The Pentabox Master Integrals with the Simplified Differential Equations approach

Costas G. Papadopoulos

INPP, NCSR "Demokritos"

UGR, Granada, April 2014, 2016

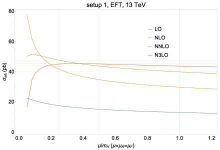


Figure 8: The dependence of the cross-section on a common renormalization and factorization scale $\mu = \mu_F = \mu_R$.

		$\Delta^{\rm scale}_{EFT,k}$
LO	$(k = 0)$	$\pm 14.8\%$
NLO	$(k = 1)$	$\pm 16.6\%$
NNLO	$(k = 2)$	$\pm 8.8\%$
$\mathrm{N^3LO}$	$(k = 3)$	$\pm 1.9\%$

Table 5: Scale variation of the cross-section as defined in eq. (3.11) for a common renormalization and factorization scale $\mu = \mu_F = \mu_R$.

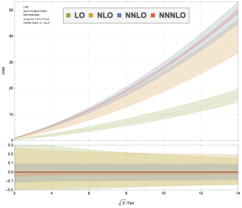
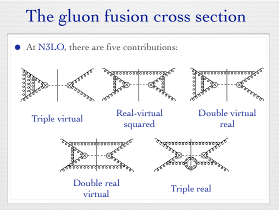


FIG. 3: The gluon fusion cross-section at all perturbative orders through $\mathrm{N^3LO}$ in the scale interval $[\frac{m_H}{4}, m_H]$ as a function of the center-of-mass energy $\sqrt{S}$.

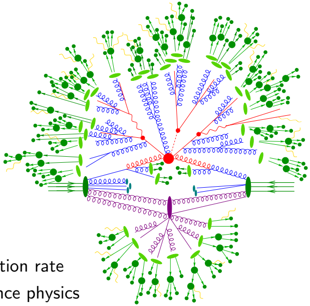


C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, T. Gehrmann, F. Herzog, A. Lazopoulos and B. Mistlberger, arXiv:1602.00695

Factorization

Collins,Soper,Sterman'85–'89

- Calculate
  - Scattering probability
  - Gluon emission probability
- Measure
  - Long distance interactions
  - Particle decay rates



Divide et Impera

- Quantity of interest: Total interaction rate
- Convolution of short & long distance physics

$$\sigma_{p_1 p_2 \to X} = \sum_{i,j \in \{q,g\}} \int \mathrm{d}x_1 \mathrm{d}x_2 \underbrace{f_{p_1,i}(x_1,\mu_F^2) f_{p_2,j}(x_2,\mu_F^2)}_{\text{long distance physics}} \underbrace{\hat{\sigma}_{ij\to X}(x_1 x_2, \mu_F^2)}_{\text{short distance physics}}$$

QCD as a perturbative quantum field theory: Fixed-order calculations

Taming the beast ...

From Feynman graphs ...

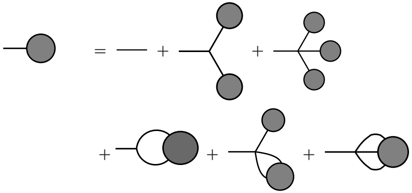
$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

Taming the beast ...

From Feynman graphs ...

$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

to Dyson-Schwinger recursion! `Helac-Phegas`

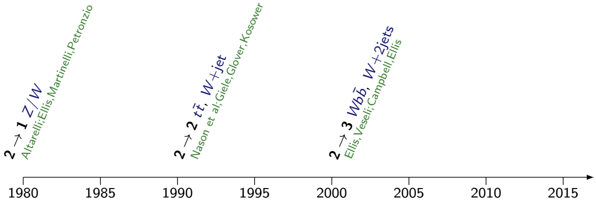


$gg \to ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

NLO Revolution

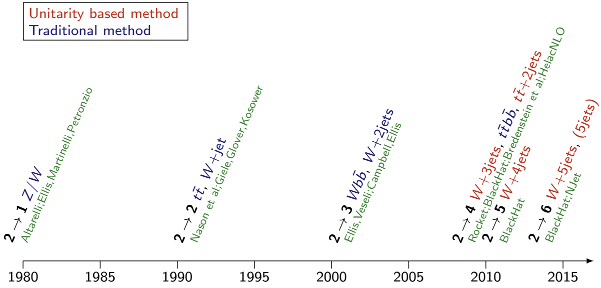
G. P. Salam, PoS ICHEP **2010**, 556 (2010) [arXiv:1103.1318 [hep-ph]]

The NLO revolution



NLO Revolution

The NLO revolution



BlackHat → Berger,Bern,Dixon,Febres Cordero,Forde,Ita,Kosower,Maître
HelacNLO → Bevilacqua,Czakon,Papadopoulos,Pittau,Worek
NJet → Badger,Biedermann,Uwer,Yundin
Rocket → Ellis,Melnikov,Zanderighi

NLO Revolution

PRL **116,** 052003 (2016) PHYSICAL REVIEW LETTERS week ending 5 FEBRUARY 2016

Top Quark Pair Production in Association with a Jet with Next-to-Leading-Order QCD Off-Shell Effects at the Large Hadron Collider

G. Bevilacqua,[1] H. B. Hartanto,[2] M. Kraus,[2] and M. Worek[2]
[1]*INFN, Laboratori Nazionali di Frascati, Via E. Fermi 40, I-00044 Frascati, Italy*
[2]*Institut für Theoretische Teilchenphysik und Kosmologie, RWTH Aachen University, D-52056 Aachen, Germany*
(Received 2 October 2015; revised manuscript received 1 December 2015; published 5 February 2016)

We present a complete description of top quark pair production in association with a jet in the dilepton channel. Our calculation is accurate to next-to-leading order (NLO) in QCD and includes all nonresonant diagrams, interferences, and off-shell effects of the top quark. Moreover, nonresonant and off-shell effects due to the finite *W* gauge boson width are taken into account. This calculation constitutes the first fully realistic NLO computation for top quark pair production with a final state jet in hadronic collisions. Numerical results for differential distributions as well as total cross sections are presented for the Large Hadron Collider at 8 TeV. With our inclusive cuts, NLO predictions reduce the unphysical scale dependence by more than a factor of 3 and lower the total rate by about 13% compared to leading-order QCD predictions. In addition, the size of the top quark off-shell effects is estimated to be below 2%.

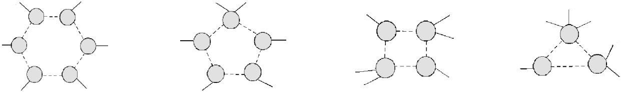
DOI: 10.1103/PhysRevLett.116.052003

$$pp \to W^+W^- b\bar{b}g$$

The one-loop calculation in a nutshell

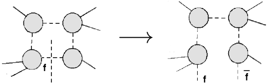
. The computation of $pp(p\bar{p}) \to e^{+}\nu_e\mu^{-}\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, HELAC evaluates numerically the numerators $N_i^6(q), N_i^5(q), \ldots$. with the values of the loop momentum q provided by CutTools

- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

→ MadGraph, RECOLA, OpenLoops

The one-loop calculation in a nutshell

Institute of Nuclear Physics "Demokritos" Bergische Universitat Wuppertal Institute of Nuclear Physics PAN RWTH Aachen University

Content

HELAC-NLO & Associated Tools

Projects

HELAC-PHEGAS - A generator for all parton level processes in the Standard Model

HELAC-DIPOLES - Dipole formalism for the arbitrary helicity eigenstates of the external partons

HELAC-1LOOP - A program for numerical evaluation of QCD virtual corrections to scattering amplitudes

ONELOOP - A program for the evaluation of one-loop scalar functions

CUTTOOLS - A program implementing the OPP reduction method to compute one-loop amplitudes

PARNI - A program for importance sampling and density estimation

KALEU - A general-purpose parton-level phase space generator

HELAC-ONIA - An automatic matrix element generator for heavy quarkonium physics

:: top ::

People

Giuseppe Bevilacqua
Michal Czakon
Maria Vittoria Garzelli
Andreas van Hameren
Adam Kardos
Yiannis Malamos
Costas G. Papadopoulos
Roberto Pittau
Malgorzata Worek
Hua-Sheng Shao

:: top ::

Contact us

If you have a question, comment, suggestion or bug report, please e-mail us at:

bevilacqua(at)physik.rwth-aachen.de
mczakon(at)physik.rwth-aachen.de
garzelli(at)to.infn.it
Andre.Hameren(at)ifj.edu.pl
kardam(at)t-online.hu
J.Malamos(at)science.ru.nl
Costas.Papadopoulos(at)cern.ch
pittau(at)ugr.es
Malgorzata.Worek(at)cern.ch
erdissshaw(at)gmail.com

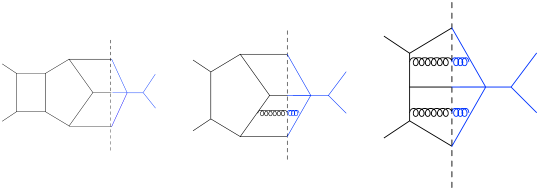
:: top ::

Last modified by Malgorzata Worek
Thursday, January 10th, 2013

Proof of concept: the first NLO public code

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$



What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NNLO} \;\to\; & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left|M_m^{(1)}\right|^2\right) J_m(\Phi) && VV \\
+ & \int_{m+1} d\Phi_{m+1} \left(2Re\left(M_{m+1}^{(0)*} M_{m+1}^{(1)}\right)\right) J_{m+1}(\Phi) && RV \\
+ & \int_{m+2} d\Phi_{m+2} \left|M_{m+2}^{(0)}\right|^2 J_{m+2}(\Phi) && RR
\end{aligned}$$

$RV + RR \to$ Antenna-S, Colorfull-S, STRIPPER

A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047

P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059

M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152

OPP at two loops

$$
\begin{aligned}
\frac{N(q)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i0}\bar{D}_{i1}\bar{D}_{i2}\bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i0}\bar{D}_{i1}\bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i0}\bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007)

- Write the "OPP-type" equation at two loops

$$\frac{N(l_1, l_2; \{p_i\})}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$$\sum \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}} \rightarrow \text{spurious} \oplus \text{ISP} - \text{irreducible integrals}$$

OPP AT TWO LOOPS

- Write the "OPP-type" equation at two loops

$$\frac{N(l_1, l_2; \{p_i\})}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$$\sum \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}} \rightarrow \text{spurious} \oplus \text{ISP} - \text{irreducible integrals}$$

ISP-irreducible integrals $\rightarrow$ use IBPI to Master Integrals

Libraries in the future: `QCD2LOOP`, `TwOLOop`

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, Phys. Lett. B **718** (2012) 173

J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D **83** (2011) 045012

H. Ita, arXiv:1510.05626 [hep-th].

C. G. Papadopoulos, R. H. P. Kleiss and I. Malamos, PoS Corfu **2012** (2013) 019.

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k, l\} + p_i)^2 - M_i^2$
-
$$F[a_1, \dots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \dots D_N^{a_N}}$$
$$\int d^d k d^d l \quad \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$
- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Find a better IBP algorithm ... Generating function technique, Baikov ?
- Or numerical: `SecDec`, Weinzierl

IBPI: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $$F[a_1,\dots,a_N] = \int d^dk d^dl \frac{1}{D_1^{a_1}\dots D_N^{a_N}}$$ $$\int d^dk d^dl \; \frac{\partial}{\partial\{k^\mu, l^\mu\}}\left(\frac{\{k^\mu, l^\mu, \upsilon^\mu\}}{D_1^{a_1}\dots D_N^{a_N}}\right) = 0$$
- IBP Laporta: FIRE, AIR, Reduze reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Find a better IBP algorithm ... Generating function technique, Baikov ?
- Or numerical: SecDec, Weinzierl

IBPI: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $$F[a_1,\ldots,a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1}\ldots D_N^{a_N}}$$
  $$\int d^d k d^d l \;\; \frac{\partial}{\partial\{k^\mu, l^\mu\}}\left(\frac{\{k^\mu, l^\mu, \upsilon^\mu\}}{D_1^{a_1}\ldots D_N^{a_N}}\right) = 0$$

F. V. Tkachov, Phys. Lett. B **100** (1981) 65.

K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159.

- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Find a better IBP algorithm ... Generating function technique, Baikov ?
- Or numerical: `SecDec`, Weinzierl

IBPI: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k, l\} + p_i)^2 - M_i^2$
- $$F[a_1, \ldots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \ldots D_N^{a_N}}$$
$$\int d^d k d^d l \quad \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \ldots D_N^{a_N}} \right) = 0$$
- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI

S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087

C. Anastasiou and A. Lazopoulos, JHEP **0407** (2004) 046

C. Studerus, Comput. Phys. Commun. **181** (2010) 1293

A. V. Smirnov, Comput. Phys. Commun. **189** (2014) 182

- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Find a better IBP algorithm ... Generating function technique, Baikov ?
- Or numerical: SecDec, Weinzierl

IBPI: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $$F[a_1,\ldots,a_N] = \int d^dk d^dl \frac{1}{D_1^{a_1}\ldots D_N^{a_N}}$$
  $$\int d^dk d^dl \; \frac{\partial}{\partial\{k^\mu, l^\mu\}}\left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1}\ldots D_N^{a_N}}\right) = 0$$
- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations

Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

V. A. Smirnov, Phys. Lett. B **460** (1999) 397

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Find a better IBP algorithm ... Generating function technique, Baikov ?
- Or numerical: `SecDec`, Weinzierl

IBPI: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k, l\} + p_i)^2 - M_i^2$
- $$F[a_1, \dots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \dots D_N^{a_N}}$$
$$\int d^d k d^d l \ \ \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$
- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Find a better IBP algorithm ... Generating function technique, Baikov ?

P. A. Baikov, Nucl. Instrum. Meth. A **389** (1997) 347

V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **672** (2003) 199

- Or numerical: `SecDec`, Weinzierl

IBPI: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $$F[a_1,\dots,a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1}\dots D_N^{a_N}}$$
$$\int d^d k d^d l \;\; \frac{\partial}{\partial\{k^\mu, l^\mu\}}\left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1}\dots D_N^{a_N}}\right) = 0$$
- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Find a better IBP algorithm ... Generating function technique, Baikov ?
- Or numerical: `SecDec`, Weinzierl

S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **196** (2015) 470

S. Becker, C. Reuschle and S. Weinzierl, JHEP **1012** (2010) 013

Differential Equations Approach

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\})$$
$$\partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0$$

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Boundary conditions: expansion by regions or regularity conditions.

B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012) 2139 [arXiv:1206.0546 [hep-ph]].

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$p_j^{\mu} \frac{\partial}{\partial p_i^{\mu}} G[a_1, \ldots, a_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\})$$
$$\partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0$$

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Boundary conditions: expansion by regions or regularity conditions.

B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012) 2139 [arXiv:1206.0546 [hep-ph]].

Differential Equations Approach

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\})$$
$$\partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0$$

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Boundary conditions: expansion by regions or regularity conditions.

B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012) 2139 [arXiv:1206.0546 [hep-ph]].

Differential Equations Approach

- Iterated Integrals K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n, \ldots, a_1, x\right) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}\left(a_{n-1}, \ldots, a_1, t\right)$$

with the special cases, $\mathcal{G}(x) = 1$ and

$$\mathcal{G}\left(\underbrace{0, \ldots 0}_{n}, x\right) = \frac{1}{n!} \log^n(x)$$

- Shuffle algebra

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

C. Bogner and F. Brown

- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n,\ldots,a_1,x\right) = \int_0^x dt \frac{1}{t-a_n} \mathcal{G}\left(a_{n-1},\ldots,a_1,t\right)$$

with the special cases, $\mathcal{G}(x) = 1$ and

$$\mathcal{G}\left(\underbrace{0,\ldots 0}_{n},x\right) = \frac{1}{n!}\log^n\left(x\right)$$

- Shuffle algebra

Differential Equations Approach

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n,\ldots,a_1,x\right)=\int_0^x dt\frac{1}{t-a_n}\mathcal{G}\left(a_{n-1},\ldots,a_1,t\right)$$

with the special cases, $\mathcal{G}(x)=1$ and

$$\mathcal{G}\left(\underbrace{0,\ldots 0}_{n},x\right)=\frac{1}{n!}\log^n\left(x\right)$$

- Shuffle algebra

$$\mathcal{G}\left(a_1,a_2;x\right)\mathcal{G}\left(b_1;x\right)=\mathcal{G}\left(a_1,a_2,b_1;x\right)+\mathcal{G}\left(a_1,b_1,a_2;x\right)+\mathcal{G}\left(b_1,a_1,a_2;x\right)$$

The Simplified Differential Equations Approach

C. G. Papadopoulos, JHEP **1407** (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\ldots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k+p_1+p_2)^2 \ldots (k+p_1+p_2+\ldots+p_n)^2}$$

- Now the integral becomes a function of x, which allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + x p_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

- and using IBPI we obtain, for instance for the one-loop 3 off-shell legs

$$\begin{aligned} m_1 x G_{121} + \frac{1}{x} G_{021} \quad &= \left(\frac{1}{x-1} + \frac{1}{x-m_3/m_1}\right)\left(\frac{d-4}{2}\right) G_{111} \\ &+ \frac{d-3}{m_1-m_3}\left(\frac{1}{x-1} - \frac{1}{x-m_3/m_1}\right)\left(\frac{G_{101}-G_{110}}{x}\right) \end{aligned}$$

The Simplified Differential Equations Approach

C. G. Papadopoulos, JHEP **1407** (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\ldots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k+p_1+p_2)^2 \ldots (k+p_1+p_2+\ldots+p_n)^2}$$

- Now the integral becomes a function of x, which allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + x p_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

- and using IBPI we obtain, for instance for the one-loop 3 off-shell legs

$$m_1 x G_{121} + \frac{1}{x} G_{021} = \left(\frac{1}{x-1} + \frac{1}{x - m_3/m_1}\right)\left(\frac{d-4}{2}\right) G_{111} + \frac{d-3}{m_1 - m_3}\left(\frac{1}{x-1} - \frac{1}{x - m_3/m_1}\right)\left(\frac{G_{101} - G_{110}}{x}\right)$$

The Simplified Differential Equations Approach

C. G. Papadopoulos, JHEP **1407** (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\ldots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\, p_1)^2\,(k+p_1+p_2)^2 \ldots (k+p_1+p_2+\ldots+p_n)^2}$$

- Now the integral becomes a function of x, which allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + x p_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

- and using IBPI we obtain, for instance for the one-loop 3 off-shell legs

$$\begin{aligned} m_1 x G_{121} + \tfrac{1}{x} G_{021} &= \left(\tfrac{1}{x-1} + \tfrac{1}{x - m_3/m_1}\right)\left(\tfrac{d-4}{2}\right) G_{111} \\ &+ \tfrac{d-3}{m_1 - m_3}\left(\tfrac{1}{x-1} - \tfrac{1}{x-m_3/m_1}\right)\left(\tfrac{G_{101} - G_{110}}{x}\right) \end{aligned}$$

The Simplified Differential Equations Approach

- The integrating factor M is given by

$$M = x\,(1-x)^{\frac{4-d}{2}}\,(-m_3+m_1 x)^{\frac{4-d}{2}}$$

- and the DE takes the form, $d = 4 - 2\varepsilon$,

$$\frac{\partial}{\partial x} M G_{111} = c_\Gamma \frac{1}{\varepsilon}\,(1-x)^{-1+\varepsilon}\,(-m_3+m_1 x)^{-1+\varepsilon}\left(\left(-m_1 x^2\right)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)$$

  - Integrating factors $\epsilon = 0$ do not have branch points
  - DE can be straightforwardly integrated order by order $\rightarrow$ GPs.

The Simplified Differential Equations Approach

- The integrating factor M is given by

$$M = x\,(1-x)^{\frac{4-d}{2}}\,(-m_3 + m_1 x)^{\frac{4-d}{2}}$$

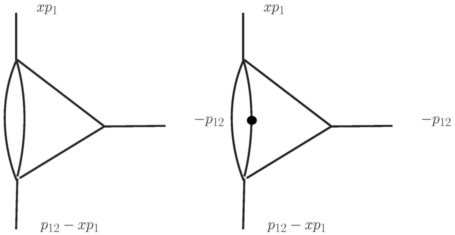
- and the DE takes the form, $d = 4 - 2\varepsilon$,

$$\frac{\partial}{\partial x} M G_{111} = c_\Gamma \frac{1}{\varepsilon}\,(1-x)^{-1+\varepsilon}\,(-m_3 + m_1 x)^{-1+\varepsilon}\left(\left(-m_1 x^2\right)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)$$

• Integrating factors $\epsilon = 0$ do not have branch points
• DE can be straightforwardly integrated order by order $\rightarrow$ GPs.

The Simplified Differential Equations Approach

The two-loop 3-off-shell-legs triangle



The Simplified Differential Equations Approach

We are interested in $G_{0101011}$. The DE involves also the MI $G_{0201011}$, so we have a system of two coupled DE, as follows:

$$\begin{aligned}
\frac{\partial}{\partial x} f(x) &= \frac{A_3(2-3\varepsilon)(1-x)^{-2\varepsilon} x^{-1+\varepsilon} (m_1 x - m_3)^{-2\varepsilon}}{2\varepsilon(2\varepsilon-1)} \\
&+ \frac{m_1 \varepsilon (1-x)^{-2\varepsilon} (m_1 x - m_3)^{-2\varepsilon}}{2\varepsilon - 1} g(x)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x} g(x) &= \frac{A_3(3\varepsilon-2)(3\varepsilon-1)(-m_1)^{-2\varepsilon}(1-x)^{2\varepsilon-1} x^{-3\varepsilon} (m_1 x - m_3)^{2\varepsilon-1}}{2\varepsilon^2} \\
&+ (2\varepsilon - 1)(3\varepsilon - 1)(1 - x)^{2\varepsilon-1} (m_1 x - m_3)^{2\varepsilon-1} f(x)
\end{aligned}$$

where $f(x) \equiv M_{0101011} G_{0101011}$ and $g(x) \equiv M_{0201011} G_{0201011}$, $M_{0201011} = (1-x)^{2\varepsilon} x^{\varepsilon+1} (m_1 x - m_3)^{2\varepsilon}$ and $M_{0101011} = x^{\varepsilon}$

- Solve sequentially in ε expansion
- Reproduce limit $\varepsilon \to 0$

The Simplified Differential Equations Approach

The singularity at $x=0$ is proportional to $x^{-1+\varepsilon}$ and can easily be integrated by the following decomposition

$$\int_0^x dt\ t^{-1+\varepsilon} F(t) = F(0) \int_0^x dt\ t^{-1+\varepsilon} + \int_0^x dt\ \frac{F(t)-F(0)}{t} t^{\varepsilon}$$
$$= F(0)\,\frac{x^{\varepsilon}}{\varepsilon} + \int_0^x dt\ \frac{F(t)-F(0)}{t} \left(1+\varepsilon \log(t) + \frac{1}{2}\varepsilon^2 \log^2(t) + \ldots\right)$$

Reproduce correctly boundary term $x=0$

The Simplified Differential Equations Approach

General setup

m: number of denominators

$$\partial_x G_{m+1} = H\left(\{s_{ij}\}, \epsilon; x\right) G_{m+1} + \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'},$$

$m_0 = 3$ in the case of two loops

The Simplified Differential Equations Approach

General setup

m: number of denominators

$$\partial_x G_{m+1} = H\left(\{s_{ij}\}, \epsilon; x\right) G_{m+1} + \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'},$$

$m_0 = 3$ in the case of two loops

$$\partial_x M = -MH$$

$$\partial_x (M G_{m+1}) = M \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'}.$$

The Simplified Differential Equations Approach

General setup

m: number of denominators

$$\partial_x G_{m+1} = H\left(\{s_{ij}\}, \epsilon; x\right) G_{m+1} + \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'},$$

$m_0 = 3$ in the case of two loops

$$\partial_x M = -MH$$

$$\partial_x (M G_{m+1}) = M \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'}.$$

$$M \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'} =: \sum_i x^{-1+\beta_i \epsilon} \tilde{I}_{sin}^{(i)}(\{s_{ij}\}, \epsilon) + \tilde{I}_{reg}(\{s_{ij}\}, \epsilon; x).$$

The Simplified Differential Equations Approach

$$MG_{m+1} = C(\{s_{ij}\},\epsilon) + \sum_i \frac{x^{\beta_i \epsilon}}{\beta_i \epsilon} \tilde{I}^{(i)}_{sin}(\{s_{ij}\},\epsilon) + \int_0^x dx' \tilde{I}_{reg}(\{s_{ij}\},\epsilon; x'),$$

- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed.*

The Simplified Differential Equations Approach

$$MG_{m+1} = C(\{s_{ij}\},\epsilon) + \sum_i \frac{x^{\beta_i \epsilon}}{\beta_i \epsilon} \tilde{I}^{(i)}_{sin}(\{s_{ij}\},\epsilon) + \int_0^x dx' \tilde{I}_{reg}(\{s_{ij}\},\epsilon;x'),$$

- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed.*

The Simplified Differential Equations Approach

$$MG_{m+1} = C(\{s_{ij}\},\epsilon) + \sum_i \frac{x^{\beta_i \epsilon}}{\beta_i \epsilon} \tilde{I}^{(i)}_{sin}(\{s_{ij}\},\epsilon) + \int_0^x dx' \tilde{I}_{reg}(\{s_{ij}\},\epsilon;x'),$$

- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed.*

The Simplified Differential Equations Approach

$$MG_{m+1} = C(\{s_{ij}\},\epsilon) + \sum_i \frac{x^{\beta_i \epsilon}}{\beta_i \epsilon} \tilde{I}^{(i)}_{sin}(\{s_{ij}\},\epsilon) + \int_0^x dx' \tilde{I}_{reg}(\{s_{ij}\},\epsilon; x'),$$

- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed.*

The Simplified Differential Equations Approach

$$MG_{m+1} = C(\{s_{ij}\}, \epsilon) + \sum_i \frac{x^{\beta_i \epsilon}}{\beta_i \epsilon} \tilde{I}_{sin}^{(i)}(\{s_{ij}\}, \epsilon) + \int_0^x dx' \tilde{I}_{reg}(\{s_{ij}\}, \epsilon; x'),$$

- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\}, \epsilon) = 0$: *no independent calculation of boundary terms needed.*

The Simplified Differential Equations Approach

$$MG_{m+1} = C(\{s_{ij}\},\epsilon) + \sum_i \frac{x^{\beta_i \epsilon}}{\beta_i \epsilon} \tilde{I}^{(i)}_{sin}(\{s_{ij}\},\epsilon) + \int_0^x dx' \tilde{I}_{reg}(\{s_{ij}\},\epsilon; x'),$$

- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed.*

The Simplified Differential Equations Approach

When the DE are coupled

$$\partial_x \vec{G}_{m+1} = \mathbf{H}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m+1} + \sum_{m' \geq m_0}^{m} \mathbf{R}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m'},$$

- $\mathbf{M}_D$: $\partial_x \mathbf{M}_D = -\mathbf{M}_D \mathbf{H}_D$, where $\mathbf{H}_D$ is the diagonal part of $\mathbf{H}$.
- $\tilde{\mathbf{H}} =: \mathbf{M}_D \left(\mathbf{H} - \mathbf{H}_D\right) \mathbf{M}_D^{-1}$ of the reduced system of DE is then a *strictly triangular matrix* at order ϵ^0 and the system becomes effectively uncoupled.
- Problem: In very few specific cases, $\sim C\, x^{-2+\beta_i \epsilon}$ appears in the matrix $\tilde{\mathbf{H}}$,
- Solution: $x \to 1/x$ back to $x^{-1+\beta_i \epsilon}$ in the *inhomogeneous part* of the DE.

The Simplified Differential Equations Approach

When the DE are coupled

$$\partial_x \vec{G}_{m+1} = \mathbf{H}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m+1} + \sum_{m' \geq m_0}^{m} \mathbf{R}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m'},$$

- $\mathbf{M}_D \,:\, \partial_x \mathbf{M}_D = -\mathbf{M}_D \mathbf{H}_D$, where $\mathbf{H}_D$ is the diagonal part of $\mathbf{H}$.
- $\tilde{\mathbf{H}} =: \mathbf{M}_D \left(\mathbf{H} - \mathbf{H}_D\right) \mathbf{M}_D^{-1}$ of the reduced system of DE is then a *strictly triangular matrix* at order ϵ^0 and the system becomes effectively uncoupled.
- Problem: In very few specific cases, $\sim C\, x^{-2+\beta_i \epsilon}$ appears in the matrix $\tilde{\mathbf{H}}$,
- Solution: $x \to 1/x$ back to $x^{-1+\beta_i \epsilon}$ in the *inhomogeneous part* of the DE.

The Simplified Differential Equations Approach

When the DE are coupled

$$\partial_x \vec{G}_{m+1} = \mathbf{H}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m+1} + \sum_{m' \geq m_0}^{m} \mathbf{R}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m'},$$

- $\mathbf{M}_D$: $\partial_x \mathbf{M}_D = -\mathbf{M}_D \mathbf{H}_D$, where $\mathbf{H}_D$ is the diagonal part of $\mathbf{H}$.
- $\tilde{\mathbf{H}} =: \mathbf{M}_D \left(\mathbf{H} - \mathbf{H}_D\right) \mathbf{M}_D^{-1}$ of the reduced system of DE is then a *strictly triangular matrix* at order ϵ^0 and the system becomes effectively uncoupled.
- Problem: In very few specific cases, $\sim C\, x^{-2+\beta_i \epsilon}$ appears in the matrix $\tilde{\mathbf{H}}$,
- Solution: $x \to 1/x$ back to $x^{-1+\beta_i \epsilon}$ in the *inhomogeneous part* of the DE.

The Simplified Differential Equations Approach

When the DE are coupled

$$\partial_x \vec{G}_{m+1} = \mathbf{H}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m+1} + \sum_{m' \geq m_0}^{m} \mathbf{R}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m'},$$

- $\mathbf{M}_D$: $\partial_x \mathbf{M}_D = -\mathbf{M}_D \mathbf{H}_D$, where $\mathbf{H}_D$ is the diagonal part of $\mathbf{H}$.
- $\tilde{\mathbf{H}} =: \mathbf{M}_D \left(\mathbf{H} - \mathbf{H}_D\right) \mathbf{M}_D^{-1}$ of the reduced system of DE is then a *strictly triangular matrix* at order ϵ^0 and the system becomes effectively uncoupled.
- Problem: In very few specific cases, $\sim C\, x^{-2+\beta_i \epsilon}$ appears in the matrix $\tilde{\mathbf{H}}$,
- Solution: $x \to 1/x$ back to $x^{-1+\beta_i \epsilon}$ in the *inhomogeneous part* of the DE.

The Simplified Differential Equations Approach

When the DE are coupled

$$\partial_x \vec{G}_{m+1} = \mathbf{H}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m+1} + \sum_{m' \geq m_0}^{m} \mathbf{R}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m'},$$

- $\mathbf{M}_D \,:\, \partial_x \mathbf{M}_D = -\mathbf{M}_D \mathbf{H}_D$, where $\mathbf{H}_D$ is the diagonal part of $\mathbf{H}$.
- $\tilde{\mathbf{H}} =: \mathbf{M}_D \left(\mathbf{H} - \mathbf{H}_D\right) \mathbf{M}_D^{-1}$ of the reduced system of DE is then a *strictly triangular matrix* at order ϵ^0 and the system becomes effectively uncoupled.
- Problem: In very few specific cases, $\sim C\, x^{-2+\beta_i \epsilon}$ appears in the matrix $\tilde{\mathbf{H}}$,
- Solution: $x \to 1/x$ back to $x^{-1+\beta_i \epsilon}$ in the *inhomogeneous part* of the DE.

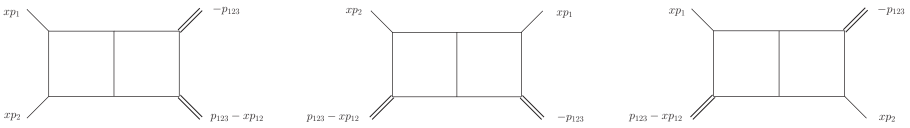


Figure : The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

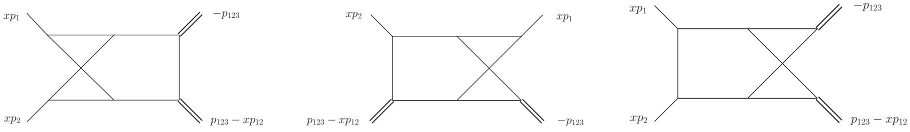


Figure : The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

Two-loop, four-point, two off-shell legs

$$p(q_1)p'(q_2) \to V_1(-q_3)V_2(-q_4), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = M_4^2.$$

Two-loop, four-point, two off-shell legs

$$p(q_1)p'(q_2) \to V_1(-q_3)V_2(-q_4), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = M_4^2.$$

$$q_1 = xp_1, \quad q_2 = xp_2, \quad q_3 = p_{123} - xp_{12}, \quad q_4 = -p_{123}, \quad p_i^2 = 0,$$
$$s_{12} := p_{12}^2, \qquad s_{23} := p_{23}^2, \qquad q := p_{123}^2,$$

Two-loop, four-point, two off-shell legs

$$p(q_1)p'(q_2) \to V_1(-q_3)V_2(-q_4), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = M_4^2.$$

$$q_1 = xp_1, \quad q_2 = xp_2, \quad q_3 = p_{123} - xp_{12}, \quad q_4 = -p_{123}, \quad p_i^2 = 0,$$
$$s_{12} := p_{12}^2, \qquad s_{23} := p_{23}^2, \qquad q := p_{123}^2,$$

$$S = (q_1 + q_2)^2 \qquad T = (q_1 + q_3)^2$$

$$S = s_{12}x^2, \quad T = q - (s_{12} + s_{23})x, \quad M_3^2 = (1-x)(q - s_{12}x), \quad M_4^2 = q.$$

$$U = (q_1 + q_4)^2 : \; S + T + U = M_3^2 + M_4^2.$$

Triangle rule:

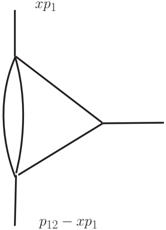


Figure : Required parametrization for off mass-shell triangles after possible pinching of internal line(s).

Two-loop, four-point, two off-shell legs

Planar topologies

$$G^{P_{12}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int\frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-xp_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{P_{13}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int\frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{P_{23}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int\frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+p_{123}-xp_2)^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-p_1)^{2a_6}(k_2+xp_2-p_{123})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

Two-loop, four-point, two off-shell legs

Planar topologies

P_{12} : {010000011, 001010001, 001000011, 100000011, 101010010, 101010100, 101000110, 010010101, 101000011, 101000012, 100000111, 100000112, 001010011, 001010012, 010000111, 010010011, 101010110, 111000011, 101000111, 101010011, 011010011, 011010012, 110000111, 110000112, 010010111, 010010112, 111010011, 111000111, 111010111, 111m10111, 11101m111},

P_{13} : {000110001, 001000011, 001010001, 001101010, 001110010, 010000011, 010101010, 010110010, 001001011, 001010011, 001010012, 001011011, 001101001, 001101011, 001110001, 001110002, 001110011, 001111001, 001111011, 001211001, 010010011, 010110001, 010110011, 011010011, 011010021, 011110001, 011110011, 011111011, m11111011},

P_{23} : {001010001, 001010011, 010000011, 010000101, 010010011, 010010101, 010010111, 011000011, 011010001, 011010010, 011010011, 011010012, 011010100, 011010101, 011010111, 011020011, 012010011, 021010011, 100000011, 101000011, 101010010, 101010011, 101010100, 110000111, 111000011, 111010011, 111010111, 111m10111}.

Two-loop, four-point, two off-shell legs

Non-planar topologies

$$G^{N12}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_2)^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{N13}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2-xp_{12})^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_1)^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{N34}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_{12}-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}.$$

Non-planar topologies

N_{12} : {100001010, 000110010, 000110001, 000101010, 000101001, 101010010, 100110010, 100101020, 100101010, 100101001, 001110010, 001110002, 001110001, 001101001, 101110020, 101110010, 101101002, 101101001, 100111020, 100111010, 100102011, 100101011, 001120011, 001111002, 001111001, 001110011, 000111011, 101011011, 100111011, 1m0111011, 001111011, 0m1111011, 101111011, 1m1111011, 1m1111m11},

N_{13} : {010000110, 000110010, 001000101, 001000110, 001010001, 010110100, 001110100, 001010102, 001110002, 000110110, 001010101, 001010110, 001100110, 001110001, 001110010, 010100110, 010110101, 002010111, 001120011, 001210110, 011010102, 001110120, 001010111, 001110210, 001110011, 001110101, 001110110, 002110110, 011000111, 011010101, 011100110, 011110001, 011110110, m11010111, 010110111, m01110111, 0m1110111, 00111m111, 001110111, 011010111, 011110101, 011110111, m11110111},

N_{34} : {001001010, 001010010, 010010010, 100000110, 100010010, 000010111, 010010110, 001010102, 001010101, 010010101, 001020011, 010000111, 001010011, 010010011, 101010020, 101010010, 101010100, 101000011, 110010120, 110010110, 010010112, 010010121, 010010111, 010020111, 020010111, 011010102, 001010111, 011010101, 110000211, 011020011, 110000111, 011010011, 111000101, 111010010, 101010101, 101010011, 111010110, 111010101, 101010111, 11m010111, 110m10111, 11001m111, 110010111, m11010111, 011m10111, 01101m111, 011010111, 111000111, 111010011, 111010111, 111m10111}.

GP-indices

$$I(P_{12}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12}}{q}, \frac{q}{q-s_{23}}, 1-\frac{s_{23}}{q}, 1+\frac{s_{23}}{s_{12}}, \frac{s_{12}}{s_{12}+s_{23}}\right\},$$

$$I(P_{13}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12}+s_{23}}{s_{12}}, \frac{q}{q-s_{23}}, \xi_-, \xi_+, \frac{q(q-s_{23})}{q^2-(q+s_{12})s_{23}}\right\},$$

$$I(P_{23}) = \left\{0, 1, \frac{q}{s_{12}}, 1+\frac{s_{23}}{s_{12}}, \frac{q}{q-s_{23}}, \frac{q}{s_{12}+s_{23}}, \frac{q-s_{23}}{s_{12}}\right\},$$

$$\xi_\pm = \frac{qs_{12} \pm \sqrt{qs_{12}s_{23}(-q+s_{12}+s_{23})}}{qs_{12}-s_{12}s_{23}}.$$

$$I(N_{12}) = I(P_{23}),$$

$$I(N_{34}) = I(P_{12}) \cup I(P_{23}) \cup \left\{\frac{s_{12}}{q-s_{23}}, \frac{s_{12}+s_{23}}{q}, \frac{q^2-qs_{23}-s_{12}s_{23}}{s_{12}(q-s_{23})}, \frac{s_{12}^2+qs_{23}+s_{12}s_{23}}{s_{12}(s_{12}+s_{23})}\right\},$$

$$I(N_{13}) = I(P_{23}) \cup \left\{\xi_-, \xi_+, 1+\frac{q}{s_{12}}+\frac{q}{-q+s_{23}}\right\}.$$

Example

$$G^{P_{13}}_{011111011}(x,s,\epsilon)=\frac{A_3(\epsilon)}{x^2 s_{12}(-q+x(q-s_{23}))^2}\left\{\frac{-1}{2\epsilon^4}+\frac{1}{\epsilon^3}\left(-GP\left(\frac{q}{s_{12}};x\right)+2\,GP\left(\frac{q}{q-s_{23}};x\right)\right.\right.$$

$$\left.+2\;GP(0;x)-GP(1;x)+\log\left(-s_{12}\right)+\frac{9}{4}\right)+\frac{1}{4\epsilon^2}\left(18\;GP\left(\frac{q}{s_{12}};x\right)-36\;GP\left(\frac{q}{q-s_{23}};x\right)\right.$$

$$\left.-8\;GP\left(0,\frac{q}{s_{12}};x\right)+16\;GP\left(0,\frac{q}{q-s_{23}};x\right)+8\;GP\left(\frac{s_{23}}{s_{12}}+1,\frac{q}{q-s_{23}};x\right)+\cdots\right)$$

$$+\frac{1}{\epsilon}\left(9\left(GP\left(0,\frac{q}{s_{12}};x\right)+GP(0,1;x)\right)-4\left(GP\left(0,0,\frac{q}{s_{12}};x\right)+GP(0,0,1;x)\right)+\cdots\right)$$

$$\left.+6\left(GP\left(0,0,1,\xi_-;x\right)+GP\left(0,0,1,\xi_+;x\right)\right)-2\,GP\left(0,0,\frac{q}{q-s_{23}},\frac{q\left(q-s_{23}\right)}{q^2-s_{23}\left(q+s_{12}\right)};x\right)+\cdots\right\}.$$

$$A_3(\epsilon)=-e^{2\gamma_E\epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{\Gamma(3-3\epsilon)}.$$

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072

Physical region

$$S > \left(\sqrt{M_3^2} + \sqrt{M_4^2}\right)^2, \quad T < 0, \quad U < 0,$$

$$M_3^2 > 0, \quad M_4^2 > 0, \quad q_\perp^2 = \frac{TU - M_3^2 M_4^2}{S} > 0,$$

$$x > 1, \quad \frac{q - s_{12}}{s_{23}} > 1, \quad x s_{12} > q, \quad q > 0.$$

$$x > 1, \quad \begin{cases} s_{23} < 0, & s_{12} + s_{23} > q, \quad q > 0 \\ s_{23} > 0, & s_{12} + s_{23} < q, \quad s_{12} > q/x. \end{cases}$$

Analytic continuation

- Feynman propagator

$$D \to D + i\epsilon$$

Two-loop, four-point, two off-shell legs

Analytic continuation

- Feynman propagator

$$D \to D + i\epsilon$$

s_{ij} (s_{12}, s_{23} and q in the present study) and the parameter x,
$s_{ij} \to s_{ij} + i\delta_{s_{ij}}\eta$, $x \to x + i\delta_x\eta$, with $\eta \to 0$.

Analytic continuation

- Feynman propagator

$$D \to D + i\epsilon$$

s_{ij} (s_{12}, s_{23} and q in the present study) and the parameter x,
$s_{ij} \to s_{ij} + i\delta_{s_{ij}}\eta$, $x \to x + i\delta_x \eta$, with $\eta \to 0$.

- $\delta_{s_{ij}}$ and δ_x are determined as follows:

1) Input data:

$$G^{P12}_{001000011} \sim \left(-(-1+x)(-q+s_{12}x)\right)^{1-2\epsilon} \sim (1-x)^{1-2\epsilon}\left(1 - xs_{12}/q\right)^{1-2\epsilon}(-q)^{1-2\epsilon}$$

Two-loop, four-point, two off-shell legs

Analytic continuation

- Feynman propagator

$$D \to D + i\epsilon$$

s_{ij} (s_{12}, s_{23} and q in the present study) and the parameter x,
$s_{ij} \to s_{ij} + i\delta_{s_{ij}}\eta$, $x \to x + i\delta_x\eta$, with $\eta \to 0$.

- $\delta_{s_{ij}}$ and δ_x are determined as follows:

1) Input data:

$$G^{P12}_{001000011} \sim (-(-1+x)(-q+s_{12}x))^{1-2\epsilon} \sim (1-x)^{1-2\epsilon}\,(1-xs_{12}/q)^{1-2\epsilon}\,(-q)^{1-2\epsilon}$$

2) Second graph polynomial: $\mathcal{F}$

C. Bogner and S. Weinzierl, Int. J. Mod. Phys. A **25** (2010) 2585 [arXiv:1002.3458 [hep-ph]].

in terms of s_{ij} and x, should acquire a definite-negative imaginary part in the limit $\eta \to 0$.

Two-loop, four-point, two off-shell legs

Analytic continuation

- Feynman propagator

$$D \to D + i\epsilon$$

s_{ij} (s_{12}, s_{23} and q in the present study) and the parameter x,
$s_{ij} \to s_{ij} + i\delta_{s_{ij}}\eta$, $x \to x + i\delta_x\eta$, with $\eta \to 0$.

- $\delta_{s_{ij}}$ and δ_x are determined as follows:

1) Input data:

$$G^{P12}_{001000011} \sim (-(-1+x)(-q+s_{12}x))^{1-2\epsilon} \sim (1-x)^{1-2\epsilon}(1-xs_{12}/q)^{1-2\epsilon}(-q)^{1-2\epsilon}$$

2) Second graph polynomial: $\mathcal{F}$

C. Bogner and S. Weinzierl, Int. J. Mod. Phys. A **25** (2010) 2585 [arXiv:1002.3458 [hep-ph]].

in terms of s_{ij} and x, should acquire a definite-negative imaginary part in the limit $\eta \to 0$.

E. Panzer, Comput. Phys. Commun. **188** (2014) 148 [arXiv:1403.3385 [hep-th]].

5BOX - ONE LEG OFF-SHELL: ALL FAMILIES

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1511.09404 [hep-ph].

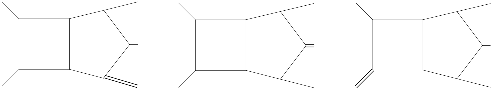


FIGURE : The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

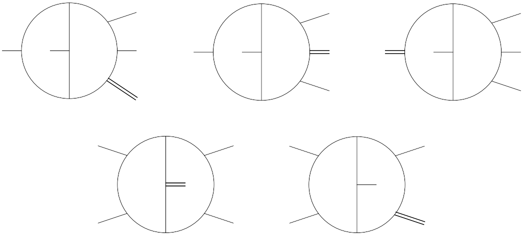


FIGURE : The five non-planar families with one external massive leg.

$$p(q_1)p'(q_2) \to V(q_3)j_1(q_4)j_2(q_5), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = q_5^2 = 0.$$

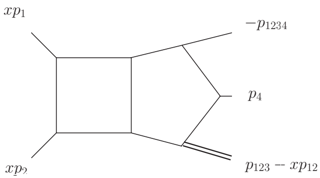


FIGURE : The parametrization of external momenta in terms of x for the planar pentabox of the family P_1. All external momenta are incoming.

$$s_{12} := p_{12}^2, \quad s_{23} := p_{23}^2, \quad s_{34} := p_{34}^2, \quad s_{45} := p_{45}^2 = p_{123}^2, \quad s_{51} := p_{15}^2 = p_{234}^2,$$

$$M_3^2 = (1-x)(s_{45} - s_{12}x).$$

5BOX - ONE LEG OFF-SHELL: P1

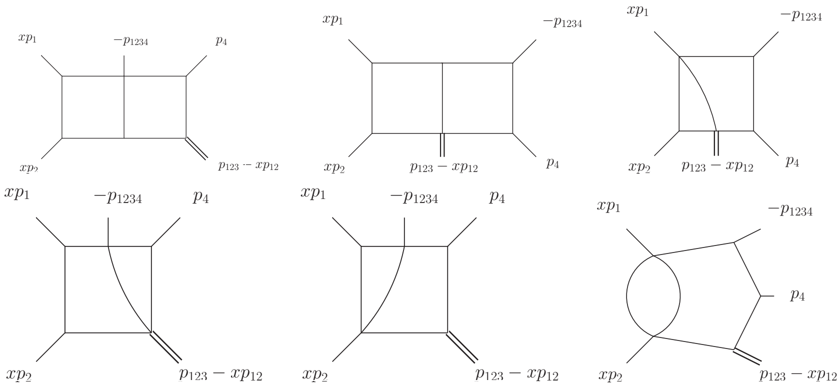


FIGURE : The five-point Feynman diagrams, besides the pentabox itself in Figure **??**, that are contained in the family P_1. All external momenta are incoming.

$$G^{P_1}_{a_1\cdots a_{11}}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$

$$\times\frac{1}{(k_1+p_{1234})^{2a_5}k_2^{2a_6}(k_2-xp_1)^{2a_7}(k_2-xp_{12})^{2a_8}(k_2-p_{123})^{2a_9}(k_2-p_{1234})^{2a_{10}}(k_1+k_2)^{2a_{11}}},$$

P_1 : {10000000101, 01000000101, 00100000101, 10000001001, 01000000011, 00100000011, 10100001100, 10100001010, 10100101000, 01000101001, 10100100100, 10100000102, 10100000101, 10100000011, 10000001102, 10000001101, 10000001011, 01000100101, 01000001101, 01000001011, 00100100102, 00100100101, 11100000101, 11100000011, 11000001102, 11000001101, 11000001012, 11000001011, 11000000111, 10100000112, 10000001111, 01100100102, 01100100101, 01100100011, 01100000111, 01000101102, 01000101101, 01000101011, 01000100111, 01000001111, 00100100111, 10100101100, 10100100101, 10100001101, 10100001011, 10100000111, 111m0000111, 110000m1111, 11000001111, 10100101110, 10100100111, 10100001111, 011001m0111, 01100100111, 010m0101111, 01000101111, 11100100101, 11100001101, 11100001011, 11100000111, 111m0101101, 111001m1101, 11100101101, 1110m101011, 11100101011, 111m0100111, 11100100111, 111000m1111, 111m0001111, 11100001111, 111001m0111, 11100101111, 111001m1111, 111m0101111},

Choosing m= -1 or 2

5box P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\left\{s_{ij}\right\}, \varepsilon, x\right) \mathbf{G}$$

$$\left(M_D\right)_{IJ} = \delta_{IJ} M_{II}\left(\varepsilon = 0\right), I, J = 1 \ldots 74$$

$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G}$, $\mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right)$ and $\mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}$.

$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{\left(x - l_i\right)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\begin{aligned}
\Delta_1 &= (s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2+4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})\\
\Delta_2 &= (s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2-4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})\\
\Delta_3 &= -(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45}))
\end{aligned}$$

5box P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right)\mathbf{G}$$

$$\left(M_D\right)_{IJ} = \delta_{IJ}\, M_{II}\left(\varepsilon = 0\right), I, J = 1\ldots 74$$

$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G}$, $\mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right)$ and $\mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}$.

$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\Delta_1 = (s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2 + 4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})$$
$$\Delta_2 = (s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2 - 4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})$$
$$\Delta_3 = -(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45}))$$

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right)\mathbf{G}$$

$$\left(M_D\right)_{IJ} = \delta_{IJ}\, M_{II}\left(\varepsilon = 0\right), I, J = 1 \dots 74$$

$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G}$, $\mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right)$ and $\mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}$.

$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{\left(x - l_i\right)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}+s_{34}-s_{51}\right)},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\Delta_1 = \left(s_{12}\left(s_{51}-s_{23}\right)+s_{23}s_{34}+s_{45}\left(s_{51}-s_{34}\right)\right)^2+4s_{12}s_{45}s_{51}\left(s_{23}+s_{34}-s_{51}\right)$$
$$\Delta_2 = \left(s_{12}\left(-s_{23}+s_{45}+s_{51}\right)+s_{23}s_{34}+s_{45}\left(s_{51}-s_{34}\right)\right)^2-4s_{12}s_{45}s_{51}\left(-s_{23}+s_{45}+s_{51}\right)$$
$$\Delta_3 = -\left(s_{12}s_{34}s_{45}\left(s_{12}-s_{34}-s_{45}\right)\right)$$

5box P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\left\{s_{ij}\right\}, \varepsilon, x\right) \mathbf{G}$$

$$\left(M_D\right)_{IJ} = \delta_{IJ}\, M_{II}\left(\varepsilon = 0\right), I, J = 1 \dots 74$$

$$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G},\ \mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right) \text{ and } \mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}.$$

$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{\left(x - l_i\right)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}+s_{34}-s_{51}\right)},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\begin{aligned}
\Delta_1 &= \left(s_{12}\left(s_{51}-s_{23}\right)+s_{23}s_{34}+s_{45}\left(s_{51}-s_{34}\right)\right)^2+4s_{12}s_{45}s_{51}\left(s_{23}+s_{34}-s_{51}\right)\\
\Delta_2 &= \left(s_{12}\left(-s_{23}+s_{45}+s_{51}\right)+s_{23}s_{34}+s_{45}\left(s_{51}-s_{34}\right)\right)^2-4s_{12}s_{45}s_{51}\left(-s_{23}+s_{45}+s_{51}\right)\\
\Delta_3 &= -\left(s_{12}s_{34}s_{45}\left(s_{12}-s_{34}-s_{45}\right)\right)
\end{aligned}$$

5box P1 - DE

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\int_0^x dt \frac{1}{(t-a_n)^2}\mathcal{G}(a_{n-1},\ldots,a_1,t) \qquad \int_0^x dt\ t^m\ \mathcal{G}(a_{n-1},\ldots,a_1,t)$$

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\ G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\mathbf{G} \to (\mathbf{I}-\mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x\mathbf{K}_i - \mathbf{K}_i\mathbf{M})(\mathbf{I}-\mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$$\partial_x \mathbf{G} = \left(\varepsilon\sum_{a=1}^{19}\frac{\mathbf{M}_a}{(x-l_a)}\right)\mathbf{G}$$

5box P1 - DE

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\ G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})\,(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

5box P1 - DE

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon)$, $G_I \to n_I(\varepsilon) G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$\mathbf{M}(\varepsilon = 0)$ contains $(x - l_i)^{-2}$ and x^0

$$(\mathbf{K}_1)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon=0))_{IJ} & I,J \neq 69,74 \\ 0 & I,J = 69,74 \end{cases}$$

$$(\mathbf{K}_2)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon=0))_{IJ} & I,J \neq 74 \\ 0 & I,J = 74 \end{cases}$$

$$(\mathbf{K}_3)_{IJ} = \int dx (\mathbf{M}(\varepsilon=0))_{IJ}$$

M.A. Barkatou and E.Pflügel, Journal of Symbolic Computation, **44** (2009),1017

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x-l_a)} \right) \mathbf{G}$$

5box P1 - DE

Fuchsian
$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\ G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20} \sum_{j=1}^{2} \sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1} \sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j \right).$$

$$\mathbf{G} \to (\mathbf{I} - \mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M})(\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

5box P1 - solution

- Solution:

$$
\begin{aligned}
\mathbf{G} &= \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
&+ \varepsilon^0 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
&+ \varepsilon \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
&+ \varepsilon^2 \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)}\right. \\
&+ \left. \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}
$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$
\mathbf{G} \underset{x \to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}
$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

$\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a \varepsilon}$, n_a positive integers, $x \to l_a$.

5box P1 - solution

- Solution:

$$\begin{aligned}
\mathbf{G} &= \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
&+ \varepsilon^{0}\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
&+ \varepsilon\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
&+ \varepsilon^{2}\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)}\right. \\
&+ \left.\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$\mathbf{G} \underset{x \to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

  $\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a \varepsilon}$, n_a positive integers, $x \to l_a$.

5BOX - BOUNDARY TERMS

- Resummed

$$G_{res} = \lim_{x \to 0} G = \sum_j c_j x^{i_0 + j\epsilon} + d_j x^{i_0+1+j\epsilon} + \mathcal{O}(x^{i_0+2}),$$

- DE: using the above and equating terms $x^{i+j\epsilon}$, linear equations for c_i and d_i
- *bottom-up*: MI with homogeneous DE treated exactly
- MI needing special treatment (20)
  - Expansion by regions (11)

$$\{(10100000101), (10100000102), (11000001012), (11000001011), (01000101011), (10100100111), (10100001111), (111m0100111), (111000m1111), (11100001111), (111001m0111)\}.$$

  - Shifted boundary point (6)

$$\begin{array}{rl} \infty : & \{(10100000011), (10000001011), (11100000011), (01100100011), (10100100111)\} \\ (s_{12} - s_{34} + s_{51})/s_{12} : & \{(01000001011)\} \end{array}$$

  - Extraction from known integrals (3)

$$\begin{aligned} G_{11100001011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100100101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{11100101011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{111m0101011}(x, s_{12}, s_{34}, s_{51}) &= G_{111m0101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ s'_{12} = x^2 s_{12}, \quad s'_{23} = x s_{51}, \quad & s'_{45} = -x s_{12} + x s_{34} + x^2 s_{12}. \end{aligned} \tag{1}$$

5BOX - BOUNDARY TERMS

- Resummed

$$G_{res} = \lim_{x \to 0} G = \sum_j c_j x^{i_0 + j\epsilon} + d_j x^{i_0 + 1 + j\epsilon} + \mathcal{O}(x^{i_0+2}),$$

- DE: using the above and equating terms $x^{i+j\epsilon}$, linear equations for c_i and d_i
- *bottom-up*: MI with homogeneous DE treated exactly
- MI needing special treatment (20)
  - Expansion by regions (11)

$$\{(10100000101), (10100000102), (11000001012), (11000001011), (01000101011), (10100100111), (10100001111), (111m0100111), (111000m1111), (11100001111), (111001m0111)\}.$$

  - Shifted boundary point (6)

$$\begin{aligned} \infty : &\quad \{(10100000011), (10000001011), (11100000011), (01100100011), (10100100111)\} \\ (s_{12} - s_{34} + s_{51})/s_{12} : &\quad \{(01000001011)\} \end{aligned}$$

  - Extraction from known integrals (3)

$$\begin{aligned} G_{11100001011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100100101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{11100101011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{111m0101011}(x, s_{12}, s_{34}, s_{51}) &= G_{111m0101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ s'_{12} = x^2 s_{12}, \qquad s'_{23} = x s_{51}, &\qquad s'_{45} = -x s_{12} + x s_{34} + x^2 s_{12}. \end{aligned} \tag{1}$$

5box - boundary terms

- Resumed

$$G_{res} = \lim_{x \to 0} G = \sum_j c_j x^{i_0 + j\epsilon} + d_j x^{i_0+1+j\epsilon} + \mathcal{O}(x^{i_0+2}),$$

- DE: using the above and equating terms $x^{i+j\epsilon}$, linear equations for c_i and d_i
- *bottom-up*: MI with homogeneous DE treated exactly
- MI needing special treatment (20)
  - Expansion by regions (11)

$$\{(10100000101), (10100000102), (11000001012), (11000001011), (01000101011), (10100100111), (10100001111), (111m0100111), (111000m1111), (11100001111), (111001m0111)\}.$$

  - Shifted boundary point (6)

$$\begin{aligned} \infty : \quad & \{(10100000011), (10000001011), (11100000011), (01100100011), (10100100111)\} \\ (s_{12} - s_{34} + s_{51})/s_{12} : \quad & \{(01000001011)\} \end{aligned}$$

  - Extraction from known integrals (3)

$$\begin{aligned} G_{11100001011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100100101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{11100101011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{111m0101011}(x, s_{12}, s_{34}, s_{51}) &= G_{111m0101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ s'_{12} = x^2 s_{12}, \qquad s'_{23} = x s_{51}, & \qquad s'_{45} = -x s_{12} + x s_{34} + x^2 s_{12}. \end{aligned} \tag{1}$$

5BOX - BOUNDARY TERMS

- Resumed

$$G_{res} = \lim_{x\to 0} G = \sum_j c_j x^{i_0+j\epsilon} + d_j x^{i_0+1+j\epsilon} + \mathcal{O}(x^{i_0+2}),$$

- DE: using the above and equating terms $x^{i+j\epsilon}$, linear equations for c_i and d_i
- *bottom-up*: MI with homogeneous DE treated exactly
- MI needing special treatment (20)
  - Expansion by regions (11)

$$\{(10100000101), (10100000102), (11000001012), (11000001011), (01000101011), (10100100111), (10100001111), (111m0100111), (111000m1111), (11100001111), (111001m0111)\}.$$

  - Shifted boundary point (6)

$$\begin{aligned} \infty : \quad & \{(10100000011), (10000001011), (11100000011), (01100100011), (10100100111)\} \\ (s_{12} - s_{34} + s_{51})/s_{12} : \quad & \{(01000001011)\} \end{aligned}$$

  - Extraction from known integrals (3)

$$\begin{aligned} G_{11100001011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100100101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{11100101011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{111m0101011}(x, s_{12}, s_{34}, s_{51}) &= G_{111m0101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ s'_{12} = x^2 s_{12}, \qquad s'_{23} = x s_{51}, \qquad & s'_{45} = -x s_{12} + x s_{34} + x^2 s_{12}. \end{aligned} \tag{1}$$

5BOX - ON-SHELL

- $x = 1$ corresponds to l_2

$$\mathbf{G} = \sum_{n \geq -2} \varepsilon^n \sum_{i=0}^{n+2} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i (1-x)$$

- with $\mathbf{M}_2$ the residue matrix at $x = 1$ and
- $\mathbf{G}_{trunc} \equiv \mathbf{G}_{reg}(x = 1)$

$$\mathbf{G}_{x=1} = \left(\mathbf{I} + \frac{3}{2}\mathbf{M}_2 + \frac{1}{2}\mathbf{M}_2^2 \right) \mathbf{G}_{trunc}$$

- $x = 1$ corresponds to l_2
- with $\mathbf{M}_2$ the residue matrix at $x = 1$ and

$$\mathbf{c}_i^{(n)} = \mathbf{M}_2 \mathbf{c}_{i-1}^{(n-1)} \quad i \geq 1$$

$$\mathbf{G}_{reg} = \sum_{n \geq -2} \varepsilon^n \mathbf{c}_0^{(n)}.$$

characteristic polynomial: $x^{61}(1+x)^9(2+x)^4$

$$\mathbf{G} = \mathbf{G}_{reg} + \frac{\left((1-x)^{-2\varepsilon} - 1\right)}{(-2\varepsilon)} \mathbf{X} + \frac{\left((1-x)^{-\varepsilon} - 1\right)}{(-\varepsilon)} \mathbf{Y}$$

$$\mathbf{X} = \sum_{n \geq -1} \varepsilon^n \mathbf{X}^{(n)} \quad \mathbf{Y} = \sum_{n \geq -1} \varepsilon^n \mathbf{Y}^{(n)}.$$

$$(-1)^n {\mathbf{M}_2}^n = {\mathbf{M}_2}^2 \left(2^{n-1} - 1\right) + \mathbf{M}_2 \left(2^{n-1} - 2\right), \quad n \geq 1.$$

minimal polynomial: $x(x+1)(x+2)$

- $\mathbf{G}_{trunc} \equiv \mathbf{G}_{reg}(x = 1)$

$$\mathbf{G}_{x=1} = \left(\mathbf{I} + \frac{3}{2}\mathbf{M}_2 + \frac{1}{2}\mathbf{M}_2^2\right) \mathbf{G}_{trunc}$$

5BOX - ON-SHELL

- $x = 1$ corresponds to l_2

$$\mathbf{G} = \sum_{n \geq -2} \varepsilon^n \sum_{i=0}^{n+2} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i(1-x)$$

- with $\mathbf{M}_2$ the residue matrix at $x = 1$ and

$$\mathbf{G} = \mathbf{G}_{reg} + \frac{\left((1-x)^{-2\varepsilon} - 1\right)}{(-2\varepsilon)} \mathbf{X} + \frac{\left((1-x)^{-\varepsilon} - 1\right)}{(-\varepsilon)} \mathbf{Y}$$

- $\mathbf{G}_{trunc} \equiv \mathbf{G}_{reg}(x = 1)$

$$\mathbf{G}_{x=1} = \left(\mathbf{I} + \frac{3}{2}\mathbf{M}_2 + \frac{1}{2}\mathbf{M}_2^2\right) \mathbf{G}_{trunc}$$

5box - numerical checks

- $\mathcal{O}(3,000)$ GPs for all 74 MI
- Directly computed by using **GiNaC**
- All invariants negative Euclidean: perfect agreement with `SecDec`
- $\mathcal{O}(10)$ secs.

  `HyperInt` analytic extraction of imaginary parts before numerics: increasing efficiency by $\mathcal{O}(100)$
- Physical region awaiting tests for 5boxes. Direct timing $\mathcal{O}(1000)$ secs.

5BOX - NUMERICAL CHECKS

- $\mathcal{O}(3,000)$ GPs for all 74 MI
- Directly computed by using **GiNaC**

J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005) 177

- All invariants negative Euclidean: perfect agreement with `SecDec`
- $\mathcal{O}(10)$ secs.
  `HyperInt` analytic extraction of imaginary parts before numerics: increasing efficiency by $\mathcal{O}(100)$
- Physical region awaiting tests for 5boxes. Direct timing $\mathcal{O}(1000)$ secs.

5BOX - NUMERICAL CHECKS

- $\mathcal{O}(3,000)$ GPs for all 74 MI
- Directly computed by using **GiNaC**
- All invariants negative Euclidean: perfect agreement with `SecDec`
- $\mathcal{O}(10)$ secs.
  `HyperInt` analytic extraction of imaginary parts before numerics: increasing efficiency by $\mathcal{O}(100)$
- Physical region awaiting tests for 5boxes. Direct timing $\mathcal{O}(1000)$ secs.

5BOX - NUMERICAL CHECKS

- $\mathcal{O}(3,000)$ GPs for all 74 MI
- Directly computed by using **GiNaC**
- All invariants negative Euclidean: perfect agreement with `SecDec`
- $\mathcal{O}(10)$ secs.
  `HyperInt` analytic extraction of imaginary parts before numerics: increasing efficiency by $\mathcal{O}(100)$

E. Panzer, Comput. Phys. Commun. **188** (2014) 148

- Physical region awaiting tests for 5boxes. Direct timing $\mathcal{O}(1000)$ secs.

- $\mathcal{O}(3,000)$ GPs for all 74 MI
- Directly computed by using **GiNaC**
- All invariants negative Euclidean: perfect agreement with `SecDec`
- $\mathcal{O}(10)$ secs.
  `HyperInt` analytic extraction of imaginary parts before numerics: increasing efficiency by $\mathcal{O}(100)$
- Physical region awaiting tests for 5boxes. Direct timing $\mathcal{O}(1000)$ secs.

Discussion

1. SDE: proven reliable and efficient: evolving
2. IBP: better understanding
3. Complete massless MI with up to 8 denominators, at least 3 of-shell legs
4. Including internal masses
5. Feynman parametrization - MB vs DE: pros and cons
6. Integrand reduction at two loops: implementation

Discussion

1. SDE: proven reliable and efficient: evolving
2. IBP: better understanding
3. Complete massless MI with up to 8 denominators, at least 3 of-shell legs
4. Including internal masses
5. Feynman parametrization - MB vs DE: pros and cons
6. Integrand reduction at two loops: implementation

Discussion

1. SDE: proven reliable and efficient: evolving
2. IBP: better understanding
3. Complete massless MI with up to 8 denominators, at least 3 of-shell legs
4. Including internal masses
5. Feynman parametrization - MB vs DE: pros and cons
6. Integrand reduction at two loops: implementation

Discussion

1. SDE: proven reliable and efficient: evolving
2. IBP: better understanding
3. Complete massless MI with up to 8 denominators, at least 3 of-shell legs
4. Including internal masses
5. Feynman parametrization - MB vs DE: pros and cons
6. Integrand reduction at two loops: implementation

Discussion

1. SDE: proven reliable and efficient: evolving
2. IBP: better understanding
3. Complete massless MI with up to 8 denominators, at least 3 of-shell legs
4. Including internal masses
5. Feynman parametrization - MB vs DE: pros and cons
6. Integrand reduction at two loops: implementation

Discussion

1. SDE: proven reliable and efficient: evolving
2. IBP: better understanding
3. Complete massless MI with up to 8 denominators, at least 3 of-shell legs
4. Including internal masses
5. Feynman parametrization - MB vs DE: pros and cons
6. Integrand reduction at two loops: implementation

SUMMARY

1. Understanding QFT and provide precise calculations for analysis of experimental data
2. NLO revolution: plethora of highly automated codes/software
3. LHC physics benefits: unprecedented
4. Moving beyond NLO: NNLO and N3LO
5. NNLO revolution: ante portas ?

Summary

1. Understanding QFT and provide precise calculations for analysis of experimental data
2. NLO revolution: plethora of highly automated codes/software
3. LHC physics benefits: unprecedented
4. Moving beyond NLO: NNLO and N3LO
5. NNLO revolution: ante portas ?

Summary

1. Understanding QFT and provide precise calculations for analysis of experimental data
2. NLO revolution: plethora of highly automated codes/software
3. LHC physics benefits: unprecedented
4. Moving beyond NLO: NNLO and N3LO
5. NNLO revolution: ante portas ?

Summary

1. Understanding QFT and provide precise calculations for analysis of experimental data
2. NLO revolution: plethora of highly automated codes/software
3. LHC physics benefits: unprecedented
4. Moving beyond NLO: NNLO and N3LO
5. NNLO revolution: ante portas ?

Summary

1. Understanding QFT and provide precise calculations for analysis of experimental data
2. NLO revolution: plethora of highly automated codes/software
3. LHC physics benefits: unprecedented
4. Moving beyond NLO: NNLO and N3LO
5. NNLO revolution: ante portas ?