On the use of stable law in cosmic ray physics.

Yoann Génolini

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A work in collaboration with Pasquale Serpico, Pierre Salati and Richard Taillet
Beautiful power law over many decades $\rightarrow \phi \propto E^\gamma$. 

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Introduction

![Graph showing the relationship between kinetic energy and the differential flux density for CREAM 2005 and AMS 2015 data.](image)
Diffusion of CRs across the Galaxy

⇒ In the low turbulence regime $K \propto R^\delta$. 
Pure diffusive propagation equation

\[
\frac{\partial \Psi}{\partial t} - \nabla_r \cdot (K \nabla_r \Psi) = Q(r, E)
\]

⇒ We solve the time independent equation!
Pure diffusive propagation equation

\[- \nabla_r (K \nabla_r \Psi) = Q(r, E)\]

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⇒ We assume a continuous production in space and time!
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\[Q(r, t) = \sum_{i}^{N} q_i \delta(r_i - r) \delta(t_i - t)\]
Pure diffusive propagation equation

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\[\Rightarrow \text{We solve the time independent equation!}\]

\[\Rightarrow \text{We assume a continuous production in space and time!}\]

\[Q(r, t) = \sum_{i}^{N} q_i \delta(r_i - r) \delta(t_i - t)\]

\[\langle Q(r, t) \rangle = \left\langle \sum_{i}^{N} q_i \delta(r_i - r) \delta(t_i - t) \right\rangle\]
Pure diffusive propagation equation

\[ - \nabla_r \cdot (K \nabla_r \Psi) = Q(r, E) \]

⇒ We solve the time independent equation!

⇒ We assume a continuous production in space and time!

\[ Q(r, t) = \sum_{i}^{N} q_i \delta(r_i - r) \delta(t_i - t) \]

\[ \langle Q(r, t) \rangle \approx \frac{q \nu}{V_{MW}} \Theta(h - |z|) \Theta(R_{gal} - r) \]

\[ V_{MW} = 2h \pi R^2 \text{ and } \nu \approx 3 \text{ SNs/century} \]

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Pure diffusive propagation equation

\[ V_{MW} = 2 \ h \ \pi R^2 \]
\[ \nu \approx 3 \ SNs/\text{century} \]

Figure: Two zones diffusion box commonly used.
The mean of the cosmic-ray flux

\[- \nabla_r (K \nabla_r \Psi) = Q(r, E) \Rightarrow \langle \Psi \rangle = \underbrace{Q \left( \frac{h L}{K} \right)}_{\text{injection rate}} \]

\[\langle \Psi \rangle = \frac{q \nu}{V_{MW}} \frac{h L}{K} \]

with \(\nu\) the explosion rate of SNRs.

\[= \frac{q}{2 L \pi R^2} \nu \frac{L^2}{K} \]

\[\equiv \langle \psi \rangle \propto N\]
...sources are discrete in space and time!

[Büsching et al. 2005]

→ Stochastic behaviour!
One question would be:

What is the probability that a particular configuration of the sources explains the break features?

$\Phi_p E_k \times 10^{2.7} [GeV \cdot m^{-2} \cdot s^{-1} \cdot sr^{-1}]$

$P(\Psi)$?
Statistical treatment of the flux

The flux from $N$ sources writes:

$$\Psi = \sum_{i=1}^{N} \psi_i \quad \Rightarrow \quad \langle \Psi \rangle = \sum_{i=1}^{N} \langle \psi \rangle = N \langle \psi \rangle$$

One can expect to compute $\langle \psi \rangle$ from $p(\psi)$:

$$\langle \psi \rangle = \int_{0}^{\infty} \psi \, d\psi \, p(\psi)$$

With:

$$p(\psi) = \int_{V} D(r_s, t_s) \, dr_d \, dt_d$$

Normalized distribution in space and time for one source

Integration over the domain of space and time that gives a flux between $\psi$ and $\psi + d\psi$. 

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Probability to measure $\psi$ from one source

\[ p(\psi) = \int_{\mathcal{V}_\psi} D(r_s, t_s) \, dr_s \, dt_s \]  

(2)

\( \mathcal{V}_\psi \): domain of space and time that gives a flux between $\psi$ and $\psi + d\psi$.

Surface equation in pure diffusive regime:

\[ \psi = \frac{q}{(4 \pi K t)^{3/2}} \exp\left(\frac{r^2}{4 K t}\right) \]

$D(r_s, t_s)$ can assume two limiting behaviours, 2D or 3D!

For: $\psi \gg \langle \psi \rangle$ we have, $p(\psi) \propto \begin{cases} \psi^{-8/3} & \text{3D} \\ \psi^{-7/3} & \text{2D} \end{cases}$
Variance of the total flux

\[ \Psi = \sum_{i=1}^{N} \psi_i \quad \Rightarrow \quad p(\psi) \to P(\Psi) \]

Central limit theorem?

\[ \sigma_{\Psi}^2 = \langle \Psi^2 \rangle - \langle \Psi \rangle^2 = N \sigma^2_{\psi} = N \langle \psi^2 \rangle - \frac{\langle \Psi \rangle^2}{N} \]

\[ \langle \psi^2 \rangle = \int_{0}^{\infty} \psi^2 \, p(\psi) \, d\psi \propto \begin{cases} \left[ \psi^{1/3} \right]_{cte}^\infty = \infty & 3D \\ \left[ \psi^{2/3} \right]_{cte}^\infty = \infty & 2D \end{cases} \]

The variance diverges
But actually the pdf exists! 
Generalised central limit theorem?

The heavy tail behaviour conditioned the stable law limit!

\[ \forall \psi \geq 0, \quad C(\psi) \equiv \int_{\psi}^{\infty} p(\psi') \, d\psi' \to \lim_{\psi \to \infty} \psi^\alpha C(\psi) = \eta > 0 \]

For \( N \) sufficiently large:

\[ P(\Psi) \to \frac{1}{\sigma_N} S[\alpha; 1, 1, 0; 1] \left( \frac{\Psi - \langle \Psi \rangle}{\sigma_N} \right) \]

With: \( \alpha = \begin{cases} \frac{5}{3} & 3D \\ \frac{4}{3} & 2D \end{cases} \) and \( \sigma_N = \left( \frac{\eta \pi N}{2\Gamma(\alpha) \sin(\alpha \pi/2)} \right)^{1/\alpha} \)
But actually the pdf exists!

\[ \sigma_N = 1, \alpha = 5/3 \rightarrow 3D, \alpha = 4/3 \rightarrow 2D \]

\[ \Rightarrow \text{So one can define confidence intervals, pvalues...} \]

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On the form of $p(\psi)$

The diffusive propagator is not causal for some region in space and time...

▶ **Loophole**: reevaluation of

$$p(\psi) = \int_{\mathcal{V}_\text{causal}} D(r_s, t_s) \, dr_s \, dt_s$$

▶ $p(\psi) \propto \begin{cases} 
\psi^{-8/3} & \text{for}: \psi < \psi_c \\
\psi^{-11/3} & \text{for}: \psi > \psi_c 
\end{cases}$

**The variance converges again!**

Shall we use the central limit theorem?...*Not really* if $\psi_c$ is very large.
Small deviation from the stable law

For: \( p_c(\psi) = (1 + \epsilon) \, p(\psi) \, \Theta(\psi_c - \psi) \)

Then \( P(\Psi) \) writes:

\[
P(\Psi) = (1 + \epsilon)^N \int_{\Psi_1} \ldots \int_{\Psi_N} p_c(\psi_1) \ldots p_c(\psi_N) \, \Theta(\psi_c - \psi_1) \ldots \Theta(\psi_c - \psi_N) \, \delta \left( \sum_{i=1}^{N} \psi_i - \Psi \right) \, d\psi_1 \ldots d\psi_N
\]

For \( \Psi < \psi_c \, (\star) \), each value \( \psi_i \) needs to be such as \( \psi_i < \psi_c \). Hence we get:

\[
P(\Psi) = (1 + \epsilon)^N \int_{\Psi_1} \ldots \int_{\Psi_N} p(\psi_1) \ldots p(\psi_N) \, \delta \left( \sum_{i=1}^{N} \psi_i - \Psi \right) \, d\psi_1 \ldots d\psi_N
\]

For \( N \) sufficiently large:

\[
P(\Psi) \to (1 + \epsilon)^N \, \frac{1}{\sigma_N} \, S[\alpha, 1, 1, 0; 1] \left( \frac{\Psi - \langle \Psi \rangle}{\sigma_N} \right) \quad \text{for} \quad \Psi < \psi_c \, (\star)
\]

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Simulation check

For example at 1TeV:

Simulation generated $10^6$ configurations of galaxies.

Transition from the 2D to the 3D regime!

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Is the constraint $\Psi < \psi_c$ still interesting?
Local environment information

For the purpose of the proton flux...

- No sources on Earth ⇒ define a $\psi_{max}$

- Example from the catalogue of local sources from [Green D. A., 2014] [Delahaye et al. 2010]:

  \( \tau > \tau_c = 2,7 \text{ kyr} \text{ and } r > r_c = 0,06 \text{ kpc} \)

  \[ p(\psi) = (1 + \epsilon) p_D(\psi) \Theta(\psi_{max} - \psi) \]
Which constraint $\Psi < \psi_c$ or $\Psi < \psi_{\text{max}}$ does dominate?

\begin{align*}
\text{MIN (dotted), MED (dashed), MAX (solid)}
\end{align*}
One question would be..

What is the probability that a particular configuration of the sources explain the break features?

$\Rightarrow P(\Psi)$?
Probability of such an excess

We compute an upper value of the probability that a particular configuration of the sources gives a flux $\Psi$ at 12.8 TeV:

$$p_{value} = \int_{\Psi_{exp}}^{\infty} d\psi_{exp} \int_{0}^{+\infty} d\psi_{th} p(\psi_{exp} | \psi_{th}) \cdot p(\psi_{th} | Model),$$

Example for the benchmark models:

<table>
<thead>
<tr>
<th>Models</th>
<th>MIN</th>
<th>MED</th>
<th>MAX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probabilities (Stable law 4/3)</td>
<td>0.031</td>
<td>0.0082</td>
<td>0.0013</td>
</tr>
</tbody>
</table>

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The break feature of the protons

[Kachelrieß et al., 2015]

- Local flux dominated by a 2Myr old SNR in order to explain the knee.

- $D_\perp \ll D_\parallel$

- At $E = 1$ TeV, $\Psi \approx 2.86\langle\Psi\rangle$

$\Rightarrow P(\Psi)$ can be used for an homogeneous diffusion model.

<table>
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<td>0.0072</td>
<td>0.0012</td>
<td>0.00016</td>
</tr>
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</table>
The break feature of the protons

[Tomassetti et al., 2015]

- Two component model, without prior on their number of sources

- Homogeneous diffusion

- At $E = 10\text{GeV}$, $\Psi \approx 3.3\langle \Psi \rangle$

$\Rightarrow P(\Psi)$ can be used!

The probability @10GeV is $8.6 \times 10^{-5}$!
Conclusion:

- The variance of the cosmic ray flux does not diverge!
- The probability law to measure a particular flux follows a stable law up to \( \approx 20 \text{ TeV} \).
- We propose a generic way to deal with heavy tail probability distributions.
- A lot of applications and possible improvements!

⇒ On the arxiv soon!

Thanks for listening