On the use of stable law in cosmic ray physics.

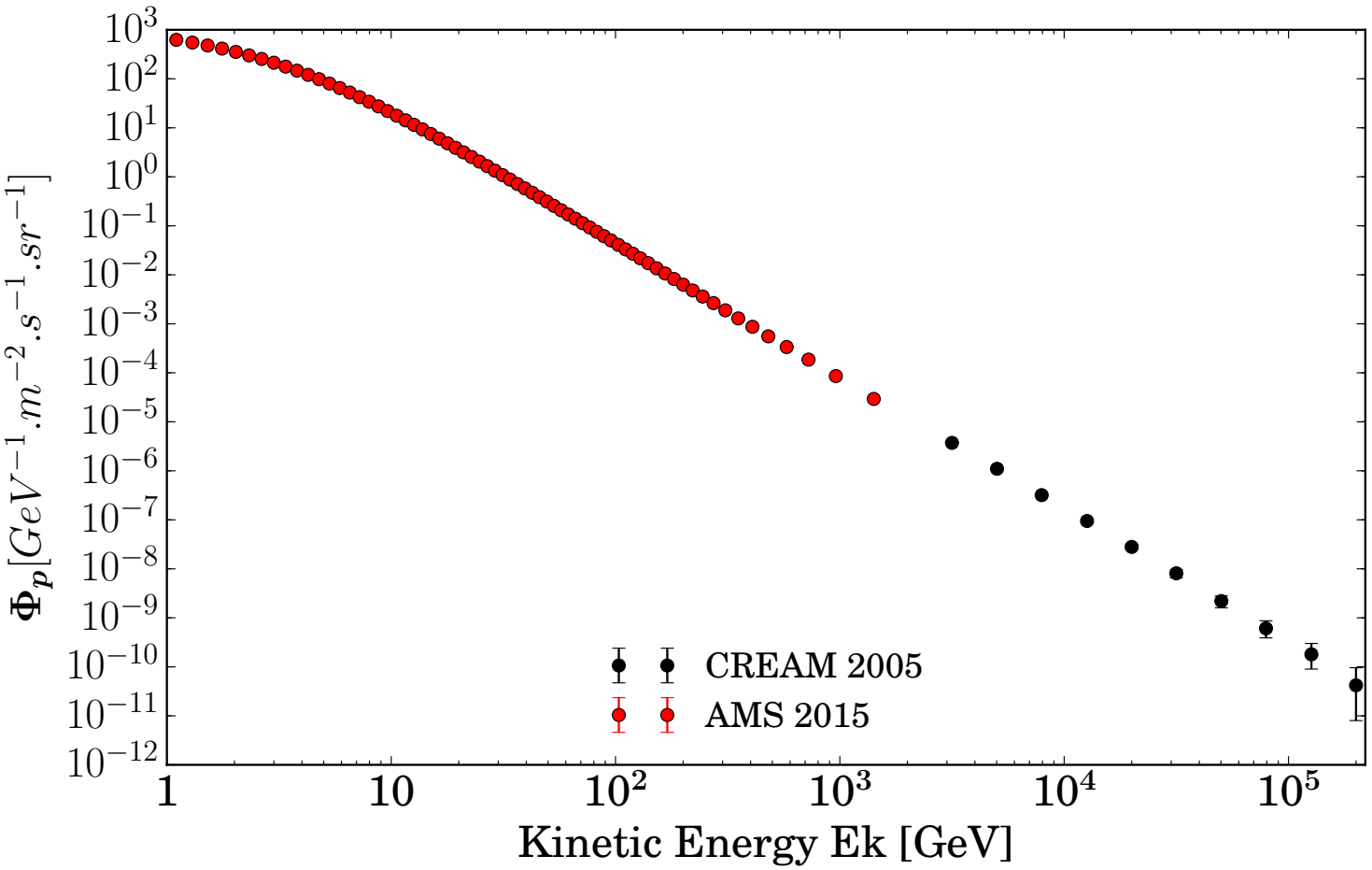
Yoann Génolini

LAPTh

TeVPa September 13th, 2016

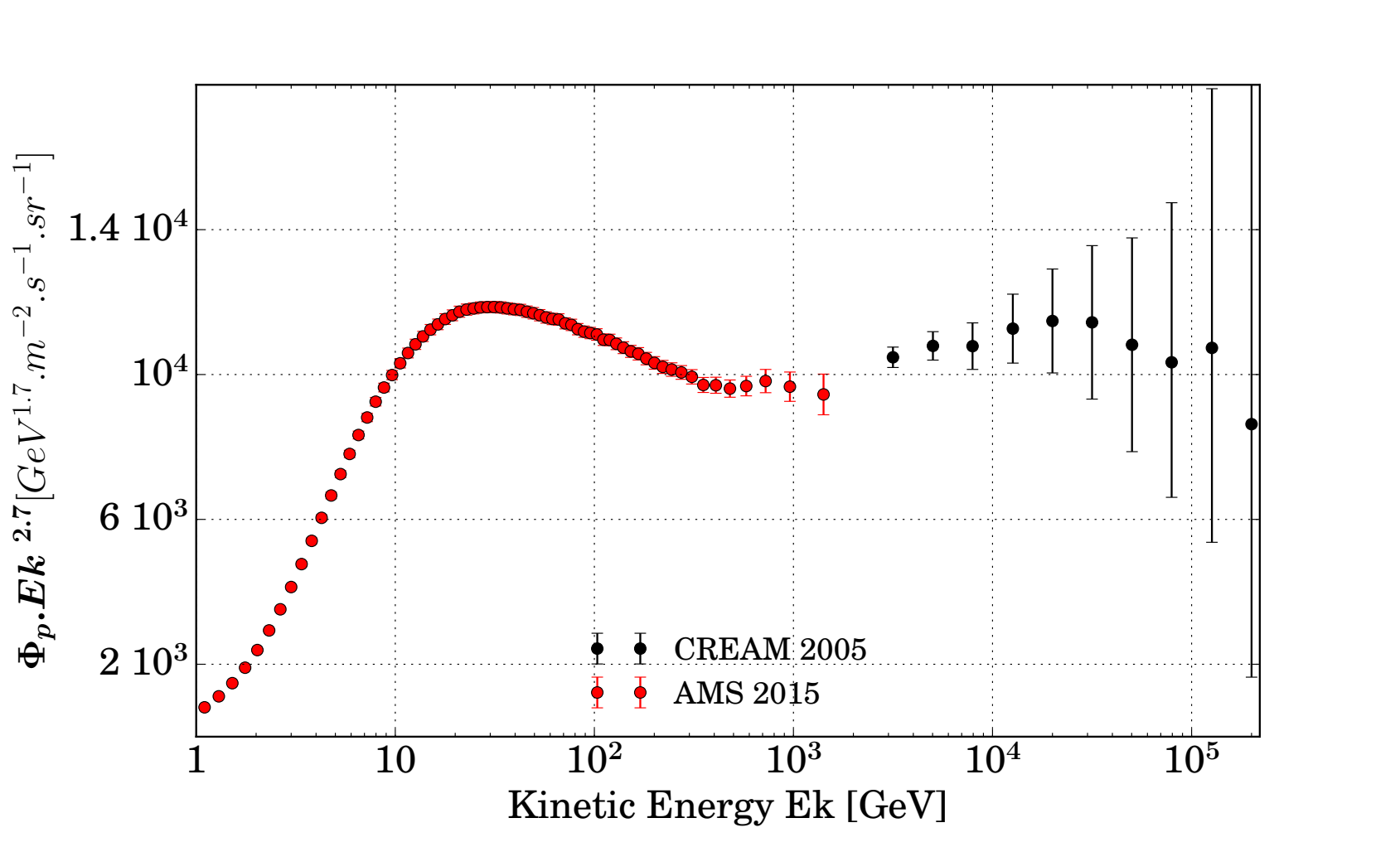
A work in collaboration with Pasquale Serpico, Pierre Salati and Richard Taillet

Introduction

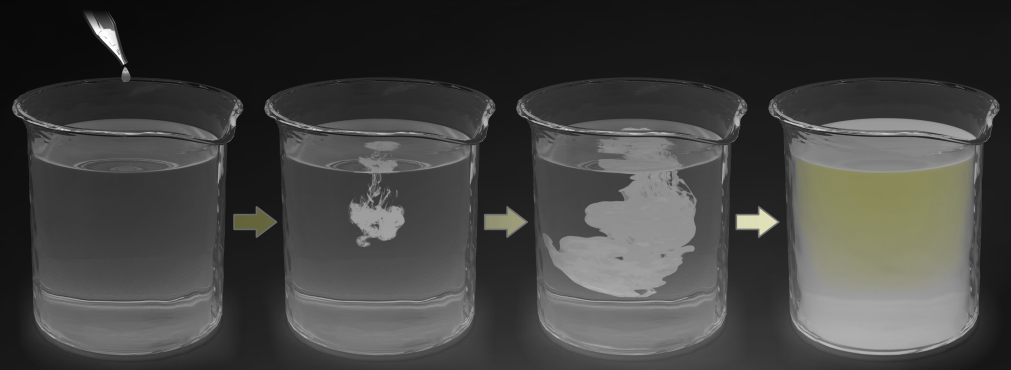


Beautiful power law over many decades $\rightarrow \phi \propto E^{\gamma}$.

Introduction

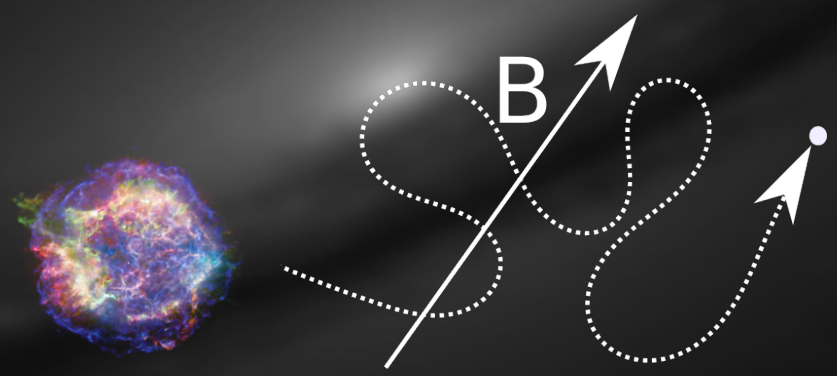


Diffusion of CRs across the Galaxy



Diffusion

$q \propto R^{\alpha}$



$\Rightarrow$ **In the low turbulence regime** $K \propto R^{\delta}$.

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Pure diffusive propagation equation

$$\frac{\partial \Psi}{\partial t} - \nabla_{\boldsymbol{r}} \cdot (K \nabla_{\boldsymbol{r}} \Psi) = Q(\boldsymbol{r}, E)$$

$\Rightarrow$ We solve the time independent equation !

Pure diffusive propagation equation

$$-\nabla_{\boldsymbol{r}}.(K\nabla_{\boldsymbol{r}}\Psi) = Q(\boldsymbol{r}, E)$$

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$\Rightarrow$ We assume a continuous production in space and time !

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$$Q(\mathbf{r}, t) = \sum_i^N q_i \, \delta(\boldsymbol{r_i} - \boldsymbol{r}) \, \delta(t_i - t)$$

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$$Q(\mathbf{r}, t) = \sum_i^N q_i\, \delta(\boldsymbol{r_i} - \boldsymbol{r})\, \delta(t_i - t)$$

$$\langle Q(\mathbf{r}, t)\rangle = \left\langle \sum_i^N q_i\, \delta(\boldsymbol{r_i} - \boldsymbol{r})\, \delta(t_i - t)\right\rangle$$

Pure diffusive propagation equation

$$-\nabla_{\boldsymbol{r}}.(K\nabla_{\boldsymbol{r}}\Psi) = Q(\boldsymbol{r}, E)$$

$\Rightarrow$ We solve the time independent equation !

$\Rightarrow$ We assume a continuous production in space and time !

$$Q(\mathbf{r}, t) = \sum_i^N q_i\, \delta(\boldsymbol{r_i} - \boldsymbol{r})\, \delta(t_i - t)$$

$$\langle Q(\mathbf{r}, t)\rangle \simeq \frac{q\,\nu}{V_{\mathrm{MW}}}\, \Theta(h - |z|)\, \Theta(R_{gal} - r)$$

$V_{\mathrm{MW}} = 2\, h\, \pi R^2$ and $\nu \approx 3\ SNs/century$

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Pure diffusive propagation equation

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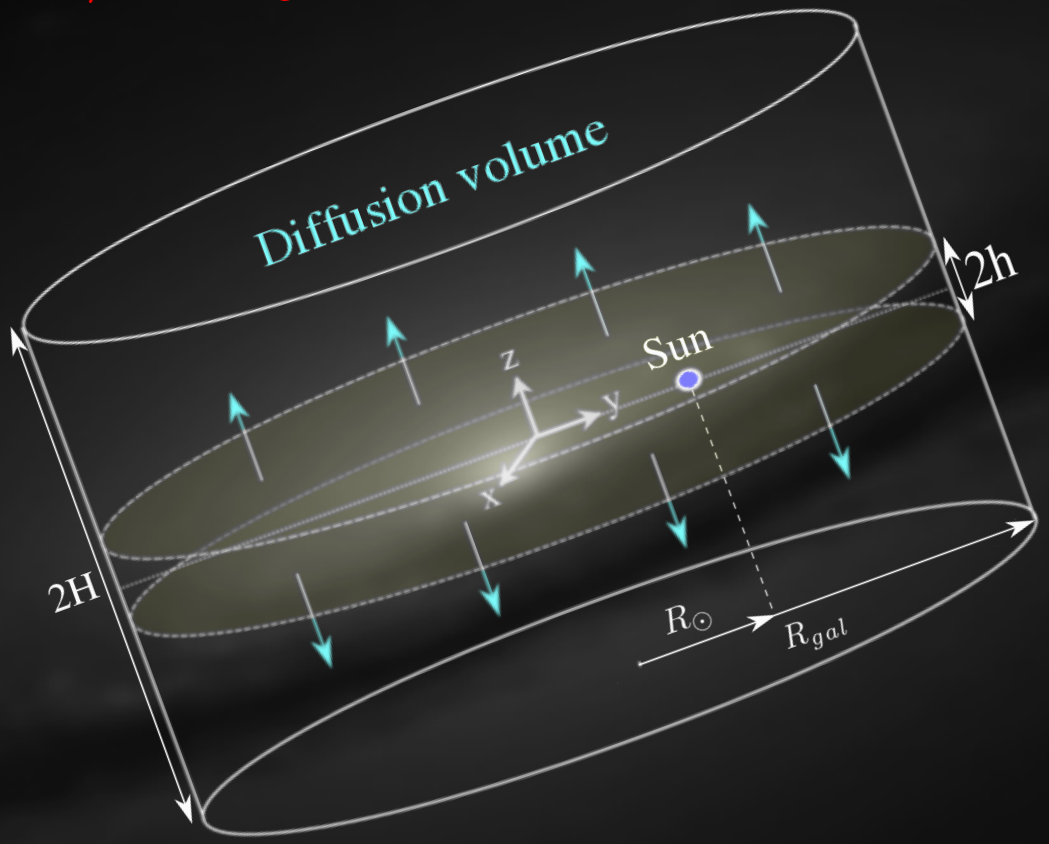


Figure: Two zones diffusion box commonly used.

The mean of the cosmic-ray flux

$$-\nabla_{\boldsymbol{r}}.(K\nabla_{\boldsymbol{r}}\Psi) = Q(\boldsymbol{r}, E) \quad \Rightarrow \quad \langle\Psi\rangle = \underbrace{Q}_{\text{injection rate}} \frac{h\,L}{K}$$

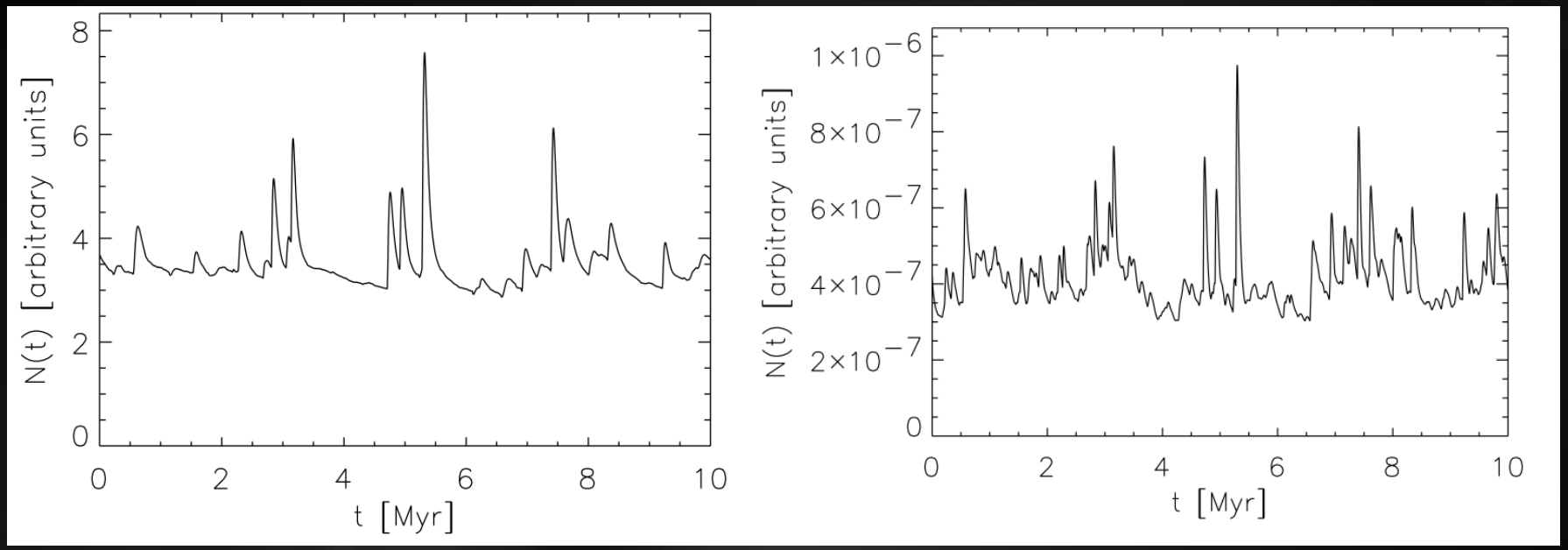
$$\langle\Psi\rangle = \frac{q\,\nu}{V_{MW}}\frac{h\,L}{K}$$ with ν the explosion rate of SNRs.

$$= \underbrace{\frac{q}{2\,L\,\pi R^2}}_{\langle\psi\rangle}\,\underbrace{\nu\,\frac{L^2}{K}}_{N}$$

$$\equiv \langle\psi\rangle \quad N$$

...sources are discrete in space and time !

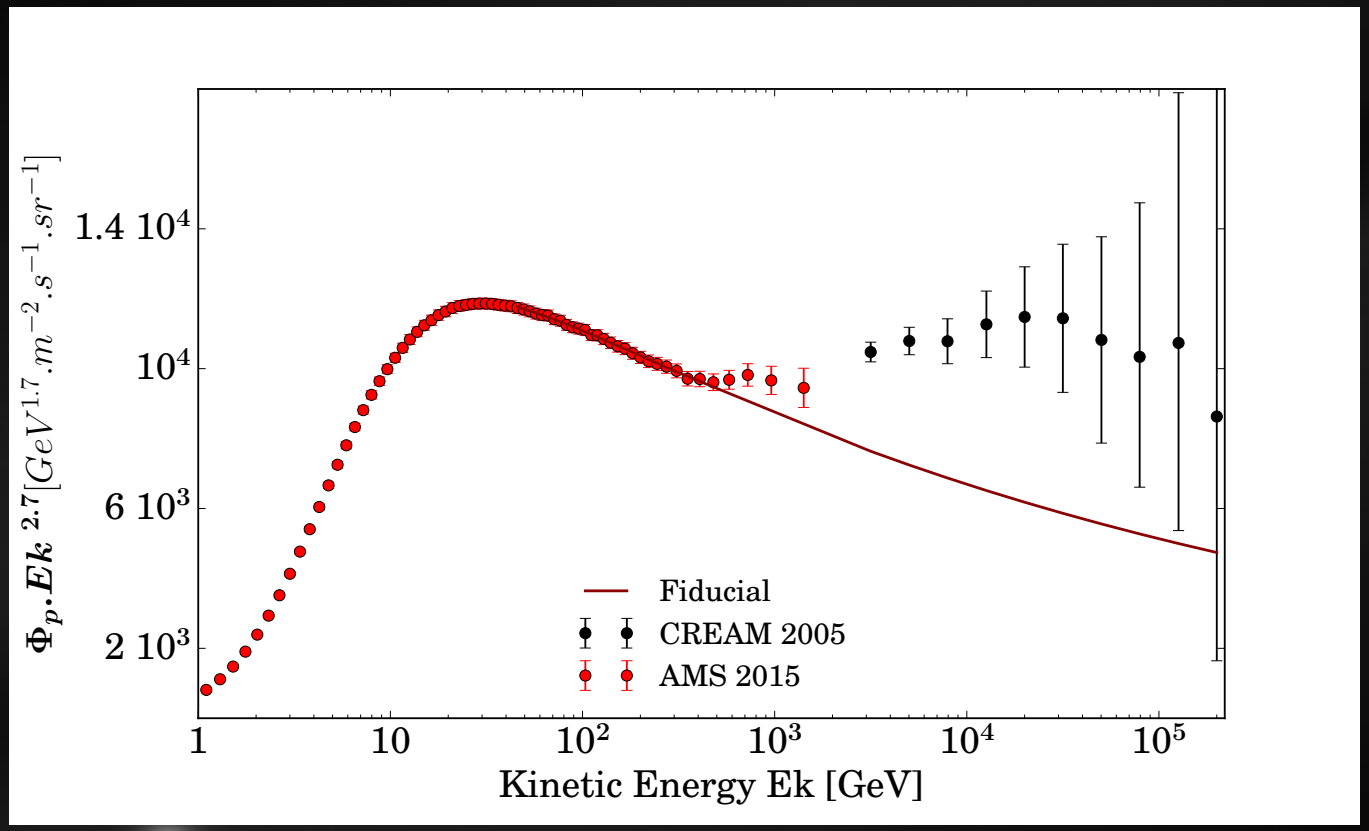
[Büsching et al. 2005]



→ Stochastic behaviour !

One question would be..

What is the probability that a particular configuration of the sources explains the break features ?



$\Rightarrow \quad P(\Psi)$?

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Statistical treatment of the flux

The flux from N sources writes :

$$\Psi = \sum_{i=1}^{N} \psi_i \quad \Rightarrow \quad \langle \Psi \rangle = \sum_{i=1}^{N} \langle \psi \rangle = N \, \langle \psi \rangle$$

One can expect to compute $\langle \psi \rangle$ from $p(\psi)$:

$$\langle \psi \rangle = \int_0^\infty d\psi \, \psi \, p(\psi)$$

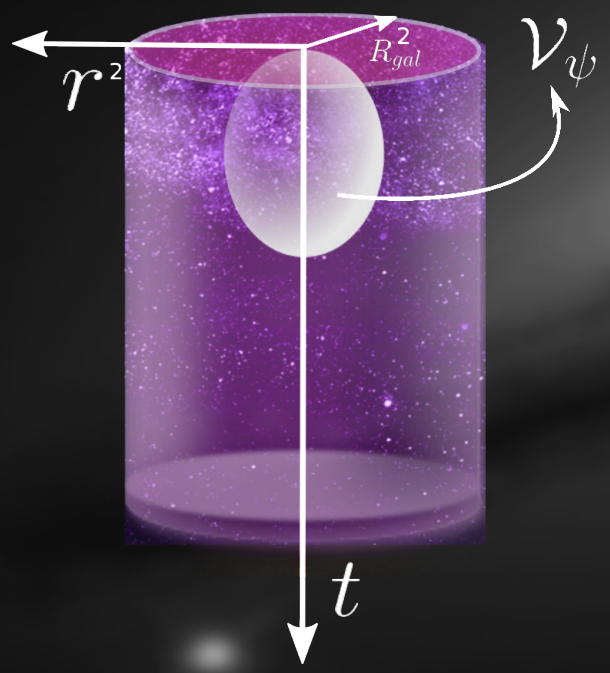
With :

$$p(\psi) = \int_{\mathcal{V}_\psi} \underbrace{\mathcal{D}(\boldsymbol{r}_s, t_s)}_{\text{Normalized distribution in space and time for one source}} \quad \mathrm{d}\boldsymbol{r}_s \, \mathrm{d}t_s \qquad (1)$$

Integration over the domain of space and time that gives a flux between ψ and $\psi + \mathrm{d}\psi$.

Probability to measure ψ from one source

$$p(\psi) = \int_{\mathcal{V}_\psi} \mathcal{D}(\boldsymbol{r}_s, t_s)\, \mathrm{d}\boldsymbol{r}_s\, \mathrm{d}t_s \qquad (2)$$



$\mathcal{V}_\psi$: domain of space and time that gives a flux between ψ and $\psi + \mathrm{d}\psi$.

Surface equation in pure diffusive regime :

$$\psi = \frac{q}{(4\,\pi\,K\,t)^{3/2}}\, \exp\left(\frac{r^2}{4\,K\,t}\right)$$

$\mathcal{D}(\boldsymbol{r}_s, t_s)$ can assume two limiting behaviours, 2D or 3D !

For : $\psi \gg \langle\psi\rangle$ we have, $\boldsymbol{p(\psi) \propto \begin{cases} \psi^{-8/3} & 3D \\ \psi^{-7/3} & 2D \end{cases}}$

Variance of the total flux

$$\Psi = \sum_{i=1}^{N} \psi_i \quad \Rightarrow \quad p(\psi) \to P(\Psi)$$

Central limit theorem ?

$$\sigma_\Psi^2 = \langle \Psi^2 \rangle - \langle \Psi \rangle^2 = N\,\sigma_\psi^2 = N \langle \psi^2 \rangle - \frac{\langle \Psi \rangle^2}{N}$$

$$\langle \psi^2 \rangle = \int_0^\infty \psi^2\, p(\psi) d\psi \;\propto \begin{cases} \left[\psi^{1/3}\right]_{cte}^{\infty} = \infty & 3D \\ \left[\psi^{2/3}\right]_{cte}^{\infty} = \infty & 2D \end{cases}$$

The variance diverges

But actually the pdf exists !

Generalised central limit theorem ?

The heavy tail behaviour conditioned the stable law limit !

$$\forall \psi \geqslant 0, \;\; C(\psi) \equiv \int_{\psi}^{\infty} p(\psi')\, \mathrm{d}\psi' \;\rightarrow \quad \lim_{\psi \to \infty} \psi^{\alpha} C(\psi) = \eta > 0$$

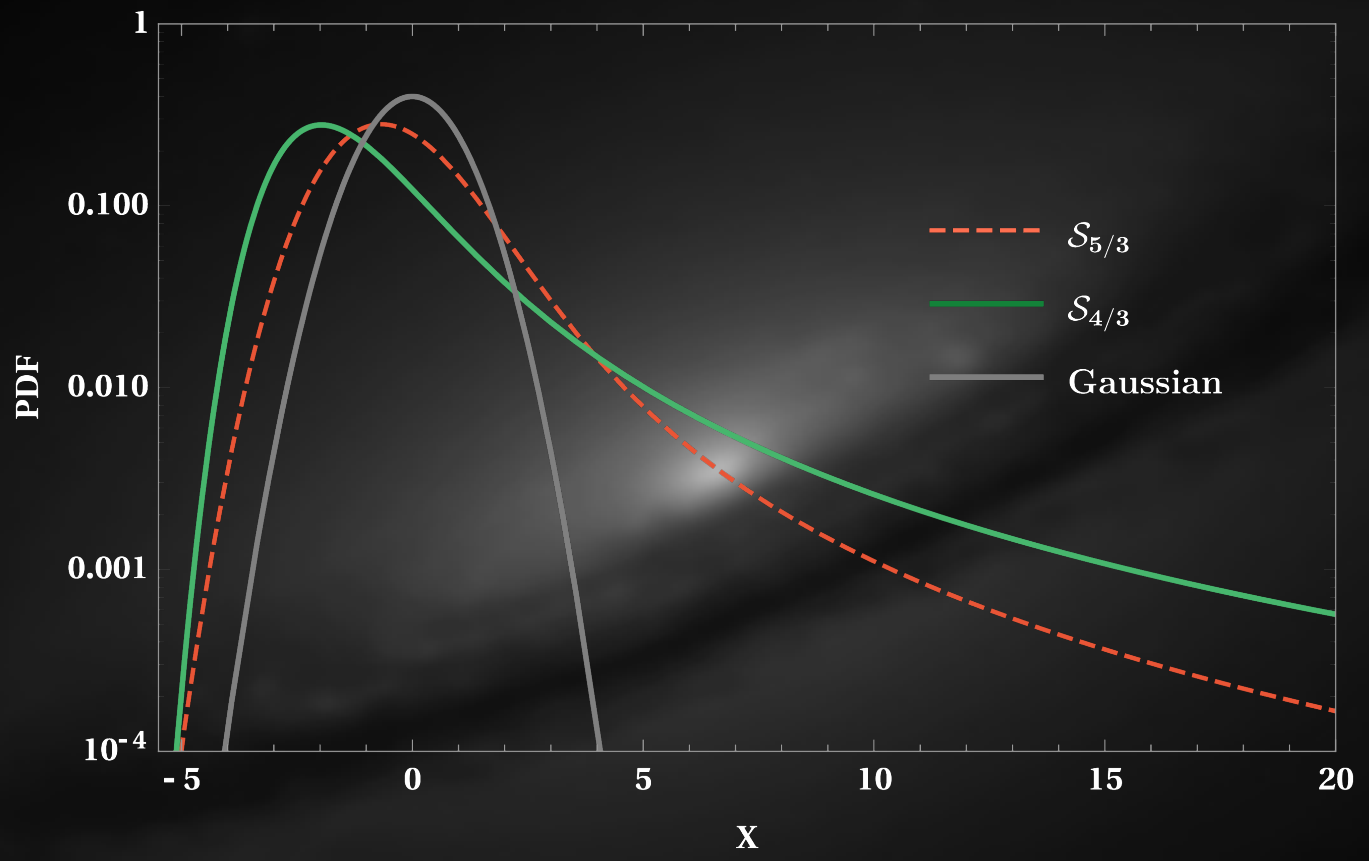
For N *sufficiently* large :

$$P(\Psi) \rightarrow \frac{1}{\sigma_N}\, S[\alpha, 1, 1, 0; 1] \left(\frac{\Psi - \langle \Psi \rangle}{\sigma_N} \right)$$

With : $\alpha = \begin{cases} 5/3 & 3D \\ 4/3 & 2D \end{cases}$ and $\sigma_N = \left(\frac{\eta\, \pi\, N}{2\Gamma(\alpha) \sin(\alpha\, \pi/2)} \right)^{1/\alpha}$

But actually the pdf exists !

$\sigma_N = 1$, $\alpha = 5/3 \rightarrow 3D$, $\alpha = 4/3 \rightarrow 2D$



$\Rightarrow$ So one can define confidence intervals, pvalues...

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On the form of $p(\psi)$

The diffusive propagator is not causal for some region in space and time...

- Loophole : reevaluation of

$$p(\psi) = \int_{\mathcal{V}_\psi^{\text{causal}}} \mathcal{D}(\boldsymbol{r}_s, t_s)\, \mathrm{d}\boldsymbol{r}_s\, \mathrm{d}t_s$$

- $p(\psi) \propto \begin{cases} \psi^{-8/3} & \text{for : } \psi < \psi_c \\ \psi^{-11/3} & \text{for : } \psi > \psi_c \end{cases}$

The variance converges again !

Shall we use the central limit theorem ?...Not really if ψ_c is very large.

Small deviation from the stable law

For : $p_c(\psi) = (1+\epsilon)\, p(\psi)\, \Theta(\psi_c - \psi)$

Then $P(\Psi)$ writes :

$$P(\Psi) = (1+\epsilon)^N \int_{\psi_1} \dots \int_{\psi_N} p_c(\psi_1)..p_c(\psi_N)\, \Theta(\psi_c - \psi_1)..\Theta(\psi_c - \psi_N)\, \delta\left(\sum_i^N \psi_i - \Psi\right)\, \mathrm{d}\psi_1..\mathrm{d}\psi_N$$

For $\Psi < \psi_c$ $(\star)$, each value ψ_i needs to be such as $\psi_i < \psi_c$. Hence we get :

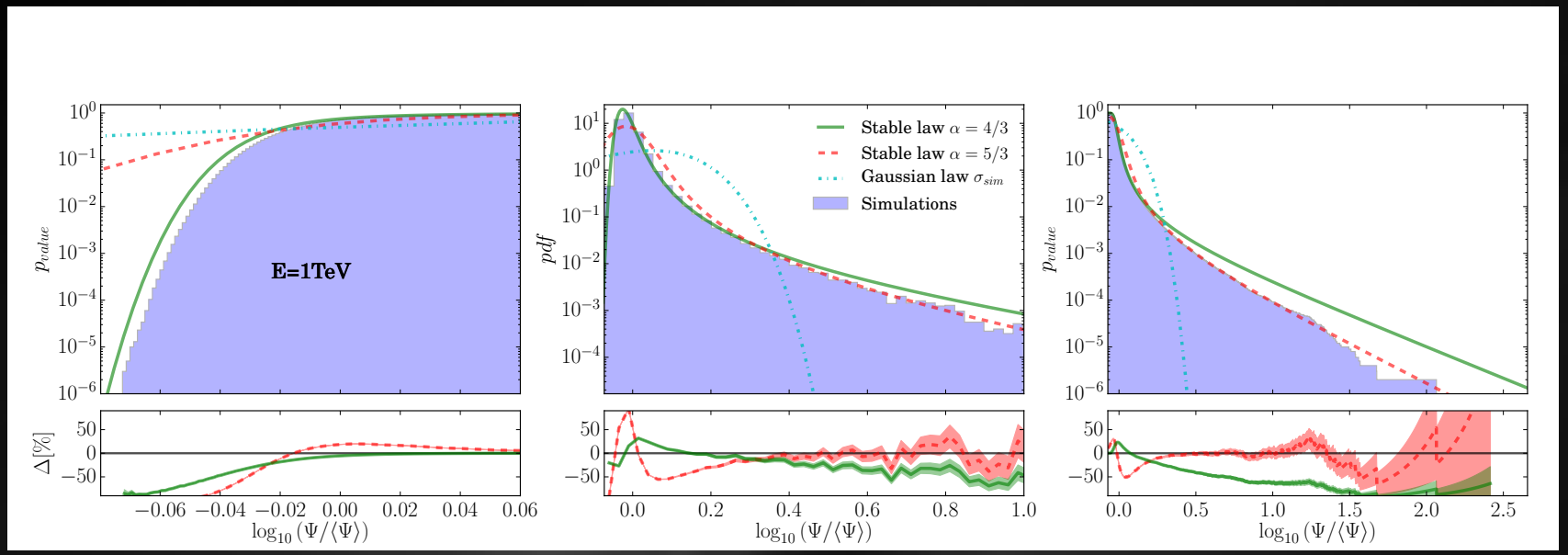
$$P(\Psi) = (1+\epsilon)^N \int_{\psi_1} \dots \int_{\psi_N} p(\psi_1)...p(\psi_N)\, \delta\left(\sum_i^N \psi_i - \Psi\right)\, \mathrm{d}\psi_1...\mathrm{d}\psi_N$$

For N *sufficiently* large :

$$\boldsymbol{P(\Psi) \to (1+\epsilon)^N \frac{1}{\sigma_N} S[\alpha, 1, 1, 0; 1]\left(\frac{\Psi - \langle\Psi\rangle}{\sigma_N}\right)} \quad \text{for} \quad \boldsymbol{\Psi < \psi_c\ (\star)}$$

Simulation check

For example at 1TeV :



Simulation generated 10^6 configurations of galaxies.

Transition from the 2D to the 3D regime !

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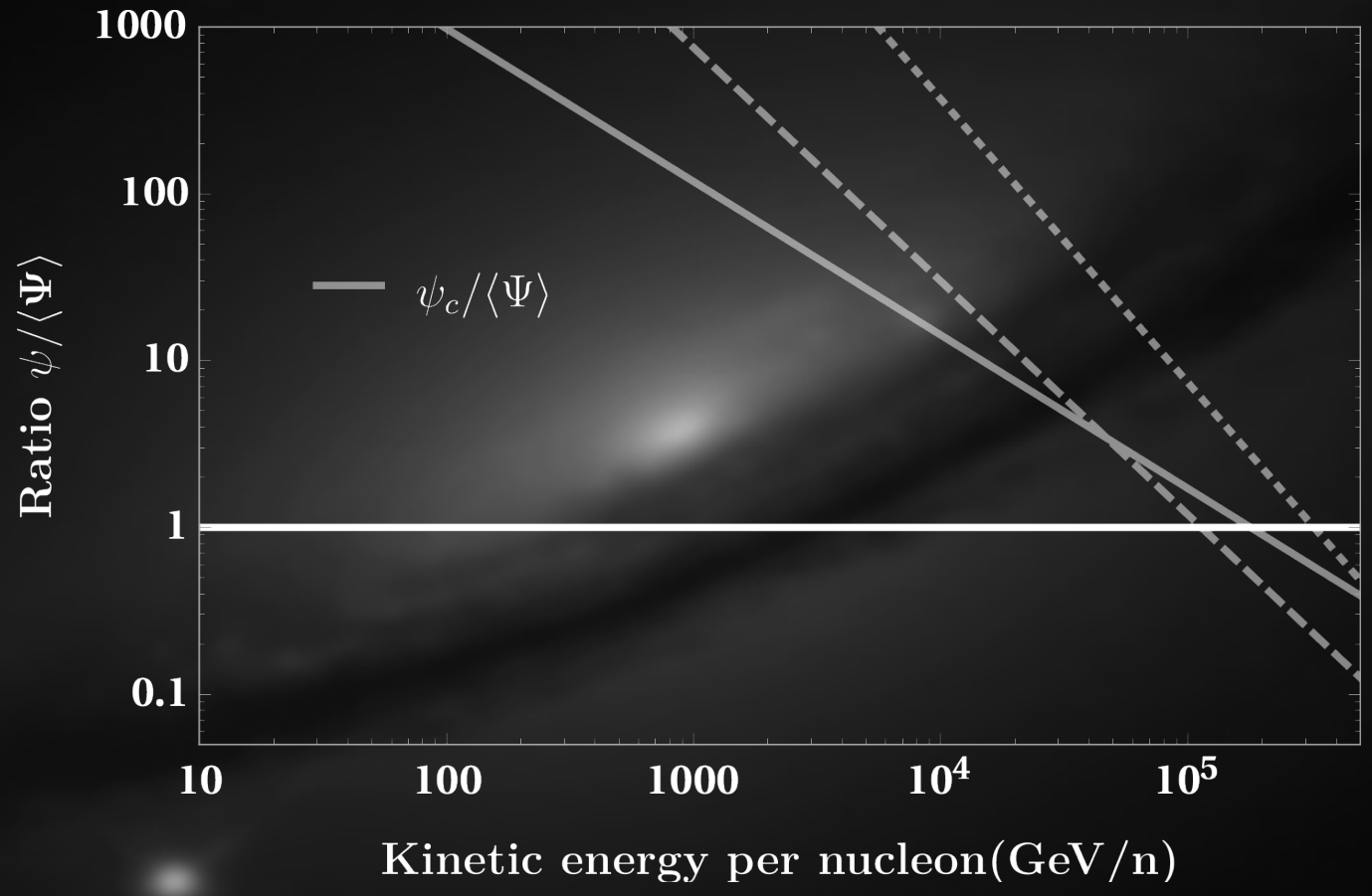
Is the constraint $\Psi < \psi_c$ still interesting ?



MIN(dotted), MED(dashed), MAX(solid)

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Local environment information

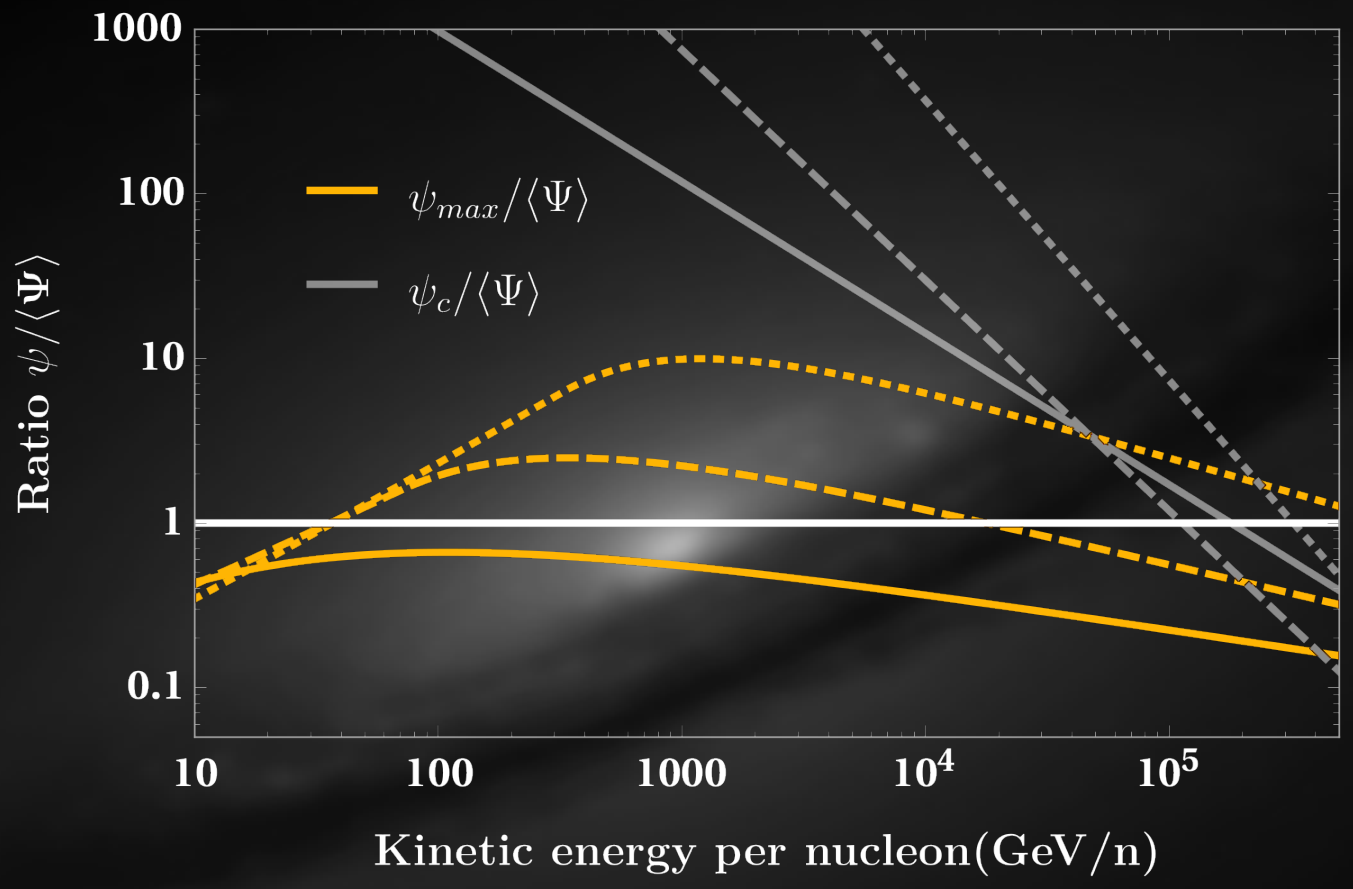
For the purpose of the proton flux...

- No sources on Earth $\Rightarrow$ define a ψ_{max}
- Example from the catalogue of local sources from [Green D. A., 2014] [Delahaye et al. 2010] :

$$\tau > \tau_c = 2,7 \text{ kyr and } r > r_c = 0,06 \text{ kpc}$$

$$\Rightarrow \quad p(\psi) = (1+\epsilon)\, p_D(\psi)\, \Theta(\psi_{max} - \psi)$$

Which constraint $\Psi < \psi_c$ or $\Psi < \psi_{max}$ does dominate ?



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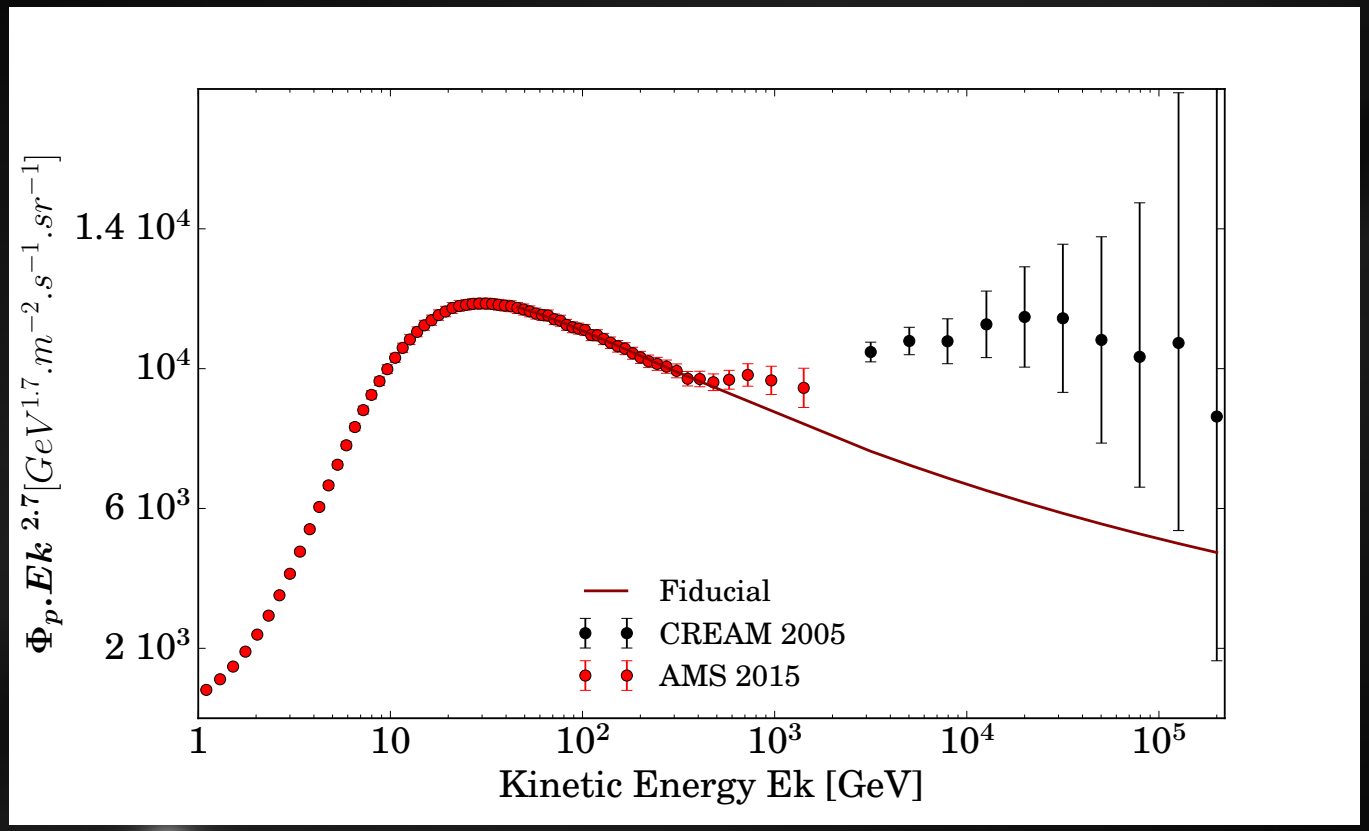
What is the probability that a particular configuration of the sources explain the break features ?



$\Rightarrow \quad P(\Psi)$?

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Probability of such an excess

We compute an upper value of the probability that a particular configuration of the sources gives a flux Ψ at 12.8TeV :

$$p_{value} = \int_{\Psi_{exp}}^{\infty} d\psi_{exp} \int_{0}^{+\infty} d\psi_{th}\; p(\psi_{exp}|\psi_{th})\; p(\psi_{th}|Model),$$

Example for the benchmark models :

Models	MIN	MED	MAX
Probabilities(Stable law 4/3)	0.031	0.0082	0.0013

The break feature of the protons

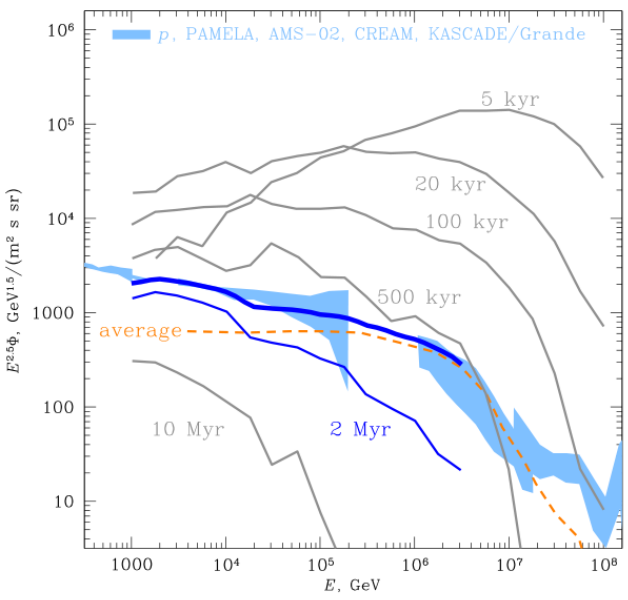

FIG. 1: Proton flux of the local source at different times. The average Galactic proton flux is shown as a thin orange line, the measured spectra of protons (light blue) from PAMELA [8], CREAM [9], KASCADE and KASCADE-Grande [11] as a band including experimental uncertainties. The sum of the average flux and the 2 Myr old source is shown by the blue thick line.

[Kachelrieß et al., 2015]

- Local flux dominated by a 2Myr old SNR in order to explain the knee..
- $D_\perp \ll D_\parallel$
- At $E = 1$TeV, $\Psi \approx 2.86\langle\Psi\rangle$

$\Rightarrow$ $P(\Psi)$ can be used for an homogeneous diffusion model.

Models	MIN	MED	MAX
Probabilities	0.0072	0.0012	0.00016

The break feature of the protons

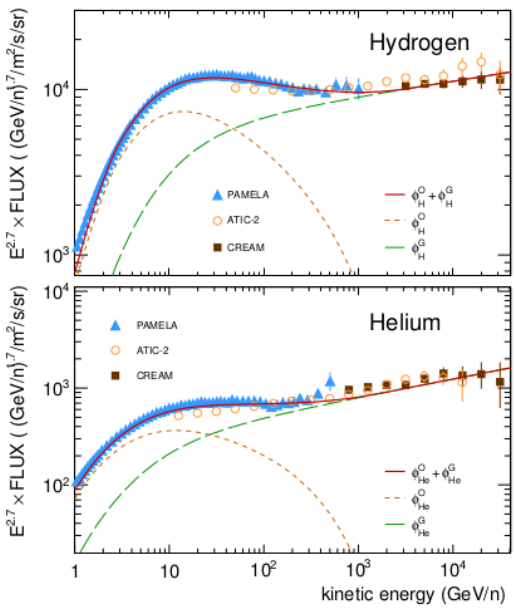

FIG. 2.— Energy spectra of H (top) and He (bottom) multiplied by $E^{2.7}$. The solid lines indicate the model calculations. The contribution arising from OSNR (short-dashed lines) and from GSNR (long-dashed lines) are shown. The data are from PAMELA (Adriani et al. 2011), ATIC-2 (Panov et al. 2009) and CREAM (Yoon et al. 2010).

[Tomassetti et al., 2015]

- Two component model, without prior on their number of sources
- Homogeneous diffusion
- At $E = 10$GeV, $\Psi \approx 3.3\langle\Psi\rangle$

$\Rightarrow P(\Psi)$ can be used !

The probability @10GeV is 8.6×10^{-5} !

Conclusion :

- **The variance of the cosmic ray flux does not diverge !**
- The probability law to measure a particular flux follows a **stable law** up to $\approx 20TeV$.
- We propose a generic way to deal with **heavy tail probability distributions.**
- A lot of applications and possible improvements !

$\Rightarrow$ **On the arxiv soon !**

Thanks for listening