p^2}\frac{1 - |S_0|^2}{4}$$

DM-DM elastic scattering — DM-DM annihilation

We have checked our formula is consistent with

- $\sigma_{ann} = \sigma_0 \times S(v)$ at the leading order of σ_0
- $\sigma_{sc} \propto v^0, \sigma_{ann} \propto v^{-1}$ for large v.

A formula of annihilation cross section

$$\sigma v \simeq \frac{\sigma v_0\, S(v)}{\left|1+\left(\eta\sqrt{\frac{\mu^2\sigma_{sc,0}}{4\pi}-\left(\frac{\mu^2\sigma v_0}{4\pi}\right)^2}-i\frac{\mu^2\sigma v_0}{4\pi}\right)(T(v)+iS(v))\,v\right|^2}$$

Short-range physics

- σv_0 : LO annihilation cross section
- $\sigma_{sc,0}$: LO DM-DM elastic scattering cross section
- $\eta = \mathrm{sgn} k_{p_0}^{-1}$ It is determined by matching the scattering amplitude. It is relevant that short-range force is attractive / replusive.

Long-range physics

- $S(v) = \frac{1}{p}\mathrm{Im} g_p'(0)$ ("usual" Sommerfeld factor)
- $T(v) = \frac{1}{p}[\mathrm{Re} g_p'(0) - \mathrm{Re} g_{p_0}'(0)]$

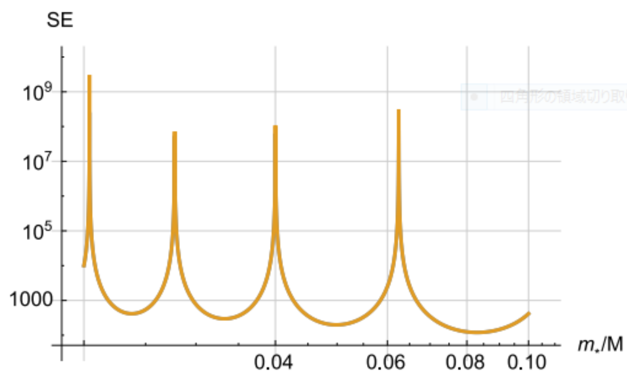
where $\left(-\frac{1}{2\mu}\frac{d^2}{dr^2}+V(r)-\frac{p^2}{2\mu}\right)g_p(r)=0,\ g_p(0)=1$

Example 1 (small bare cross section)

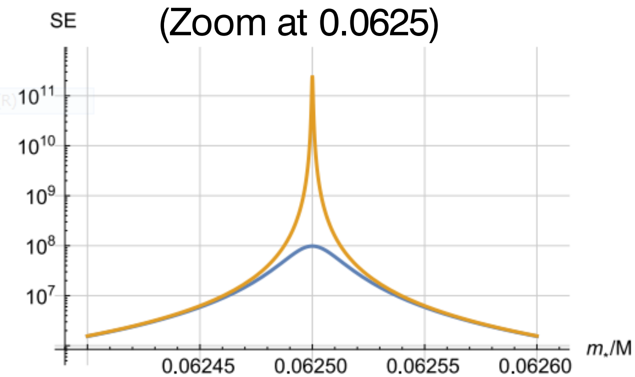
Hulthen potential : $V(r) = -\dfrac{\alpha m_* e^{-m_* r}}{1 - e^{-m_* r}}$ (Good approximation of $V(r) = -\frac{\alpha e^{-mr}}{r}, m_* = \frac{\pi^2}{6} m$)

$$\alpha = 1, \qquad \sigma_{ann} v = \frac{1}{32\pi M^2}, \qquad \sigma_{sc} = \frac{\mu^2}{4\pi} (\sigma_{ann} v)^2$$

$v = 10^{-6}$



(Zoom at 0.0625)



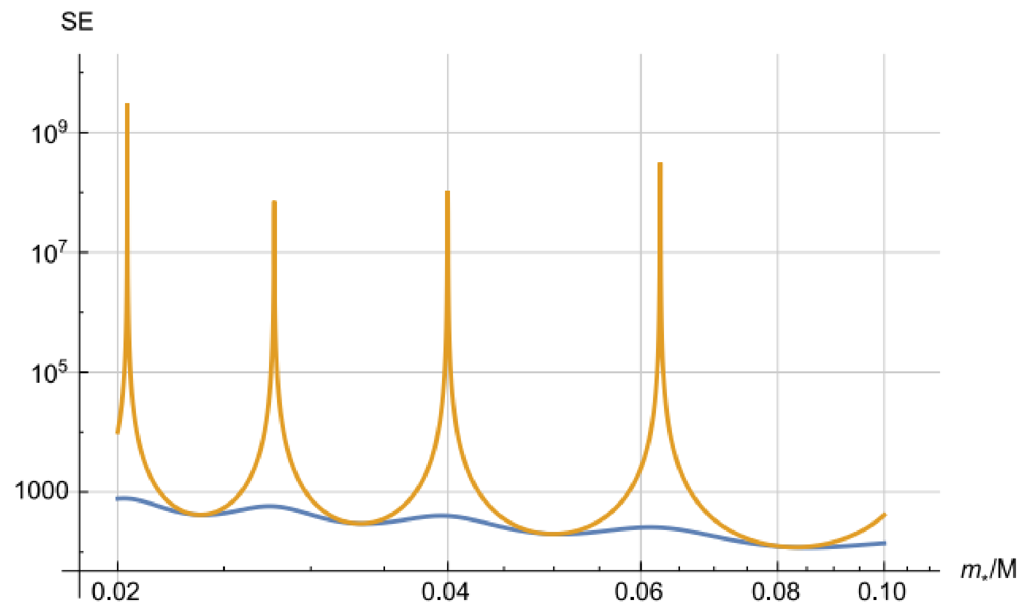
Yellow : usual formula
Blue : our formula

Example 2 (large bare cross section)

Hulthen potential : $V(r) = -\frac{\alpha m_* e^{-m_* r}}{1 - e^{-m_* r}}$ (Good approximation of $V(r) = -\frac{\alpha e^{-mr}}{r}, m_* = \frac{\pi^2}{6} m$)

$\alpha = 1, \qquad \sigma v = \frac{2\pi}{M^2}, \qquad \sigma_{sc} = \frac{\mu^2}{4\pi}(\sigma v)^2$

$v = 10^{-6}$ ⇒



Yellow : usual formula

Blue : our formula

Summary

- Solved Schrodinger Eq with long-range potential and delta function potential.
- Constructed DM annihilation cross section from non-relativistic QM.
- Our formula satisfies the unitarity bound.

$$S_0 = \frac{d_p}{d_p^*} \frac{k_{p_0} - \mathrm{Re}\, g'_{p_0}(0) + {g'_p}^*(0)}{k_{p_0} - \mathrm{Re}\, g'_{p_0}(0) + g'_p(0)}$$

$$\Downarrow$$

$$\sigma_{sc} = \frac{4\pi}{p^2} \left| \frac{1 - S_0}{2i} \right|^2 , \qquad \sigma_{ann} = \frac{4\pi}{p^2} \frac{1 - |S_0|^2}{4}$$

Backup

Large momentum limit (determination of k)

$$S_0 = \frac{d_p}{d_p^*}\frac{k_{p_0} - \mathrm{Re}\,g'_{p_0}(0) + g'^*_p(0)}{k_{p_0} - \mathrm{Re}\,g'_{p_0}(0) + g'_p(0)}$$

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) + u\delta^3(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

$$\psi(r) \to S_0\frac{e^{ipr}}{r} - \frac{e^{-ipr}}{r}$$

$$\left(-\frac{1}{2\mu}\frac{d^2}{dr^2} + V(r) - \frac{p^2}{2\mu}\right)g_p(r) = 0$$

$$g_p(0) = 1, \qquad \lim_{r\to\infty} e^{-ipr}g_p(r) = d_p$$

$$\sigma_{sc} = \frac{4\pi}{p^2}\left|\frac{1-S_0}{2i}\right|^2, \qquad \sigma_{ann} = \frac{4\pi}{p^2}\frac{1-|S_0|^2}{4}$$

DM-DM elastic scattering DM-DM annihilation

Large momentum limit ($p^2 \gg |2\mu V|$)

$$d_p \to 1, \qquad g'_p \to ip$$

and for small coupling ($|k_{p_0}^{-1}| \ll p^{-1}, p_0^{-1}$),

$$\sigma_{sc} \to 4\pi\left|k_{p_0}^{-1}\right|^2, \qquad \sigma_{ann} \to \frac{4\pi}{p}\frac{\mathrm{Im}\,k_{p_0}}{\left|k_{p_0}\right|^2}$$

$$\mathrm{Re}\,k_{p_0}^{-1} \simeq \pm\sqrt{\frac{\sigma_{sc}}{4\pi} - \left(\frac{\mu\sigma_{ann}v}{4\pi}\right)^2}$$

$$\mathrm{Im}\,k_{p_0}^{-1} \simeq -\frac{\mu\sigma_{ann}v}{4\pi}$$

Relation with conventional formulae

$$S_0 = \frac{d_p}{d_p^*}\frac{k_{p_0} - \mathrm{Re}g'_{p_0}(0) + {g'_p}^*(0)}{k_{p_0} - \mathrm{Re}g'_{p_0}(0) + g'_p(0)}$$

$$\sigma_{ann} = \frac{4\pi}{p^2}\frac{1-|S_0|^2}{4} = \frac{4\pi}{p^2}\frac{\mathrm{Im}k_{p_0}\mathrm{Im}g'_p(0)}{\left|k_{p_0} - \mathrm{Re}g'_{p_0} + g'_p(0)\right|^2}$$

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) + u\delta^3(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

$$\psi(r) \to S_0\frac{e^{ipr}}{r} - \frac{e^{-ipr}}{r}$$

$$\left(-\frac{1}{2\mu}\frac{d^2}{dr^2} + V(r) - \frac{p^2}{2\mu}\right)g_p(r) = 0$$

$$g_p(0) = 1, \qquad \lim_{r\to\infty} e^{-ipr}g_p(r) = d_p$$

At the leading order of k^{-1} (or u),

$$\sigma v \simeq \frac{4\pi}{\mu}\frac{\mathrm{Im}k_{p_0}}{\left|k_{p_0}\right|^2} \times \frac{\mathrm{Im}g'_p(0)}{p}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\sigma v \simeq (\sigma v)_0 \times S(v)$$

$$W_p(r) \equiv \frac{1}{2i}\left(g_p^*(r)g'_p(r) - g_p(r){g_p^*}'(r)\right)$$

$$\left\{\begin{aligned} &W'_p(r) = 0 \\ &W_p(0) = \mathrm{Im}g'_p(0) \\ &W_p(\infty) = p\left|d_p\right|^2 \end{aligned}\right. \longrightarrow \frac{1}{p}\mathrm{Im}g'_p(0) = \left|d_p\right|^2 = S(v)$$

Finite piece of Green function

$$G_p(r,0) = \frac{g_p(r)}{4\pi r} \qquad \begin{cases} \left(-\frac{d^2}{dr^2} + 2\mu V(r) - p^2\right) g_p(r) = 0 \\ g_p(0) = 0, \qquad \lim_{r\to\infty} g_p(r) e^{-ipr} = d_p \end{cases}$$

$$2\mu V(r) = \frac{V_{-1}}{r} + V_0 + V_1 r + \cdots$$

$$\Rightarrow \begin{cases} g_p(r) = (1 + g_1 r + \cdots) + V_{-1}(r + h_2 r^2 + \cdots)\log r \\ G_p(r,0) = \frac{1}{4\pi r} + \frac{V_{-1}}{4\pi}\log r + \frac{g_1}{4\pi} + \cdots \end{cases}$$

Divergent parts   Finite parts

Finite part of $G_p(0,0)$ is only in real part and it is independent on p.

$G_p(0,0) - \mathrm{Re}\,G_{p_0}(0,0)$ is finite for any p and p_0

Regularization velocity

Full Sommerfeld factor is approximated by shifting velocity.

$$\frac{\sigma v}{\sigma v_0} \simeq S(v + v_c),$$

$$\begin{aligned} v_c &= \frac{\alpha\, m\, M\, \sigma v_0}{4\pi} \\ &\approx 2 \times 10^{-3}\, \alpha \left(\frac{M}{10\ \mathrm{TeV}}\right)^2 \left(\frac{m/M}{0.1}\right) \left(\frac{\sigma v_0}{3 \times 10^{-26}\ \mathrm{cm}^3/\mathrm{sec}}\right) \end{aligned}$$